THE EFFECTIVE CONE OF THE KONTSEVICH MODULI SPACE

IZZET COSKUN, JOE HARRIS, AND JASON STARR

Abstract. In this paper we prove that the cone of effective divisors on the Kontsevich moduli spaces of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ stabilize when $r \geq d$. We give a complete characterization of the effective divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$: They are non-negative linear combinations of boundary divisors and the divisor of maps with degenerate image.

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1. Introduction

The ample and effective cones of divisors play a crucial role in the birational geometry of a variety. The study of these cones for the moduli spaces of stable curves has been especially fruitful leading to the proof that the moduli space of stable curves $\overline{M}_g$ is of general type when $g > 23$ (see [HM], [H], [EH]). Recently, inspired by the work of G. Farkas, D. Khosla and M. Popa, there has been renewed interest in constructing divisors of small slope on $\overline{M}_g$ in order to understand the effective cone of $\overline{M}_g$ and to determine the Kodaira dimension of $\overline{M}_g$ in the remaining cases (see [FaP], [Far]).

The aim of this paper is to describe the classes of effective divisors on a related moduli space, the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. For $d > 1$, the scheme parameterizing smooth, degree d, rational curves in $\mathbb{P}^r$ is not proper. The Kontsevich moduli space gives a useful compactification. For integers $n, d \geq 0$,

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2000 *Mathematics Subject Classification.* Primary:14D20, 14E99, 14H10.

During the preparation of this article the second author was partially supported by the NSF grant DMS-0200659. The third author was partially supported by the NSF grant DMS-0353692 and a Sloan Research Fellowship.

the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is a smooth, proper, Deligne-Mumford stack parameterizing the data $(C, (p_1, ..., p_n), f)$ of,

(i) C, a proper, connected, at-worst-nodal curve of arithmetic genus 0,
(ii) $p_1, \ldots, p_n$, an ordered sequence of distinct, smooth points of C,
(iii) and $f : C \to \mathbb{P}^r$, a morphism with $\deg(f^*\mathcal{O}_{\mathbb{P}^r}(1)) = d$ satisfying the following stability condition: every irreducible component of C mapped to a point under f contains at least 3 special points, i.e., marked points p_i and nodes of C.

In this paper we will determine the classes of all effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ when $r \geq d$.

In [Pa] R. Pandharipande proves that when $r \geq 2$, the divisor class $\mathcal{H}$, and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$ generate the group of $\mathbb{Q}$-Cartier divisors of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. We recall that

(1) $\mathcal{H}$ is the class of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$ (provided $r > 1$ and $d > 0$).
(2) $\Delta_{k,d-k}$, $1 \leq k \leq \lfloor d/2 \rfloor$, is the class of the boundary divisor consisting of maps with reducible domains, where the map has degree k on one component and degree $d - k$ on the other component.

The main problem we would like to address is the following:

Problem 1.1. Describe the cone of effective divisor classes on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ in terms of these generators of the Picard group.

Denote by P_d the $\mathbb{Q}$-vector space of dimension $\lfloor d/2 \rfloor + 1$ with basis labeled $\mathcal{H}$ and $\Delta_{k,d-k}$ for $k = 1, \ldots, \lfloor d/2 \rfloor$. For each $r \geq 2$, there is a $\mathbb{Q}$-linear map

$$u_{d,r} : P_d \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$$

that is an isomorphism of $\mathbb{Q}$-vector spaces.

Definition 1.2. For every integer $r \geq 2$, denote by $\mathrm{Eff}_{d,r} \subset P_d$ the inverse image under $u_{d,r}$ of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

A more precise version of Problem 1.1 is to describe $\mathrm{Eff}_{d,r}$. A first result is that for a fixed degree d, there is an inclusion between these cones as r increases. Furthermore, the cones stabilize for $r \geq d$.

Proposition 1.3. *For every integer $r \geq 2$, $\mathit{Eff}_{d,r}$ is contained in $\mathit{Eff}_{d,r+1}$. For every integer $r \geq d$, $\mathit{Eff}_{d,r}$ equals $\mathit{Eff}_{d,d}$.*

In view of Proposition 1.3 it is especially interesting to understand $\mathrm{Eff}_{d,d}$. Most of our paper will concentrate on this case.

