QUOT FUNCTORS FOR DELIGNE-MUMFORD STACKS

MARTIN OLSSON AND JASON STARR

ABSTRACT. Given a separated and locally finitely-presented Deligne-Mumford stack $\mathcal{X}$ over an algebraic space S, and a locally finitely-presented $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, we prove that the Quot functor $\mathrm{Quot}(\mathcal{F}/\mathcal{X}/S)$ is represented by a separated and locally finitely-presented algebraic space over S. Under additional hypotheses, we prove that the connected components of $\mathrm{Quot}(\mathcal{F}/\mathcal{X}/S)$ are quasi-projective over S.

CONTENTS

1. STATEMENT OF RESULTS

Let $p : \mathcal{X} \to S$ be a separated, locally finitely-presented 1-morphism from a Deligne-Mumford stack $\mathcal{X}$ to an algebraic space S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module (on the étale site of $\mathcal{X}$) such that $\mathcal{F}$ is locally finitely-presented. Define a contravariant functor

$$Q = Q(\mathcal{F}/\mathcal{X}/S) : S - \text{schemes} \to \text{Sets} \tag{1}$$

as follows. For each S-scheme $f : Z \to S$, define $\mathcal{X}_Z$ to be $\mathcal{X} \times_S Z$, and define $\mathcal{F}_Z$ to be the pullback of $\mathcal{F}$ to $\mathcal{X}_Z$. We define $Q(Z)$ to be the set of $\mathcal{O}_{\mathcal{X}_Z}$-module quotients $\mathcal{F}_Z \to G$ which satisfy

(1) G is a quasi-coherent $\mathcal{O}_{\mathcal{X}_Z}$-module which is locally finitely-presented,
(2) G is flat over Z,
(3) the support of G is proper over Z.

All of these properties are preserved by base-change on Z, and therefore pullback makes Q into a contravariant functor.

Theorem 1.1. *Q is represented by an algebraic space which is separated and locally finitely-presented over S. If $\mathcal{F}$ has proper support over S, then $Q \to S$ satisfies the valuative criterion for properness.*

Date: April 24, 2002.

Remark: Under the hypothesis that $\mathcal{F}$ has proper support over S, we are not claiming that $Q \to S$ is proper, because we do not show that $Q \to S$ is quasi-compact (in general it need not be).

We give a better description of Q under additional hypotheses on $\mathcal{X}$. Our first hypothesis is that $\mathcal{X}$ is a global quotient.

Definition 1.2. *Let S be an algebraic space. A* global quotient stack *over S is an (Artin) algebraic stack $\mathcal{X}$ over S which is isomorphic to a stack of the form $[Z/G]$, where Z is an algebraic space which is finitely-presented over S, and G is a flat, finitely-presented group scheme over S which is a subgroup scheme of the general linear group scheme $GL_{n,S}$ for some n.*

This is essentially [4, definition 2.9] (with the Noetherian hypotheses replaced by a finite-presentation hypothesis). We remind the reader of the following characterization in [4, remark 2.11].

Lemma 1.3. *Suppose that $\mathcal{X}$ is a global quotient stack over S which is isomorphic to $[Z/G]$ as above. Then the diagonal action of G on $Z \times_S GL_{n,S}$ is free and the quotient $Z' = [Z \times_S GL_{n,S}/G]$ is an algebraic space with a right action of $GL_{n,S}$. The quotient stack $[Z'/GL_{n,S}]$ is isomorphic to the original stack $[Z/G]$.*

So every quotient stack is isomorphic to a stack of the form $[Z'/GL_{n,S}]$.

Our second hypothesis is that $\mathcal{X}$ is a *tame* Deligne-Mumford stack.

Definition 1.4. *A Deligne-Mumford stack $\mathcal{X}$ is* tame *if for any algebraically closed field k and any 1-morphism $\zeta : Spec\ k \to \mathcal{X}$, the stabilizer group $Aut(\zeta)(Spec\ k)$ has order prime to $char(k)$, where $Aut(\zeta)$ is the finite k-group scheme defined to be the Cartesian product of the diagram*

$$\begin{array}{ccc} & & Spec\ k \\ & & \downarrow{\scriptstyle (\zeta,\zeta)} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}. \end{array} \tag{2}$$

Here $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal morphism.

Theorem 1.5. *Suppose that S is an affine scheme, and let $f : \mathcal{X} \to S$ be a separated 1-morphism from a tame Deligne-Mumford stack to S such that $\mathcal{X}$ is a global quotient over S and such that the coarse moduli space X of $\mathcal{X}$ is a quasi-projective S-scheme (resp. projective S-scheme). Then the connected components of Q are quasi-projective S-schemes (resp. projective S-schemes).*

Remark: The existence of the coarse moduli space for $\mathcal{X}$ follows from [7].

Remark: The condition that S be an affine scheme is required because the property of being quasi-projective is not Zariski local on the base.

2. Representability by an algebraic space

In this section we prove theorem 1.1.

Note first that Q is a sheaf for the fppf-topology by descent theory, and is limit preserving. In addition, for each open substack $\mathcal{U} \subset \mathcal{X}$ there is a natural open immersion

$Q(\mathcal{F}|_{\mathcal{U}}/\mathcal{U}/S) \subset Q$. Moreover, Q is the union over finitely presented open substacks $\mathcal{U} \subset \mathcal{X}$ of the $Q(\mathcal{F}|_{\mathcal{U}}/\mathcal{U}/S)$. We may therefore assume that $\mathcal{X}$ is of finite presentation over S. Since the question of representability of Q is étale local on S we may assume that S is an affine scheme, and by a standard limit argument we may assume that S is of finite type over $Spec(\mathbb{Z})$.

Under these assumptions we prove the theorem by verifying the conditions of theorem 5.3 of [2].

Commutation with inverse limits. We need the Grothendieck existence theorem for Deligne-Mumford stacks:

Proposition 2.1. *Let A be a complete noetherian local ring, $\mathcal{X}/A$ a Deligne-Mumford stack, and let $A_n = A/\mathfrak{m}_A^{n+1}$, $\mathcal{X}_n = \mathcal{X} \otimes_A A_n$. Then the natural functor*

$$(\textit{coherent sheaves on } \mathcal{X} \textit{ with support proper over } A)$$
$$\downarrow$$
$$(\textit{compatible families of coherent sheaves on the } \mathcal{X}_n \textit{ with proper support})$$

is an equivalence of categories.

Proof. This proposition is perhaps best viewed in a context of formal algebraic stacks, but since we do not want to develop such a theory here we take a more ad-hoc approach.

Let $\widehat{\mathcal{X}}$ be the ringed topos $(\mathcal{X}_{0,et}, \mathcal{O}_{\widehat{\mathcal{X}}})$, where $\mathcal{O}_{\widehat{\mathcal{X}}}$ is the sheaf of rings which to any étale $\mathcal{X}_0$-scheme $U_0 \to \mathcal{X}_0$ associates

$$\varprojlim \Gamma(U_n, \mathcal{O}_{U_n}).$$

Here U_n denotes the unique lifting of U_0 to an étale $\mathcal{X}_n$-scheme. There is a natural morphism of ringed topoi

$$j : \widehat{\mathcal{X}} \to \mathcal{X}_{et}.$$

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules, then we denote by $\widehat{\mathcal{F}}$ the sheaf $j^*\mathcal{F}$. Note that the functor $\mathcal{F} \mapsto \widehat{\mathcal{F}}$ is an exact functor.