The crudest invariant one can associate to the effective cone is the slope of distinguished rays. For example, Harris and Morrison in [HMo] define the slope of $\overline{M}_g$ as the slope of the ray that bounds the effective cone in the subspace spanned by the Hodge class λ and the total boundary class δ. Determining the slope of $\overline{M}_g$ is a major open problem. In analogy with the case of $\overline{M}_g$, we define the slope $s(r, d)$ of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ as follows.

$$s(r,d) := \sup_{\alpha} \{ \ \alpha \ : \ \mathcal{H} - \alpha \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k) \ \Delta_{k,d-k} \ \text{ is on the effective cone } \}.$$

It is possible to determine the slope for the Kontsevich moduli spaces in the stable range.

Theorem 1.4. *If $r \geq d$, then the slope $s(r,d)$ of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is equal to*

$$s(r,d) = \frac{1}{d+1}.$$

When $r = d$, the effective divisor that achieves the extremal slope has a simple description. Let D_{deg} denote the locus parameterizing stable maps $f : C \to \mathbb{P}^d$ of degree d whose set theoretic image does not span $\mathbb{P}^d$. D_{deg} is a divisor in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ and has the desired slope.

D_{deg} plays a crucial role in describing the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. The following theorem, which describes the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ completely, is the main theorem of our paper.

Theorem 1.5. *The class of a divisor lies in the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ if and only if it is a non-negative linear combination of the class of D_{deg} and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.*

Theorem 1.4 follows immediately from Theorem 1.5. However, since it is easy to give an independent proof and since the curves that span the null-space of the divisor D_{deg} are interesting in their own right, we will give a simple proof of it in §2. Combining Theorem 1.5 with Proposition 1.3 and Lemma 2.1 we obtain the following corollary.

Corollary 1.6. *When $r \geq d$, the class of a divisor lies in the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ if and only if it is a non-negative linear combination of the class*

$$\mathcal{H} - \frac{1}{d+1} \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}$$

and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.

The space of curves of a given degree and genus has many distinguished subvarieties defined by imposing geometric conditions on the curves. Examples of such subvarieties are given by curves that have an unexpected secant linear space or curves with an unexpected osculating linear space or curves with a point of unexpected ramification sequence. An informal way of restating Theorem 1.5 is to say that "geometric conditions" do not give new divisors on the space of rational curves of degree d in $\mathbb{P}^d$. Rational normal curves are too predictable.

We now briefly outline the proof of Theorem 1.5. Since D_{deg} and the boundary divisors are effective, any non-negative rational linear combination of these divisors lies in the effective cone. The main content of the theorem is to show that there are no other effective divisor classes.

Definition 1.7. A reduced, irreducible curve C on a scheme X is a *moving curve* if the deformations of C cover a Zariski open subset of X. More precisely, a curve C is a moving curve if there exists a flat family of curves $\pi : \mathcal{C} \to T$ on X such that $\pi^{-1}(t_0) = C$ for $t_0 \in T$ and for a Zariski open subset $U \subset X$ every point $x \in U$ is contained in $\pi^{-1}(t)$ for some $t \in T$. We call the class of a moving curve a *moving curve class.*

An obvious observation is that the intersection pairing between the class of an effective divisor and a moving curve class is always positive. Intersecting divisors with a moving curve class gives an inequality for the coefficients of an effective divisor class. The strategy for the proof of Theorem 1.5 is to produce enough moving curves to force the effective divisor classes to be a non-negative linear combination of $D_{\deg}$ and the boundary classes.

Moving curves in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ are easy to recognize by the following lemma.

Lemma 1.8. *If $C \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is a reduced, irreducible curve that intersects the complement in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ of the boundary divisors and the divisor of maps whose image is degenerate, then C is a moving curve.*

Proof. The automorphism group of $\mathbb{P}^d$ acts transitively on rational normal curves. An irreducible curve of degree d that spans $\mathbb{P}^d$ is a rational normal curve. Hence, a curve $C \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ that intersects the complement in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ of the boundary divisors and the divisor of maps whose image is degenerate, contains a point that represents a map that is an embedding of $\mathbb{P}^1$ as a rational normal curve. The translations of C by $\mathbb{P}GL(d+1)$ cover a Zariski open set of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. $\square$

In §3 using certain linear systems on blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$ we will construct one-parameter families of rational curves whose general member is a rational normal curve. By Lemma 1.8 these will be moving curves in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. These moving curves will give us enough inequalities on the effective cone to deduce Theorem 1.5.