Lemma 2.2. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on $\mathcal{X}$ with proper support over A, then for every integer n the natural map*

$$Ext^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \longrightarrow Ext^n_{\mathcal{O}_{\widehat{\mathcal{X}}}}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}})$$

is an isomorphism.

Proof. Observe first that the natural map

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})^\wedge \to \mathcal{E}xt^1_{\mathcal{O}_{\widehat{\mathcal{X}}}}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}})$$

is an isomorphism. Indeed this can be verified locally and so follows from the theory of formal schemes. From this and the local-to-global spectral sequence for Ext it follows that it suffices to show that for any coherent sheaf $\mathcal{F}$ with proper support, the natural map

$$H^n(\mathcal{X}, \mathcal{F}) \longrightarrow H^n(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}) \tag{3}$$

is an isomorphism for every n. We prove this by induction on the dimension of $\mathcal{X}$.

The key observation is that if $a : U \to \mathcal{X}$ is a finite morphism from a scheme to $\mathcal{X}$, then the map 3 is known to be an isomorphism for all n in the case when $\mathcal{F}$ is equal to $a_*\mathcal{F}'$ for some coherent sheaf $\mathcal{F}'$ on U (since a is finite and [5]. III.5).

If the dimension of $\mathcal{X}$ is zero, then we can by ([9], 16.6) find a finite étale cover $a : U \to \mathcal{X}$ of $\mathcal{X}$ by a scheme U. If $b : U \times_{\mathcal{X}} U \to \mathcal{X}$ denotes the canonical map, then the sequences

$$\mathcal{F} \to a_*a^*\mathcal{F} \rightrightarrows b_*b^*\mathcal{F}$$
$$\widehat{\mathcal{F}} \to a_*a^*\widehat{\mathcal{F}} \rightrightarrows b_*b^*\widehat{\mathcal{F}}$$

are exact so the result is true for $n = 0$.

To prove the result for general n, assume the result is true for $n-1$ and let $\mathcal{G} = a_*a^*\mathcal{F}/\mathcal{F}$ so that we have an exact sequence of coherent sheaves

$$0 \to \mathcal{F} \to a_*a^*\mathcal{F} \to \mathcal{G} \to 0.$$

Then the commutative diagram

$$\begin{array}{ccccccc} H^{n-1}(\mathcal{X}, a_*a^*\mathcal{F}) & \to & H^{n-1}(\mathcal{X}, \mathcal{G}) & \to & H^n(\mathcal{X}, \mathcal{F}) & \to & H^n(\mathcal{X}, a_*a^*\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{n-1}(\widehat{\mathcal{X}}, a_*a^*\widehat{\mathcal{F}}) & \to & H^{n-1}(\widehat{\mathcal{X}}, \widehat{\mathcal{G}}) & \to & H^n(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}) & \to & H^n(\widehat{\mathcal{X}}, a_*a^*\widehat{\mathcal{F}}) \end{array}$$

shows that the map $H^n(\mathcal{X}, \mathcal{F}) \to H^n(\widehat{\mathcal{X}}, \widehat{\mathcal{F}})$ is injective. But then the map $H^n(\mathcal{X}, \mathcal{G}) \to H^n(\widehat{\mathcal{X}}, \widehat{\mathcal{G}})$ is also injective, and an analysis of the diagram

$$\begin{array}{ccccccc} H^{n-1}(\mathcal{X}, \mathcal{G}) & \to & H^n(\mathcal{X}, \mathcal{F}) & \to & H^n(\mathcal{X}, a_*a^*\mathcal{F}) & \to & H^n(\mathcal{X}, \mathcal{G}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{n-1}(\widehat{\mathcal{X}}, \widehat{\mathcal{G}}) & \to & H^n(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}) & \to & H^n(\widehat{\mathcal{X}}, a_*a^*\widehat{\mathcal{F}}) & \to & H^n(\widehat{\mathcal{X}}, \widehat{\mathcal{G}}) \end{array}$$

reveals that the map $H^n(\mathcal{X}, \mathcal{F}) \to H^n(\widehat{\mathcal{X}}, \widehat{\mathcal{F}})$ is an isomorphism. This completes the proof of the case when $dim(\mathcal{X}) = 0$.

To prove the result for general $\mathcal{X}$ we assume the result is true for $dim(\mathcal{X}) - 1$ and proceed by induction on n. By ([9], 16.6) there exists a finite surjective morphism $a : U \to \mathcal{X}$ which is generically étale. Let $b : U \times_{\mathcal{X}} U \to \mathcal{X}$ be the natural map and let K be the kernel of $a_*a^*\mathcal{F} \to b_*b^*\mathcal{F}$. Note that the map 3 for K is an isomorphism when $n = 0$. There is a natural map $\mathcal{F} \to K$ which is generically an isomorphism. Let $\mathcal{F}'$ be the kernel of this map and let $\mathcal{F}''$ be the cokernel. Then $\mathcal{F}'$ and $\mathcal{F}''$ have lower-dimensional support and so the map 3 is an isomorphism for these sheaves. If $\mathcal{G}$ denotes the image of $\mathcal{F} \to K$, then the map $H^0(\mathcal{X}, \mathcal{G}) \to H^0(\widehat{\mathcal{X}}, \widehat{\mathcal{G}})$ is an isomorphism by the corresponding result for K and $\mathcal{F}''$ and the exact sequence

$$0 \to \mathcal{G} \to K \to \mathcal{F}'' \to 0.$$

Then from the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{G} \to 0$$

we deduce that the map $H^0(\mathcal{X}, \mathcal{F}) \to H^0(\widehat{\mathcal{X}}, \widehat{\mathcal{F}})$ is an isomorphism.

To prove the result for n assuming the result for $n-1$, note that a similar argument to the one above shows that if the map $H^n(\mathcal{X}, K) \to H^n(\widehat{\mathcal{X}}, \widehat{K})$ is injective (resp. an isomorphism) then the map 3 is injective (resp. an isomorphism). Therefore the proof is completed by using the argument of the case $dim(\mathcal{X}) = 0$ with K replacing $\mathcal{F}$. $\square$

Now observe that the category of compatible families of coherent sheaves on the $\mathcal{X}_n$ with proper support is naturally viewed as a full subcategory of the category of sheaves of $\mathcal{O}_{\widehat{\mathcal{X}}}$-modules on $\widehat{\mathcal{X}}$ using the functor which sends a family $\{\mathcal{F}_n\}$ to the sheaf $\widehat{\mathcal{F}}$ associated to the presheaf

$$U_0 \mapsto \varprojlim \Gamma(U_0, \mathcal{F}_n).$$

The functor $\{\mathcal{F}_n\} \mapsto \widehat{\mathcal{F}}$ is fully faithful and identifies the category of compatible systems of coherent sheaves on the $\mathcal{X}_n$ with proper support with a subcategory of the category of sheaves of $\mathcal{O}_{\widehat{\mathcal{X}}}$-modules which is closed under the formation of kernels, cokernels, and extensions. Indeed, these assertions can be verified locally on $\mathcal{X}$ and hence follow from the corresponding statements for formal schemes. From this it and the lemma it follows that the functor in 2.1 is fully faithful.

Now the functor from coherent sheaves on $\mathcal{X}$ with proper support to sheaves of $\mathcal{O}_{\widehat{\mathcal{X}}}$-modules on $\widehat{\mathcal{X}}$ identifies the category of such sheaves with a full subcategory of the category of sheaves of $\mathcal{O}_{\widehat{\mathcal{X}}}$-modules which is stable under the formation of kernels, cokernels, and extensions (by lemma 2.2). Thus the following two lemmas prove proposition 2.1.

Lemma 2.3. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}' \subset \mathcal{A}$ be a full subcategory which is stable under the formation of kernels, cokernels, and extensions. Then any object of $\mathcal{A}$ which admits a morphism to an object of $\mathcal{A}'$ such that the kernel and cokernel are in $\mathcal{A}'$ is in $\mathcal{A}'$.*

Proof. By assumption such an object $\mathcal{F} \in \mathcal{A}$ sits in an exact sequence

$$0 \to K \to \mathcal{F} \to \mathcal{F}' \to Q \to 0$$

where $K, \mathcal{F}', Q \in \mathcal{A}'$. Let $K' = Ker(\mathcal{F}' \to Q)$. Then $K' \in \mathcal{A}'$, and we have an exact sequence

$$0 \to K \to \mathcal{F} \to K' \to 0.$$

Therefore $\mathcal{F} \in \mathcal{A}'$. □

Lemma 2.4. *For every compatible family of coherent sheaves $\{\mathcal{F}_n\}$ with proper support and associated sheaf $\widehat{\mathcal{F}}$ on $\widehat{\mathcal{X}}$, there exists a morphism $\widehat{\mathcal{F}} \to j^*\mathcal{G}$ for some coherent sheaf $\mathcal{G}$ on $\mathcal{X}$ with proper support such that the kernel and cokernel are isomorphic to the pullbacks of coherent sheaves on $\mathcal{X}$ with proper support.*

Proof. We proceed by induction on the dimension of the support of $\widehat{\mathcal{F}}$.

If the support of $\widehat{\mathcal{F}}$ has dimension zero, then for any étale cover $a : U \to \mathcal{X}$ the inverse image of the support of $\widehat{\mathcal{F}}$ in $\widehat{U}$ is a closed subscheme which is of dimension zero, hence is proper. Therefore, there exists a unique sheaf on U inducing the restriction of $\widehat{\mathcal{F}}$ to $\widehat{U}$ (by [5] III.5). Moreover, by the uniqueness this sheaf comes with descent datum relative to a. Hence $\widehat{\mathcal{F}}$ is induced from a coherent sheaf on $\mathcal{X}$ with proper support.

As for the general case, choose a morphism $a : U \to \mathcal{X}$ which is finite and generically étale (such a morphism exists by [9], 16.6), and let $b : U \times_{\mathcal{X}} U \to \mathcal{X}$ be the natural map. Then $a_*a^*\widehat{\mathcal{F}}$ and $b_*b^*\widehat{\mathcal{F}}$ are obtained from coherent sheaves on $\mathcal{X}$, and hence so is

$$K := Ker(a_*a^*\widehat{\mathcal{F}} \to b_*b^*\widehat{\mathcal{F}}).$$

Moreover, there is a natural map $\widehat{\mathcal{F}} \to K$ which is generically an isomorphism. Hence the kernels and cokernels have lower dimensional support and by induction are obtained from coherent sheaves on $\mathcal{X}$. □

□

Separation Conditions. These follow by the same reasoning as in ([2], page 64).

Deformation theory.

Suppose given a deformation situation (in the sense of [2])

$$A' \to A \to A_0$$

and a quotient $\mathcal{F}_A \to \mathcal{G}_A$ over $\mathcal{X}_A = \mathcal{X} \times_S Spec(A)$ giving an element of $Q(A)$. Suppose further that a map $B \to A$ is given. Then it is well-known (see for example [11], 3.4) that the map

$$Q(A' \times_B A) \longrightarrow Q(A') \times_{Q(A)} Q(B) \tag{4}$$

is a bijection.

If $M = Ker(A' \to A)$, then it follows from the bijectivity of 4 that a deformation theory in the sense of ([2]) is provided by the module $Q_{\mathcal{G}_0}(A_0[M])$ (see for example loc. cit. page 47). Here $A_0[M]$ denotes the ring with underlying module $A_0 \oplus M$ and multiplication

$$(a, m) \cdot (a', m') = (aa', am' + a'm),$$

and $Q_{\mathcal{G}_0}(A_0[M])$ denotes the set of elements in $Q(A_0[M])$ whose image in $Q(A_0)$ is the reduction $\mathcal{G}_0$ of $\mathcal{G}$. The conditions on the deformation theory of ([2], theorem 5.3) are therefore satisfied by the following lemma and standard properties of cohomology.

Lemma 2.5. *Let $\mathcal{H}_0 = Ker(\mathcal{F}_0 \to \mathcal{G}_0)$ (where $\mathcal{F}_0$ denotes the reduction of $\mathcal{F}$ to $\mathcal{X}_{A_0}$). Then there is a natural A_0-module isomorphism*

$$Q_{\mathcal{G}_0}(A_0[M]) \simeq Ext^0(\mathcal{H}_0, \mathcal{G}_0 \otimes M).$$

Proof. Let

$$0 \longrightarrow \mathcal{G}_0 \otimes M \longrightarrow \mathcal{G}_0 \otimes M \oplus \mathcal{F}_0 \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

be the sequence obtained by pushing out the sequence

$$0 \longrightarrow \mathcal{F}_0 \otimes M \longrightarrow \mathcal{F}_0 \oplus \mathcal{F}_0 \otimes M \longrightarrow \mathcal{G}_0 \longrightarrow 0$$

via the given map $\mathcal{F}_0 \to \mathcal{G}_0$. Then to give a lifting of $\mathcal{G}_0$ is equivalent to giving a sub-$\mathcal{O}_{\mathcal{X}_{A_0}}$-module $\mathcal{H} \subset \mathcal{F}_0 \oplus \mathcal{G}_0 \otimes M$ such that the induced map to $\mathcal{F}_0$ induces an isomorphism $\mathcal{H} \simeq \mathcal{H}_0$. Note that such a sub-module is automatically a sub-$\mathcal{O}_{\mathcal{X}_{A_0[M]}}$-module. In other words, the set of liftings of $\mathcal{G}_0$ is in natural bijection with the set of maps $\mathcal{H}_0 \to \mathcal{F}_0 \oplus \mathcal{G}_0 \otimes M$ lifting the inclusion $\mathcal{H}_0 \subset \mathcal{F}_0$. But to give such a map is precisely equivalent to giving a morphism $\mathcal{H}_0 \to \mathcal{G}_0 \otimes M$. The verification that this bijection is a module homomorphism is left to the reader.

□

Conditions on the obstructions.

Our understanding of the obstruction theory of Q will be in a 2-step approach (correcting a mistake in [2]). Let $A' \to A \to A_0$ be a deformation situation as above and $\mathcal{F}_A \to \mathcal{G}_A$ an object in $Q(A)$. For any quotient $\epsilon : M \to M_\epsilon$, let A_ϵ be the quotient of A' by the kernel of ϵ. For such an ϵ, the first obstruction to lifting $\mathcal{G}_A$ to A_ϵ is that the map

$$M \otimes \mathcal{F}_{A_0} \longrightarrow M_\epsilon \otimes \mathcal{G}_{A_0}$$

factors through $M\mathcal{F}_{A'}$. If we let $\mathcal{T}$ be the kernel of $M \otimes \mathcal{F}_{A_0} \to \mathcal{F}_{A'}$, then we want that the map

$$\mathcal{T} \longrightarrow M_\epsilon \otimes \mathcal{G}_{A_0} \tag{5}$$

is zero. If this is the case, then there is a canonical map $M\mathcal{F}_{A'} \to M_\epsilon \otimes \mathcal{G}_{A_0}$ and the condition that there exists an element in $Q(A_\epsilon)$ inducing $\mathcal{G}$ is equivalent to the statement that the resulting extension

$$0 \to M_\epsilon \otimes \mathcal{G}_{A_0} \to E \to \mathcal{F}_A \to 0$$

is obtained from an extension of $\mathcal{G}_A$ by $M_\epsilon \otimes \mathcal{G}_{A_0}$. In other words, if the map 5 is zero, then if we let $\mathcal{H} = Ker(\mathcal{F}_A \to \mathcal{G}_A)$ there exists a canonical obstruction in

$$\mathrm{Ext}^1_{\mathcal{X}_{A'}}(\mathcal{H}, M_\epsilon \otimes \mathcal{G}_{A_0})$$

whose vanishing is necessary and sufficient for the existence of a lifting.