Finally, we remark that most of the discussion in this paper extends to spaces of stable maps to homogeneous varieties such as the Grassmannians, and flag varieties. In a forthcoming paper we will explain these generalizations [CHS].

Acknowledgments: We would like to thank A. J. de Jong, B. Hassett and R. Miranda for useful discussions.

2. Preliminaries

In this section we prove Proposition 1.3 and collect the basic facts that we will need about the divisor $D_{\deg}$.

2.1. **The stability of the effective cone.** In this subsection we prove that $\mathrm{Eff}_{d,r}$ is contained in $\mathrm{Eff}_{d,r+1}$ and that $\mathrm{Eff}_{d,r} = \mathrm{Eff}_{d,d}$ for $r \geq d$. Recall that $\mathrm{Eff}_{d,r}$ is the image of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ when the rational Picard group of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is identified with the vector space that has a basis labeled by $\mathcal{H}$ and $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.

Proof of Proposition 1.3. Let $p \in \mathbb{P}^{r+1}$ be a point, denote $U = \mathbb{P}^{r+1} - \{p\}$, and let $\pi : U \to \mathbb{P}^r$ be a linear projection from p. This induces a smooth 1-morphism

$$\overline{\mathcal{M}}_{0,0}(\pi, d) : \overline{\mathcal{M}}_{0,0}(U, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d).$$

Let $i : U \to \mathbb{P}^{r+1}$ be the open immersion. This induces a 1-morphism

$$\overline{\mathcal{M}}_{0,0}(i,d) : \overline{\mathcal{M}}_{0,0}(U,d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1},d)$$

relatively representable by open immersions. The complement of the image of $\overline{\mathcal{M}}_{0,0}(i,d)$ has codimension r, which is greater than 2. Therefore, the pull-back morphism

$$\overline{\mathcal{M}}_{0,0}(i,d)^* : \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1},d)) \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(U,d))$$

is an isomorphism. So there is a unique homomorphism

$$h : \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r},d)) \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1},d))$$

such that

$$\overline{\mathcal{M}}_{0,0}(\pi,d)^* = \overline{\mathcal{M}}_{0,0}(i,d)^* \circ h.$$

Recalling from the introduction that $u(r,d)$ is the map that identifies the Picard group of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d))$ with the vector space spanned by $\mathcal{H}$ and the boundary divisors $\Delta_{k,d-k}$, we see that $h \circ u_{d,r}$ equals $u_{d,r+1}$. So to prove $\mathrm{Eff}_{d,r}$ is contained in $\mathrm{Eff}_{d,r+1}$, it suffices to prove that $\overline{\mathcal{M}}_{0,0}(\pi,d)$ pulls back effective divisors to effective divisors classes, which follows since $\overline{\mathcal{M}}_{0,0}(\pi,d)$ is smooth.

Next assume $r \geq d$. Let D be any effective divisor in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)$. A general point in the complement of D parameterizes a stable map $f : C \to \mathbb{P}^r$ such that $f(C)$ spans a d-plane. Denote by $j : \mathbb{P}^d \to \mathbb{P}^r$ a linear embedding whose image is this d-plane. There is an induced 1-morphism

$$\overline{\mathcal{M}}_{0,0}(j,d) : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d).$$

The map $\overline{\mathcal{M}}_{0,0}(j,d)^* \circ u_{d,r}$ equals $u_{d,d}$. By construction, $\overline{\mathcal{M}}_{0,0}(j,d)^*([D])$ is the class of the effective divisor $\overline{\mathcal{M}}_{0,0}(j,d)^{-1}(D)$, i.e., $[D]$ is in $\mathrm{Eff}_{d,d}$. Thus $\mathrm{Eff}_{d,d}$ contains $\mathrm{Eff}_{d,r}$, which in turn contains $\mathrm{Eff}_{d,d}$ by the last paragraph. Therefore $\mathrm{Eff}_{d,r}$ equals $\mathrm{Eff}_{d,d}$. □

2.2. **The divisor class $D_{\mathbf{deg}}$.** In this subsection we determine the class of the divisor of degenerate maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)$. We then give a basis of moving curves that span the null-space of D_{deg} in the cone of curves. This completes the proof of Theorem 1.4.