Now the only condition on the obstructions in ([2], 5.3) which does not follow immediately from the bijectivity of 4, is condition (5.3, [5′].c).

Thus suppose given a deformation situation as before, $\xi \in Q(A)$, and suppose further that M is free of rank n. Let K be the field of fractions of A_0, and denote by a subscript K the locations at the generic point of $Spec(A_0)$. We suppose that for every one-dimensional quotient $M_K \to M_K^*$ there is a non-trivial obstruction to lifting ξ_K to $Q(A_K^*)$, where A_K^* denotes the extension defined by M_K^*. Then we have to show that there exists a non-empty open subset $U \subset Spec(A_0)$ such that for every quotient ϵ of M of length one with support in U, ξ does not lift to A_ϵ (the extension obtained from ϵ).

Let

$$\phi \in \mathrm{Ext}^0(\mathcal{T}, M \otimes \mathcal{G}_{A_0}) \simeq \mathrm{Ext}^0(\mathcal{T}, \mathcal{G}_{A_0}) \otimes M$$

be the class defined by 5 in the case when ϵ is the identity. We reduce to the case when $\phi = 0$. Once this reduction is made the argument of ([2], page 66) will finish the proof.

To make the reduction, we can by shrinking on $Spec(A_0)$ assume that

$$\mathrm{Ext}^0(\mathcal{T}, \mathcal{G}_{A_0})$$

is a free module; say of rank r. In addition, by the argument of ([2], page 66), we can after shrinking $Spec(A_0)$ assume that for each point $s \in Spec(A_0)$, the natural map

$$\mathrm{Ext}^0(\mathcal{T}, \mathcal{G}_{A_0}) \otimes k(s) \longrightarrow \mathrm{Ext}^0(\mathcal{T}, \mathcal{G}_{A_0} \otimes k(s)) \tag{6}$$

is an isomorphism.

Choosing a basis for $\mathrm{Ext}^0(\mathcal{T}, \mathcal{G}_{A_0})$ we can think of ϕ as an element

$$\phi = (\phi_1, \dots, \phi_r) \in M^r.$$

Let $N \subset M$ be the submodule of M generated by the ϕ_i, and let $M' = M/N$. After further shrinking $Spec(A_0)$ we can assume that M' is a free module. Now note that any length-one quotient $M \to M_\epsilon$ for which the obstruction to lifting ξ goes to zero factors through M' by 6. Moreover, any such quotient which does not factor through M' is obstructed. Therefore, we may replace M by M' and hence are reduced to the case when $\phi = 0$.

Valuative criteria for properness when $\mathcal{F}$ has proper support

Let R be a discrete valuation ring with field of fractions K, and let $i : \mathcal{X}_K \hookrightarrow \mathcal{X}$ be the inclusion of the generic fiber. We suppose that we have a flat quotient $\mathcal{F}_K \to \mathcal{G}_K$ over the generic fiber which we wish to extend to $\mathcal{X}$. For this we take $\mathcal{G}$ to be the image of the map $\mathcal{F} \to i_*\mathcal{G}_K$. The image is evidently a coherent sheaf, and has proper support since $\mathcal{F}$ has proper support. It is flat because it is a subsheaf of a torsion free sheaf, and by definition $\mathcal{G}$ induces $\mathcal{G}_K$ on the generic fiber.

This completes the proof of theorem 1.1. □

3. Flattening stratifications

As a corollary of the representability result in the last section, when the locally finitely-presented sheaf $\mathcal{F}$ has proper support over S, we can construct the *flattening stratification* of $\mathcal{F}$ as an algebraic space. But in fact this algebraic space is representable and quasi-affine over S. This is a crucial step in the proof of theorem 1.5. Therefore we include a proof of the following fact:

Proposition 3.1. *Suppose $f : Y \to X$ is a finite-type, separated, quasi-finite morphism of algebraic spaces. Then f is quasi-affine, in particular f is representable by schemes.*

Proof. We need to prove that the natural morphism of X-schemes,

$$\iota : Y \to \mathrm{Spec}\ _X f_* \mathcal{O}_Y \tag{7}$$

is an open immersion. By [8, proposition II.4.18], $f_*\mathcal{O}_Y$ commutes with flat base change on X, so the formation of ι commutes with flat base change on X. Moreover one may check that a morphism of X-schemes is an open immersion after fpqc base change. Thus, without loss of generality, we may suppose that X is an affine scheme.

To prove ι is an open immersion, it suffices to check that for each point $p \in Y$, the following two conditions hold:

1. ι is étale at p, and
2. there is an open set $p \in U \subset Y$ which is disjoint from the image of $Y \times_{\iota,\iota} Y - \Delta(Y)$ under the projection $\mathrm{pr}_1 : Y \times_{\iota,\iota} Y \to Y$.

Here $Y \times_{\iota,\iota} Y$ is the fiber product of Y with itself over $\mathrm{Spec}\ _X f_*\mathcal{O}_Y$ and $\Delta : Y \to Y \times_{\iota,\iota} Y$ is the diagonal morphism. It is clear that if both (1) and (2) hold for each point $p \in Y$, then ι is an étale monomorphism, and therefore ι is an open immersion.

The claim is that for given $p \in Y$, we may check (1) and (2) after passing to an étale neighborhood of $q = f(p) \in X$, i.e. if $X' \to X$ is an étale morphism, $q' \in X'$ is a point lying over q, and if $p' \in Y' := X' \times_X Y$ is the point lying over q' and p, then it suffices to check (1) and (2) for p'. We have mentioned that the natural morphism of X'-schemes

$$\iota' : Y' \to \mathrm{Spec}\ _{X'} (f')_* \mathcal{O}_{Y'} \tag{8}$$

is the base-change of ι. The property of being étale at a point can be checked after étale (and even flat) base-change, so if (1) holds for p' then (1) holds for (p). Suppose (2) holds for p' and let U' be an open set $p' \in U' \subset Y'$ as in (2). Let $p \in U \subset Y$ be the open image of U' under $g : Y' \to Y$. We have the equality

$$g(U') \cap \mathrm{pr}_1\left(Y \times_{\iota,\iota} Y - \Delta(Y)\right) = g\left(U' \cap g^{-1}\mathrm{pr}_1\left(Y \times_{\iota,\iota} Y - \Delta(Y)\right)\right) = \tag{9}$$

$$g\left(U' \cap \mathrm{pr}'_1\left(Y' \times_{\iota',\iota'} Y' - \Delta'(Y')\right)\right). \tag{10}$$

But $U' \cap \mathrm{pr}'_1\left(Y' \times_{\iota',\iota'} Y' - \Delta'(Y')\right) = \emptyset$ by assumption. Thus $U = g(U')$ satisfies condition (2) for $p \in Y$. So it suffices to check (1) and (2) for $p' \in Y'$.