Lemma 2.1. *The class D_{deg} equals*

$$D_{deg} = \frac{1}{2d}\left[(d+1)\mathcal{H} - \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}\right]. \tag{1}$$

Proof. We will prove the equality (1) by intersecting D_{deg} by test curves. Fix a general rational normal scroll of degree i and a general rational normal curve of degree $d-i-1$ intersecting the scroll in one point p. Consider the one-parameter family C_i of degree d curves consisting of the fixed degree $d-i-1$ rational normal curve union curves in a general pencil (that has p as a base-point) of degree $i+1$ rational normal curves on the scroll. When $2 \leq i \leq \lfloor d/2 \rfloor$, C_i has the following intersection numbers with $\mathcal{H}$ and D_{deg}.

$$C_i \cdot \mathcal{H} = i, \quad C_i \cdot D_{\mathrm{deg}} = 0.$$

The curve C_i is contained in the boundary divisor $\Delta_{i+1,d-i-1}$ and has intersection number

$$C_i \cdot \Delta_{i+1,d-i-1} = -1$$

with it. The intersection number of C_i with the boundary divisors $\Delta_{i,d-i}$ and $\Delta_{1,d-1}$ is non-zero and given as follows:

$$C_i \cdot \Delta_{i,d-i} = 1, \quad C_i \cdot \Delta_{1,d-1} = i + 1.$$

Finally, the intersection number of C_i with all the other boundary divisors is zero. When $i = 1$, we have to modify the intersection number of C_1 with $\Delta_{1,d-1}$ to read $C_1 \cdot \Delta_{1,d-1} = 3$. Next consider the one-parameter family B_1 of rational curves of degree d that contain $d+2$ general points and intersect a general line. The intersection number of B_1 with all the boundary divisors but $\Delta_{1,d-1}$ is zero. Clearly $B_1 \cdot D_{\mathrm{deg}} = 0$. By the algorithm for counting rational curves in projective space given in [V] it follows that

$$B_1 \cdot \mathcal{H} = \frac{d^2+d-2}{2}, \quad B_1 \cdot \Delta_{1,d-1} = \frac{(d+2)(d+1)}{2}.$$

This determines the class of D_{deg} up to a constant multiple. In order to determine the multiple, consider the curve C that consists of a fixed degree $d-1$ curve and a pencil of lines in a general plane intersecting the curve in one point. The curve C has intersection number zero with all the boundary divisors but $\Delta_{1,d-1}$ and has the following intersection numbers:

$$C \cdot \mathcal{H} = 1, \quad C \cdot D_{\mathrm{deg}} = 1, \quad C \cdot \Delta_{1,d-1} = -1.$$

The lemma follows from these intersection numbers. □

Consider the one-parameter family B_k of rational curves of degree d in $\mathbb{P}^d$ that contain $d+2$ general fixed points and intersect a general linear space $\mathbb{P}^k$ and a general linear space $\mathbb{P}^{d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$. When $k = 1$, we omit the linear space $\mathbb{P}^{d-1}$. A general member of B_k is a rational normal curve. This follows, for example, from Lemma 14 of [FP]. By Lemma 1.8 it follows that C_k is a moving curve for every k. The only reducible elements of C_1 are unions of curves of degree 1 and $d-1$. For $k > 1$, the only reducible curves contained in C_k have degrees $(1, d-1)$ or $(k, d-k)$. Since the $d+2$ points always span $\mathbb{P}^d$, $B_k \cdot D_{\mathrm{deg}} = 0$ for every k. It follows that the moving curves C_k span the null-space of D_{deg} in the cone of curves. Observe that these curves give a proof of Theorem 1.4.