Now let $Z \to Y$ be an étale morphism of a scheme Z to Y and $r \in Z$ a point mapping to p. Then $Z \to X$ is finite-type, separated and quasi-finite. By [3, proposition 2.3.8(a)], we may find an étale morphism $X' \to X$ with X' an affine scheme, and a point $q' \in X'$ mapping to $q = f(p)$ such that if $r' \in Z'$ is the point lying over q' and r, and if Z'_0 is the connected component of Z' containing r', then $Z'_0 \to X'$ is a finite morphism of affine schemes. Let $Y'_0 \subset Y'$ be the connected component of Y' containing the image of Z'_0. Then $Z'_0 \to Y'_0$ is étale with dense image, but it is also proper. Therefore $Z'_0 \to Y'_0$ is finite,étale and surjective. So, by Knudsen's version of Chevalley's theorem [8, theorem III.4.1], Y'_0 is an affine scheme.

Let Y'_r denote the union of connected components $Y' - Y'_0$. Then we have a decomposition of $\text{Spec}\,_{X'}(f')_*\mathcal{O}_{Y'}$ into connected components, $\text{Spec}\,_{X'}(f'_0)_*\mathcal{O}_{Y'_0}$ union $\text{Spec}\,_{X'}(f'_r)_*\mathcal{O}_{Y'_r}$ and a decomposition of ι' as the "disjoint union" of the two morphisms:

$$\iota'_0 : Y'_0 \to \text{Spec}\,_{X'}(f'_0)_*\mathcal{O}_{Y'_0}, \tag{11}$$

$$\iota'_r : Y'_r \to \text{Spec}\,_{X'}(f'_r)_*\mathcal{O}_{Y'_r}. \tag{12}$$

And ι' is an isomorphism. Thus ι' is étale at p'. And defining $U' = Y'_0$, we see that U' satisfies the condition (2) for $p' \in Y'$. So (1) and (2) hold for p' and thus (1) and (2) hold for $p \in Y$. Since this holds for every $p \in Y$, we conclude that ι is an open immersion, i.e. $f : Y \to X$ is quasi-affine. □

Remark: We could not find precisely this statement in [8], although it follows easily from results proved there. If one further assumes that f is finitely-presented, then this result also follows easily from [9, théorème 16.5].

Now suppose S is an algebraic space, $f : \mathcal{X} \to S$ is a 1-morphism from a Deligne-Mumford stack to S which is separated and locally finitely-presented. Let $\mathcal{F}$ be a locally finitely-presented $\mathcal{O}_{\mathcal{X}}$-module. Consider the functor

$$\Sigma : (S - \text{schemes})^{opp} \to \text{Sets}, \tag{13}$$

which to any morphism $g : T \to S$ associates $\{*\}$ if the pullback of $\mathcal{F}$ to $T \times_S \mathcal{X}$ is flat over T, and which associates $\emptyset$ otherwise. Given a morphism of S-schemes, $h : T_1 \to T_2$, the morphism $\Sigma(h)$ is defined to be the unique map $\Sigma(T_2) \to \Sigma(T_1)$. For this to make sense, we must check that if $\Sigma(T_2)$ is nonempty, then so is $\Sigma(T_1)$. But this is clear, if the pullback of $\mathcal{F}$ to $T_2 \times_S \mathcal{X}$ is flat over T_2, then the pullback of this sheaf over $T_1 \times_{T_2} (T_2 \times_S \mathcal{X})$ is flat over T_1. And this pullback is isomorphic to the pullback of $\mathcal{F}$ to $T_1 \times_S \mathcal{X}$. So $\Sigma(T_1)$ is nonempty.

Theorem 3.2. *Let $f : \mathcal{X} \to S$, $\mathcal{F}$, and Σ be as above.*

(1) *Σ is an fpqc sheaf which is limit preserving and $\Sigma \to S$ is a monomorphism.*
(2) *If $\mathcal{F}$ has proper support over S, then Σ is an algebraic space and $\Sigma \to S$ is a surjective, finitely-presented monomorphism. In particular, $\Sigma \to S$ is quasi-affine.*

Proof. It is immediate that $\Sigma \to S$ is a monomorphism. Since one may check that a quasi-coherent sheaf on $T \times_S \mathcal{X}$ is flat over T after performing an fpqc base change of T, it follows that Σ is an fpqc sheaf. The fact that Σ is limit-preserving follows from [5, théorème IV.11.2.6]. So (1) is proved.

To prove (2), first notice that we may use (1) to reduce to the case that $S = \text{Spec}\, A$ is a Noetherian affine scheme, and $\mathcal{X}$ is the support of $\mathcal{F}$ which is a proper, finitely-presented Deligne-Mumford stack over S. By theorem 1.1 we know the Quot functor of $\mathcal{F}$ is represented by an algebraic space $Q \to S$ which is separated and locally finitely-presented. Denote by

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathrm{pr}_2^*\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{14}$$

the universal short exact sequence on $Q \times_S \mathcal{X}$. Since $\mathrm{pr}_2^*\mathcal{F}$ has proper support over Q, $\mathcal{K}$ also has proper support over Q. Define U to be the complement in Q of the image of the support of $\mathcal{K}$. The restriction of the universal short exact sequence over U is simply the identity morphism of $\mathrm{pr}_2^*\mathcal{F}$ to itself, i.e. the pullback of $\mathcal{F}$ to U is flat over U. So we have an induced morphism $U \to \Sigma$. Conversely there is an obvious morphism $\Sigma \to Q$ which factors through U, and we see that $U \to \Sigma$ is a natural isomorphism of algebraic spaces. So Σ is an algebraic space and $\Sigma \to S$ is a locally finitely-presented monomorphism.

Any module over a field is flat over that field, therefore for each field K and each morphism Spec $K \to S$, this morphism factors through $\Sigma \to S$. So $\Sigma \to S$ is surjective. The claim is that for any surjective, locally finitely-presented monomorphism $h : Y \to S$ of an algebraic space to a Noetherian affine scheme, Y is quasi-compact.

We will prove this claim by induction on the dimension of S. If $S =$ Spec K for some field K, it is obvious. If Y^{red} is quasi-compact, the same is true of Y, so we may reduce to the case that S is reduced. A finite union of quasi-compact sets is quasi-compact, so we may reduce to the case that S is integral. Now suppose the result is proved for all schemes S of dimension at most n and suppose S is an integral scheme of dimension $n+1$. Let Spec K be the generic point of S and let $U \subset Y$ be the open set where $U \to S$ is étale. Then Spec K factors through Y, and in fact it is contained in U. Thus $U \to S$ is an étale monomorphism, i.e. an open immersion which has dense image. Let the complement of U in S be C (with reduced scheme structure) and let the preimage of C be Z. Then $Z \to C$ is again a surjective, locally finitely-presented monomorphism, and C has dimension at most n. So by the induction assumption, Z is quasi-compact. Since U is an open subset of a Noetherian scheme, it is quasi-compact. Thus $Y = U \cup Z$ is quasi-compact and the claim is proved by induction.