Proof of Theorem 1.4. Since D_{deg} is an effective divisor class with slope $\frac{1}{d+1}$, the slope of $\mathrm{Eff}_{d,d}$ is at least $\frac{1}{d+1}$. On the other hand, there are moving curves that have intersection number zero with D_{deg}. It follows that the slope of $\mathrm{Eff}_{d,d}$ is at most $\frac{1}{d+1}$. □

2.3. **Open problem about the slope of $\overline{M}_g$.** Recall that the slope $s(g)$ of $\overline{M}_g$ is defined by

$$s(g) := \inf_{\alpha} \{ \ \alpha\lambda - \delta \ \text{ is on the effective cone } \},$$

where λ is the Hodge class and δ is the total boundary class. The slope of the moduli space of stable curves $\overline{M}_g$ is non-negative (see [HMo]). Currently, all known effective divisors on $\overline{M}_g$ have slope greater than 6. As the genus tends to infinity the slopes of the Brill-Noether divisors tend to 6 from above. Determining the slope, even giving a positive lower bound for it, is an important problem with applications to problems such as the Schottky problem and the Kodaira dimension of $\overline{M}_g$. One can give lower bounds on the slope by producing moving curves on $\overline{M}_g$. To the best of our knowledge, currently known moving curves give bounds on the slope that tend to zero with the genus.

The proof of Theorem 1.4 suggests a family of moving curves that might improve known bounds. Let C_g be the one-parameter family of canonical curves in $\mathbb{P}^{g-1}$ that contain the maximum possible number of general points and intersect general linear spaces such that the sum of the codimensions of all these linear spaces (including the points) add up to g^2+3g-5 plus the number of linear spaces. When $g \geq 8$, this amounts to considering canonical curves that contain $g+5$ general points and intersect a general $\mathbb{P}^{g-7}$.

For example, C_3 is the one-parameter family of genus 3 canonical curves that contain 13 general points in $\mathbb{P}^2$. It has intersection number zero with the divisor of hyperelliptic curves. C_4 is the one-parameter family of genus 4 canonical curves that contain 9 general points and intersect 5 general lines in $\mathbb{P}^3$. It has intersection number zero with the Petri divisor of curves whose canonical image lies on a singular quadric. C_5 is the one-parameter family of genus 5 canonical curves in $\mathbb{P}^4$ that contain 11 general points and intersect a general line. It has intersection number zero with the Brill-Noether divisor of trigonal curves. C_6 is the one-parameter family of genus 6 canonical curves in $\mathbb{P}^5$ that contain 11 general points and intersect a general line and a general plane. It has intersection number zero with the Petri divisor of curves that lie on a singular quintic Del Pezzo surface. When $g \leq 6$, the curves C_g give the sharp slope bound. The analogy with rational curves and these small-genus examples suggest that this family is well-worth studying. Unfortunately we do not know the intersection of C_g with the classes λ and δ in general. It would be interesting to determine these intersections.

3. The effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$

In this section we prove that every effective divisor class in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is a positive linear combination of D_{deg} and the boundary divisors. This proves Theorem 1.5.

Since D_{deg} and the boundary divisors are effective, any positive linear combination also is a class in the effective cone. In order to prove Theorem 1.5 we have to show that we can write the class of every effective divisor as

$$\alpha D_{\mathrm{deg}} + \sum_{k=1}^{\lfloor d/2 \rfloor} \beta_{k,d-k} \; \Delta_{k,d-k},$$

where α and $\beta_{k,d-k}$ are non-negative.

First, observe that if D is an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ and D has the class

$$a\mathcal{H} + \sum_{k=1}^{\lfloor d/2 \rfloor} b_{k,d-k} \Delta_{k,d-k},$$

then $a \geq 0$. Furthermore, if $a = 0$, then $b_{k,d-k} \geq 0$. Consider a general projection of the d-th Veronese embedding of $\mathbb{P}^2$ to $\mathbb{P}^d$. Consider the image of a pencil of lines in $\mathbb{P}^2$. By Lemma 1.8 this is a moving one-parameter family C of degree d rational curves that has intersection number zero with the boundary divisors. It follows from the inequality $C \cdot D \geq 0$ that $a \geq 0$.

Furthermore, suppose that $a = 0$. Consider a general pencil of $(1,1)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Take a general projection to $\mathbb{P}^d$ of the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ by the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i, d-i)$. By Lemma 1.8 the image of the pencil gives a moving one-parameter family C of degree d curves whose intersection with $\Delta_{k,d-k}$ is zero unless $k = i$. The relation $C \cdot D \geq 0$ implies that if $a = 0$, then $b_{i,d-i} \geq 0$. We conclude that Theorem 1.5 is true if $a = 0$. We can, therefore, assume that $a > 0$.

Suppose that for every $1 \leq i \leq \lfloor d/2 \rfloor$, we could construct a moving curve C_i in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ with the property that $C_i \cdot \Delta_{k,d-k} = 0$ for $k \neq i$ and that the ratio of $C_i \cdot \Delta_{i,d-i}$ to $C_i \cdot \mathcal{H}$ is given by

$$\frac{C_i \cdot \Delta_{i,d-i}}{C_i \cdot \mathcal{H}} = \frac{d+1}{i(d-i)}. \tag{2}$$

Observe that given these intersection numbers, Lemma 2.1 implies that $C_i \cdot D_{\mathrm{deg}} = 0$. Theorem 1.5 follows from the inequalities $C_i \cdot D \geq 0$.

In the rest of this section we will first give a construction of one-parameter families of C_i with these properties. However, our construction will depend on the Harbourne-Hirschowitz conjecture. We will then modify the construction to get a sequence of curves (not depending on any conjectures) that "approximate" these intersection numbers. These curves will suffice to conclude Theorem 1.5.

3.1. **Construction 1, depending on the Harbourne-Hirschowitz conjecture.** Let F_1 and F_2 denote the two fiber classes on $\mathbb{P}^1 \times \mathbb{P}^1$. We will abuse notation and denote the proper transform of the fibers in any blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ also by F_1 and F_2. Let d, j and k be positive integers subject to the condition that $2k \leq d$. Consider S the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in $j(d+1)$ general points $p_1, \ldots, p_{j(d+1)}$. Let E_i denote the i-th exceptional divisor lying over p_i. Let $L(j)$ be the following linear system on S:

$$L(j) = d\ F_1 + \frac{jk(k+1)}{2}\ F_2 - \sum_{i=1}^{j(d+1)} k\ E_i.$$

Suppose M is a linear system on S and that $M - F_2$ is non-special, that is

$$h^1(S, \mathcal{O}_S(M - F_2)) = 0.$$

Consider the exact sequence

$$0 \to \mathcal{O}_S(M - F_2) \to \mathcal{O}_S(M) \to \mathcal{O}_{F_2}(M) \to 0.$$

The long exact sequence of cohomology implies that taking the one-parameter family of proper transforms of the fiber class F_2 under the image of the linear system $|M|$ gives a one-parameter family of rational curves of degree $M \cdot F_2$ spanning $\mathbb{P}^{M\cdot F_2}$.

In particular, suppose that $L(j) - F_2$ is non-special. Then by the discussion in the previous paragraph, the linear system $L(j)$ embeds the general curve in the linear system $|F_2|$ on S as a rational normal curve of degree d in $\mathbb{P}^d$. We thus obtain a one-parameter family $C_k(j)$ that has intersection number zero with all the boundary classes except for $\Delta_{k,d-k}$. Moreover, $C_k(j) \cdot D_{\mathrm{deg}} = 0$. Hence, Theorem 1.5 would immediately follow if $L(j) - F_2$ were non-special for at least one value of j.

We recall that the celebrated conjecture due to Harbourne and Hirschowitz characterizes the linear systems that are special on a general blow-up of $\mathbb{P}^2$ as those linear systems that have a multiple (-1)-curve in their base locus. Here we will need a weaker form of the conjecture (see [CM]).