By the last paragraph, we conclude that $\Sigma \to S$ is a surjective, finitely-presented monomorphism of algebraic spaces. So by proposition 3.1, we conclude that $\Sigma \to S$ is quasi-affine. $\square$

Remarks: (1) If the support of $\mathcal{F}$ is a scheme and the morphism to S is projective, then it follows from [10, theorem, p.55] that $\Sigma \to S$ is a disjoint union of locally closed immersions. While one can find examples of surjective, finitely-presented monomorphisms $Y \to S$ not a disjoint union of locally closed immersions, we conjecture that $\Sigma \to S$ is a disjoint union of locally closed immersions whenever $\mathcal{F}$ has proper support over S.

(2) Again in the case that the support of $\mathcal{F}$ is projective over S, the methods in [10, section 8] provide a *global construction* of the flattening stratification $\Sigma \to S$ along with a partition labelled by the Hilbert polynomial of the fibers of $\mathcal{F}$. In the case that $\mathcal{X}$ is a tame, global quotient with projective coarse moduli space, we believe that there is again a *global construction* of the flattening stratification $\Sigma \to S$ along with a partition labelled by the Hilbert polynomial (as defined in section 4). However we don't know what this global construction is, and the proof of the existence of $\Sigma \to S$ given above is the one truly non-constructive step in the proof of theorm 1.5.

4. Hilbert Polynomials

Suppose k is a field and $\mathcal{X}$ is a separated, locally finitely-presented Deligne-Mumford stack over Spec k. Suppose that $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module with proper support. Then the sum

$$\chi(\mathcal{X},\mathcal{F}) = \sum_{i=0}^{\infty}(-1)^i \dim_k H^i_{\text{ét}}(\mathcal{X},\mathcal{F}) \tag{15}$$

is finite (by [9, Théorème 15.6]). For each locally free sheaf E of finite rank on $\mathcal{X}$, we also have that $E\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{F}$ is again coherent with proper support. Since $E\mapsto \chi(\mathcal{X},E\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{F})$ is additive in short exact sequences, we have a well-defined group homomorphism

$$P_{\mathcal{F}}: K^0(\mathcal{X})\to\mathbb{Z}, [E]\mapsto \chi(\mathcal{X},E\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{F}). \tag{16}$$

Definition 4.1. *Given a homomorphism of Abelian groups $i: A\to K^0(\mathcal{X})$, define the* A-Hilbert polynomial *of $\mathcal{F}$, $P_{A,\mathcal{F}}$, to be the function $P_{A,\mathcal{F}} = P_{\mathcal{F}}\circ i$.*

Remarks: (1) If $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$, $A=\mathbb{Z}[x,x^{-1}]$ and $i: A\to K^0(\mathcal{X})$ is the group homomorphism such that $i(x^d)=[\mathcal{L}^d]$, then the A-Hilbert polynomial of $\mathcal{F}$, $P_{A,\mathcal{F}}$ is the usual Hilbert polynomial of $\mathcal{F}$ with respect to $\mathcal{L}$. We will need to consider cases where $i: A\to K^0(\mathcal{F})$ cannot be reduced to this form, which is why our notion of Hilbert polynomial is so general.

(2) The most instructive example, from our point of view, is when $\mathcal{X}=BG$ for some finite, étale k-group scheme G. Then $K^0(BG)$ is naturally isomorphic to the Grothendieck group of the category of finite $k[G]$-modules, i.e. the representation ring of $k[G]$. And the Hilbert polynomial $P_{\mathcal{F}}$ determines the image $[\mathcal{F}]$ of $\mathcal{F}$ in $K^0(BG)$.

Recall that a Deligne-Mumford stack $\mathcal{X}$ is *tame* if for each algebraically-closed field k and each 1-morphism ζ : Spec $k\to\mathcal{X}$, the k-valued points of

$$G_\zeta := \text{Spec } k\times_{(\zeta,\zeta),\mathcal{X}\times\mathcal{X},\Delta}\mathcal{X} \tag{17}$$

form a group of order prime to char(k). We remind the reader of some facts about tame Deligne-Mumford stacks.

Lemma 4.2. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack, $\pi:\mathcal{X}\to X$ its coarse moduli space.*

1. *The additive functor π_* from the category of $\mathcal{O}_{\mathcal{X}}$-modules to the category of $\mathcal{O}_X$-modules maps quasi-coherent sheaves to quasi-coherent sheaves and maps locally finitely-presented sheaves to locally finitely-presented sheaves.*
2. *The additive functor π_* is exact, in particular $R^i\pi_*\mathcal{F}=0$ for $i>0$ and $\mathcal{F}$ any quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module.*
3. *Suppose $g: X\to S$ is a morphism of algebraic spaces and suppose $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{X}$ which is flat over S. Then also $\pi_*\mathcal{F}$ is flat over S.*

Proof. (1) and (2) form [1, lemma 2.3.4]. And (3) follows by the same local analysis in the proof of [1, lemma 2.3.4] since the invariant submodule M^Γ (in the notation of [1]) of an S-flat module M is a direct summand, and so it is also S-flat. ☐

Now suppose that $f:\mathcal{X}\to S$ is a 1-morphism from a tame Deligne-Mumford stack to a connected algebraic space such that f is separated and locally finitely-presented, Define $A=K^0(\mathcal{X})$ and for each field k and each morphism g : Spec $k\to S$, define $i_g: A\to K^0(\text{Spec } k\times_S\mathcal{X})$ to be the pullback map $K^0(\text{pr}_2)$.

Lemma 4.3. *Suppose that $\mathcal{F}$ is a locally finitely-presented quasi-coherent sheaf on $\mathcal{X}$ which is flat over S and which has proper support over S. Then there exists a function $P : A \to \mathbb{Z}$ such that for all g : Spec $k \to S$, $P_{A,g^*\mathcal{F}} = P$.*

Proof. We need to show that for each locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, the function

$$(g : \text{Spec } k \to S) \mapsto \chi\left(g^*\left(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}\right)\right) \tag{18}$$

is constant. Since $R^i\pi_*$ vanishes on all quasi-coherent modules for $i > 0$, we have

$$\chi\left(g^*\left(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}\right)\right) = \chi\left(g^*\pi_*\left(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}\right)\right). \tag{19}$$

And by lemma 4.2 (3), we know that $\pi_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F})$ is an S-flat, locally finitely-presented sheaf with proper support over S. So by [10, corollary, p. 50], we conclude that there is some $P([\mathcal{E}]) \in \mathbb{Z}$ such that for all g : Spec $k \to S$, we have

$$\chi\left(g^*\pi_*\left(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}\right)\right) = P([\mathcal{E}]). \tag{20}$$

$\square$

Fix an additive homomorphism $P : K^0(\mathcal{X}) \to \mathbb{Z}$ and define $Q^P = Q^P(\mathcal{F}/\mathcal{X}/S)$ to be the subfunctor of $Q(\mathcal{F}/\mathcal{X}/S)$ such that for each S-scheme, $g : Z \to S$ we have $Q^P(g : Z \to S)$ is the set of quotients $[g^*\mathcal{F} \to \mathcal{G}] \in Q(g : Z \to S]$ such that for each field k and each morphism h : Spec $k \to Z$, we have $P_{A,h^*\mathcal{G}} = P$. By lemma 4.3, we see that Q^P is an open and closed subfunctor (possibly empty) of Q and that Q is the disjoint union of Q^P as P ranges over all P.

Observe that theorem 1.5 is implied by the following refinement.