Conjecture 3.1 (Harbourne-Hirschowitz)**.** *Let M be a complete linear system on a general blow-up S of $\mathbb{P}^2$. If $E \cdot M$ is non-negative for every (-1)-curve E on S, then M is non-special.*

Since the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point is isomorphic to the blow-up of $\mathbb{P}^2$ at three points, the Harbourne-Hirschowitz Conjecture applies to the linear systems $L(j)$. The class of any (-1)-curve on S may be expressed as

$$\alpha F_1 + \beta F_2 - \sum_{i=1}^{j(d+1)} \gamma_i E_i,$$

where α, β and γ_i are non-negative integers. Since for a (-1)-curve E we have $E \cdot K = -1$, it follows that

$$-\sum_{i=1}^{j(d+1)} \gamma_i = 1 - 2\alpha - 2\beta.$$

The intersection of the (-1)-curve with $L(j)$ is

$$d\alpha + \beta\left(\frac{jk(k+1)}{2} - 1\right) - k\sum_{i=1}^{j(d+1)} \gamma_i = (d-2k)\beta + \alpha\left(\frac{jk(k+1)}{2} - 2k - 1\right) + k.$$

When $k > 3$ and $j \geq 1$; or $k = 2, 3$ and $j > 1$; or $k = 1$ and $j \geq 3$, the intersection is non-negative. We conclude the following:

Proposition 3.2. *Suppose Conjecture 3.1 holds for $L(j) - F_2$ for some $j \geq 1$. Then the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is spanned by the classes of D_{deg} and the boundary divisors. Furthermore, every codimension one face of the effective cone is the null-locus of a moving curve.*

Remark 3.3. It is easy to prove that $L(j) - F_2$ is non-special for small values of d and k and to deduce Proposition 3.2 without any conditions. However, we could not see how to prove the non-specialty of $L(j) - F_2$ in general.

3.2. **Construction 2, completing the proof.** We modify the previous construction by imposing fewer k-fold points on the linear system $d\ F_1 + \frac{jk(k+1)}{2} F_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$. If we do not impose too many k-fold points on the linear system, we can prove the non-specialty of the desired linear system. The following proposition makes this precise.

Proposition 3.4. *Let k, j and d be positive integers subject to the condition that $2k \leq d$. There exists an integer $n(k,d)$ depending only on k and d such that the linear system*

$$L'(j) = d\ F_1 + \left(\frac{jk(k+1)}{2} - 1\right)\ F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} k\ E_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $j(d+1) + n(k,d)\frac{(k-1)(k+2)}{2}$ general points is non-special for every $j >> 0$. The integer $n(k,d)$ may be taken to be

$$n(k,d) = \lceil 2(d+1)/k \rceil.$$

Proposition 3.4 implies Theorem 1.5. As in the previous subsection we consider the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in

$$j(d+1) + \frac{n(k,d)(k-1)(k+2)}{2}$$

general points. The proper transform of the fibers F_2 under the linear system

$$L'(j) = d\ F_1 + \frac{jk(k+1)}{2}\ F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} k\ E_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

gives a one-parameter family $C_k(j)$ of rational curves of degree d that has intersection number zero with D_{deg}. Letting j tend to infinity we obtain a sequence of moving curves $C_k(j)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ that has intersection zero with all the boundary divisors but $\Delta_{1,d-1}$ and $\Delta_{k,d-k}$. Unfortunately, the intersection of $C_k(j)$ with $\Delta_{1,d-1}$ is not zero and the ratio of $C_k(j) \cdot \mathcal{H}$ to $C_k(j) \cdot \Delta_{k,d-k}$ is not the one required by Equation (2). However, as j tends to infinity, the ratio of the intersection numbers $C_k(j) \cdot \Delta_{1,d-1}$ to $C_k(j) \cdot \mathcal{H}$ tends to zero and the ratio of $C_k(j) \cdot \Delta_{k,d-k}$ to $C_k(j) \cdot \mathcal{H}$ tends to the desired ratio $\frac{d+1}{k(d-k)}$. Theorem 1.5 follows.

Proof of Proposition 3.4. The specialization technique developed in §2 of [Ya] yields the proof of the proposition. We will specialize the points of multiplicity k one by one onto a point q. At each stage the k-fold point that we specialize will be in general position. We will first slide the point along a fiber f_1 in the class F_1 onto the fiber f_2 in the fiber class F_2 containing the point q. We then slide the point onto q along f_2. We will record the flat limit of this degeneration.

There is a simple checker game that describes the limits of these degenerations. This checker game for $\mathbb{P}^2$ is described in §2 of [Ya]. The details for $\mathbb{P}^1 \times \mathbb{P}^1$ are identical. The global sections of the linear system $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)$ are bi-homogeneous polynomials of bi-degree a and b in the variables x, y and z, w, respectively. A basis for the space of global sections is given by $x^i y^{a-i} z^j w^{b-j}$, where $0 \leq i \leq a$ and $0 \leq j \leq b$. We can record these monomials in a rectangular $(a+1) \times (b+1)$ grid. In this grid the box in the i-th row and the j-th column corresponds to the monomial $x^i y^{a-i} z^j w^{b-j}$.