Theorem 4.4. *Suppose that S is an affine scheme. Suppose that $\mathcal{X}$ is a tame Deligne-Mumford stack which is separated and finitely-presented over S, whose coarse moduli space is a quasi-projective S-scheme (resp. projective S-scheme), and which is a global quotient over S. Then for each locally finitely-presented quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ and for each homomorphism $P : K^0(\mathcal{X}) \to \mathbb{Z}$, the functor $Q^P(\mathcal{F}/\mathcal{X}/S)$ is represented by an algebraic space Q^P which admits a factorization $Q^P \to Q' \to S$ where Q' is projective over S and $Q^P \to Q'$ is a finitely-presented, quasi-finite monomorphism. If $\mathcal{F}$ has proper support over S, then $Q^P \to Q'$ is a finitely-presented closed immersion.*

Remark: In particular, if S is affine, then $Q^P(\mathcal{F}/\mathcal{X}/S)$ is represented by a quasi-projective S-scheme (which is projective if the support of $\mathcal{F}$ is proper over S).

We will prove theorem 4.4 in sections 5 and 6.

5. Generating sheaves

Let $\mathcal{X}$ be a tame Deligne-Mumford stack with coarse moduli space $\pi : \mathcal{X} \to X$. For each locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, define additive functors

$$F_{\mathcal{E}} : \text{Quasi-coherent}_{\mathcal{X}} \to \text{Quasi-coherent}_X, \tag{21}$$

$$G_{\mathcal{E}} : \text{Quasi-coherent}_{\mathcal{X}} \to \text{Quasi-coherent}_{\mathcal{X}} \tag{22}$$

by the formulas

$$F_{\mathcal{E}}(\mathcal{F}) = \pi_* Hom_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F}), G_{\mathcal{E}} = \pi^*\left(F_{\mathcal{E}}(\mathcal{F})\right) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E} \tag{23}$$

where $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{X}$.

By lemma 4.2, $F_{\mathcal{E}}$ is an exact functor which preserves flatness and preserves the property of being locally finitely-presented. And $G_{\mathcal{E}}$ is a right-exact functor which preserves the property of being locally finitely-presented. Moreover there is a natural transformation $\theta_{\mathcal{E}} : G_{\mathcal{E}} \Rightarrow \mathrm{Id}$ where Id is the identity functor on the category of quasi-coherent sheaves on $\mathcal{X}$ and for a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, the morphism

$$\theta_{\mathcal{E}}(\mathcal{F}) : \pi^* \left(\pi_* Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{E},\mathcal{F})\right) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E} \to \mathcal{F} \tag{24}$$

is the left adjoint to the natural morphism

$$\pi^*\pi_* Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{E},\mathcal{F}) \to Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{E},\mathcal{F}), \tag{25}$$

which is itself the left adjoint of the identity morphism

$$\pi_* Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{E},\mathcal{F}) \to \pi_* Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{E},\mathcal{F}). \tag{26}$$

Definition 5.1. *With notation as above, $\mathcal{E}$ is a* generator *for $\mathcal{F}$ if $\theta_{\mathcal{E}}(\mathcal{F})$ is surjective.*

Example: Suppose that G is a finite group and $\mathcal{X} = BG \times_{\mathrm{Spec}\ \mathbb{Z}} \mathrm{Spec}\ k$. Then the quasi-coherent $\mathcal{O}_\mathcal{X}$-modules correspond to (left-)modules over $k[G]$. Let $\mathcal{E}$ be the locally free sheaf corresponding to the left regular representation of G on $k[G]$. Then for every quasi-coherent sheaf $\mathcal{F}$, $\mathcal{E}$ is a generator for $\mathcal{F}$. For that matter, if M is any $k[G]$-module which contains every irreducible representation of G as a submodule, then the locally free sheaf corresponding to M is a generator for every $\mathcal{F}$.

The goal of this section is to prove that the previous example is typical for separated, tame, Deligne-Mumford stacks which are global quotients.

Proposition 5.2. *Suppose that $\mathcal{X}$ is a tame, separated Deligne-Mumford stack of the form $\mathcal{X} = [Y/G]$, where Y is a scheme and G is a finite group which acts on Y. Let $f : \mathcal{X} \to BG$ be the canonical 1-morphism, let E be the locally free sheaf on BG corresponding to the left regular representation, and let $\mathcal{E} = f^*E$. Then $\mathcal{E}$ is a generator for every quasi-coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$.*

Proof. Let $g : Y \to \mathcal{X}$ denote the quotient 1-morphism, and let $p : \mathcal{X} \to X$ denote the map to the coarse moduli space of $\mathcal{X}$. Observe that $\mathcal{E}$ is simply $g_*\mathcal{O}_Y$. The composition $p \circ g : Y \to X$ is a finite, surjective morphism of algebraic spaces. In particular, it is affine and for each quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, the induced morphism

$$\alpha : (p \circ g)^*(p \circ g)_*\mathcal{G} \to \mathcal{G}, \tag{27}$$

is surjective. By adjointness of p_* and p^*, there is an induced morphism

$$p^*(p \circ g)_*\mathcal{G} \to g_*\mathcal{G}, \tag{28}$$

in fact this is precisely $\theta_{\mathcal{O}_\mathcal{X}}(g_*\mathcal{G})$. Since $g_*\mathcal{G}$ is a module over $g_*\mathcal{O}_Y$, we get an induced morphism of $\mathcal{O}_\mathcal{X}$-modules

$$\phi : g_*\mathcal{O}_Y \otimes_{\mathcal{O}_\mathcal{X}} p^*(p \circ g)_*\mathcal{G} \to g_*\mathcal{G}. \tag{29}$$

The claim is that ϕ is surjective; let us assume this for a moment. The canonical injection $g^\# : \mathcal{O}_\mathcal{X} \to g_*\mathcal{O}_Y$ induces a morphism of $\mathcal{O}_\mathcal{X}$-modules

$$\psi : g_*\mathcal{O}_Y \otimes_{\mathcal{O}_\mathcal{X}} p^*p_* Hom_{\mathcal{O}_\mathcal{X}}(g_*\mathcal{O}_Y, g_*\mathcal{G}) \to g_*\mathcal{O}_Y \otimes_{\mathcal{O}_\mathcal{X}} p^*p_* Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}, g_*\mathcal{G}). \tag{30}$$

Since g is representable, finite and flat, there is a surjective trace map

$$t : g_*\mathcal{O}_Y \to \mathcal{O}_{\mathcal{X}}, \tag{31}$$

which splits the injection $g^{\#}$. Therefore ψ is surjective. And $\theta_{\mathcal{E}}(g_*\mathcal{G})$ is the composite $\phi \circ \psi$. Since ψ and ϕ are both surjective, we conclude that $\theta_{\mathcal{E}}(g_*\mathcal{G})$ is surjective. So to prove that $\theta_{\mathcal{E}}(g_*\mathcal{G})$ is surjective, it suffice to prove the claim that ϕ is surjective.