FIGURE 1. Imposing a triple point on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,6)$.

If we impose an ordinary k-fold point on the linear system at $([x:y],[z:w]) = ([0:1],[0:1])$, then the coefficients of the monomials

$$y^a w^b, xy^{a-1}w^b, \ldots, x^{k-1}y^{a-k+1}z^{k-1}w^{b-k+1}$$

must vanish. We depict this by filling in a $k \times k$ triangle of checkers into the boxes at the upper left hand corner as in Figure 1. The coefficients of the monomials represented by boxes that have checkers in them must vanish.

We first slide the k-fold point along the fiber f_1 onto the point $([x:y],[z:w]) = ([1:0],[0:1])$. This correspond to the degeneration

$$([x:y],[z:w]) \mapsto ([x:ty],[z:w]).$$

The flat limit of this degeneration is described by the vanishing of the coefficients of certain monomials (assuming none of the checkers fall out of the rectangle). The monomials whose coefficients must vanish are those that correspond to boxes with checkers in them when we let the checkers fall according to the force of gravity. The first two panels in Figure 2 depict the result of applying this procedure to a 4-fold point when there is an aligned ideal condition at the point $([x:y],[z:w]) = ([1:0],[1:0])$.

We then follow this degeneration with a degeneration that specializes the k-fold point to q by sliding along the fiber f_2. This degeneration is explicitly given by

$$([x:y],[z:w]) \mapsto ([x:y],[z:tw]).$$

The flat limit is described by the vanishing of the coefficients of the monomials that have checkers in them when we slide all the checkers as far right as possible. The last two panels of Figure 2 depict this degeneration.

Figure 2. Depicting the degenerations by checkers.

S. Yang proves that, provided none of the checkers fall out of the ambient rectangle during these moves, these checker movements do correspond to the flat limits of the linear systems under the given degenerations. If one can play this checker game with all the multiple points that one imposes on a linear system so that during the game none of the checkers fall out of the rectangle, one can conclude that the multiple points impose independent conditions on the linear system. The limit linear system has the expected dimension. In particular, it is non-special. By upper semi-continuity the original linear system must also have the expected dimension and be non-special. Unfortunately, when one plays this game, occasionally checkers may fall out of the rectangle. In that case we loose information on what the limits are. This may happen even if the original linear system has the expected dimension.

In order to conclude the proposition we need to show that if we impose at most $j(d+1)-n(k,d)$ points of multiplicity k on the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d, jk(k+1)/2)$ where $2k \leq d$, we do not lose any checkers when we specialize all the k-fold points by the degeneration just described. This suffices to conclude the proposition because general simple points always impose independent conditions.

The main observation is that if there is a safety net of empty boxes at the top of the rectangle, then the checkers will not fall out of the box. The proof of the proposition is completed by noting the following simple facts.

(1) At any stage of the degeneration the height of the checkers in the rectangle is at most k larger than the highest row full of checkers.

(2) The left most checker of a row is to the lower left of the left most checker of any row above it.

If there are at least $(k+1)(d+1)$ empty boxes in our rectangle, then by the above two observations when we specialize a k-fold point we do not lose any of the checkers. As long as $n(k,d) \geq \lceil 2(d+1)/k \rceil$, there is always at least $(k+1)(d+1)$ boxes empty. Hence until the stage where we specialize the last k-fold point we cannot lose any checkers. This concludes the proof. □

Remark 3.5. While the asymptotic approach gives a proof of Theorem 1.5 independent of the Harbourne-Hirschowitz conjecture, it does not construct a moving curve that is dual to the codimension one faces of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$.

Department of Mathematics, M.I.T., Cambridge, MA 02139
E-mail address: `coskun@math.mit.edu`

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: `harris@math.harvard.edu`

Department of Mathematics, M.I.T., Cambridge, MA 02139
E-mail address: `jstarr@math.mit.edu`