To see that ϕ is surjective, we apply g^*; since g is faithfully flat, we may check surjectivity after base-change by g. Since g is affine, the canonical morphism

$$(g^*g_*\mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{G} \to g^*g_*\mathcal{G} \tag{32}$$

is an isomorphism. Similarly, the canonical morphism

$$(g^*g_*\mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (p \circ g)^*(p \circ g)_*\mathcal{G} \to g^* \left(g_*\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{X}}} p^*(p \circ g)_*\mathcal{G}\right) \tag{33}$$

is also an isomorphism. And we have a commutative diagram

$$\begin{array}{ccc} (g^*g_*\mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (p \circ g)^*(p \circ g)_*\mathcal{G} & \xrightarrow{1\otimes\alpha} & (g^*g_*\mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{G} \\ \downarrow & & \downarrow \\ g^* \left(g_*\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{X}}} p^*(p \circ g)_*\mathcal{G}\right) & \xrightarrow{g^*\phi} & g^*g_*\mathcal{G} \end{array} \tag{34}$$

Since α is surjective, so is $1 \otimes \alpha$. Therefore $g^*\phi$ is surjective, and it follows that ϕ is surjective. This proves the claim, and we conclude that $\mathcal{E}$ is a generator for all sheaves of the form $g_*\mathcal{G}$ with $\mathcal{G}$ a quasi-coherent $\mathcal{O}_Y$-module.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. Since g is affine, the canonical morphism of $\mathcal{O}_{\mathcal{X}}$-modules

$$g_*\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F} \to g_*g^*\mathcal{F}, \tag{35}$$

is an isomorphism. Therefore $\mathcal{E}$ is a generator for $g_*\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}$. And we have a surjective morphism of $\mathcal{O}_{\mathcal{X}}$-modules,

$$t \otimes 1 : g_*\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F} \to \mathcal{F}. \tag{36}$$

Since $G_{\mathcal{E}}$ is a right-exact functor, we conclude that $\mathcal{E}$ is also a generator for $\mathcal{F}$. □

Corollary 5.3. *Suppose that $\mathcal{X}$, $\mathcal{E}$ are as in proposition 5.2. If $\mathcal{E}'$ is a locally free sheaf on $\mathcal{X}$ which generates $\mathcal{E}$, then $\mathcal{E}'$ generates $\mathcal{F}$ for every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$.*

Proof. Since p_* is exact, for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ and any quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, we have that

$$p_*(p^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}) = \mathcal{G} \otimes_{\mathcal{O}_X} p_*\mathcal{F}. \tag{37}$$

To see this, it suffices to work locally over affine opens in X, so we may suppose that $\mathcal{G}$ is a colimit of finitely-presented $\mathcal{O}_X$-modules. Since p_* commutes with colimits, we are reduced to the case that $\mathcal{G}$ is finitely-presented. Since p^* is right-exact and since p_* is exact, we are reduced to the case that $\mathcal{G} = \mathcal{O}_X$, which is trivial.

Similarly, we conclude that $\theta_{\mathcal{E}'}(p^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F})$ is just $1_{p^*\mathcal{G}} \otimes \theta_{\mathcal{E}'}(\mathcal{F})$. In particular, if $\mathcal{E}'$ generates $\mathcal{F}$, then $\mathcal{E}'$ generates $p^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}$. Therefore, $\mathcal{E}'$ generates all sheaf of the form $p^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}$. But by proposition 5.2, we conclude that every quasi-coherent sheaf $\mathcal{F}$ is a surjective image of a sheaf of the form $p^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}$. Since $G_{\mathcal{E}'}$ is right-exact, we conclude that $\mathcal{E}'$ generates $\mathcal{F}$. □

Suppose that $\mathcal{X}$ is a quotient stack of the form $[Y/GL_n]$. Let $f : \mathcal{X} \to BGL_n$ be the natural 1-morphism. Associated to each representation V of GL_n (over $\mathbb{Z}$), we have an associated quasi-coherent sheaf on BGL_n. Let $\mathcal{E}_V$ denote the pullback of this sheaf by f. We will use the previous corollary to prove that there is a GL_n representation V which is a finitely-generated free $\mathbb{Z}$-module and which generates $\mathcal{F}$ for every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-sheaf.

Lemma 5.4. *Suppose that $\mathcal{X}$ is a tame, global quotient $[Y/GL_{n,k}]$ which is equivalent to a stack BG_k for G a finite group, where k is an algebraically-closed field, Y is a k-scheme and we are given an action $m : GL_{n,k} \times_{\text{Spec } k} Y \to Y$ by k-morphisms. Then $GL_{n,k}$ acts transitively on Y, G is isomorphic to the stabilizer of a suitable k-valued point of Y, and there is a GL_n representation V which is a finitely-generated free $\mathbb{Z}$-module and which generates $\mathcal{F}$ for every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-sheaf.*

Proof. Let $g : BG_k \to [Y/GL_{n,k}]$ be a 1-morphism which gives an equivalence of categories. The composition of g with $\text{Spec } k \to BG_k$ is a 1-morphism which is finite, surjective and étale. By the definition of $[Y/GL_{n,k}]$, there is a (left) principal GL_n-bundle $\pi : P \to \text{Spec } k$ and a GL_n-equivariant morphism $h : P \to Y$ such that h is finite, surjective and étale. Since k is algebraically-closed and π is smooth, there is a section of π. Thus we may identify P with the trivial GL_n bundle, $P = GL_{n,k}$. Let $y \in Y$ be the image under h of the identity section of $GL_{n,k}$. The 1-morphism g gives us an homomorphism of the stabilizer group of $\text{Spec } k \to BG_k$ to the stabilizer group of $\text{Spec } k \to [Y/GL_{n,k}]$, i.e. a homomorphism of G to the stabilizer of y in Y. Since g is an equivalence of categories, this homomorphism is an isomorphism of groups. Since h is surjective, Y equals the orbit of y under $GL_{n,k}$. Thus we have an identification of $[Y/GL_{n,k}]$ with $[\text{Spec } \kappa(y)/G] = BG_k$. Of course this is the identification g given above.

To see the existence of V, note that the coordinate ring $\mathcal{O}_G = \Gamma(G_k, \mathcal{O}_{G_k})$ is the surjective image of the coordinate ring of $GL_{n,k}$. Now the coordinate ring of $GL_{n,k}$ is the base change of the coordinate ring of $GL_{n,\mathbb{Z}}$ which is

$$\Gamma(GL_{n,\mathbb{Z}}, \mathcal{O}_{GL_{n,\mathbb{Z}}}) = S = \bigoplus_{m \in \mathbb{Z}} S_m, \tag{38}$$

where S is a graded $\mathbb{Z}$-algebra whose homogeneous parts S_m are finite free $\mathbb{Z}$-modules which are representations of $GL_{n,Z}$, namely

$$S = \text{Sym}^{\cdot}_{\mathbb{Z}}(V^{\vee})[D^{-1}] \tag{39}$$

here $V^{\vee}$ is the dual $\text{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ where $V = \mathbb{Z}^n$ is the standard representation of $GL_{n,\mathbb{Z}}$, and $D \in \text{Sym}^n_{\mathbb{Z}}(V^{\vee})$ is the determinant function on V. As $\mathcal{O}_G$ is a finite k-algebra, there is an integer N such that $\mathcal{O}_G$ is the image of $\bigoplus_{m=-N}^{N} S_m \otimes k$. Moreover, notice that G acts on both $\mathcal{O}_G$ and on $S \otimes k$, and the surjection $S \otimes k \to \mathcal{O}_G$ is G-equivariant. In other words, if $V = \bigoplus_{m=-N}^{N} S_m$, then there is a surjection from $g^*\mathcal{E}_V$ to the locally free sheaf on BG_k corresponding to the (left) regular representation of G. So by corollary 5.3, we conclude that $g^*\mathcal{E}_V$ is a generator for every quasi-coherent $\mathcal{O}_{BG_k}$-module $\mathcal{F}$. □

Corollary 5.5. *Suppose that k is an algebraically closed field, $\mathcal{X} = [Y/GL_{n,k}]$ is a separated, tame Deligne-Mumford stack which is a global quotient over Spec k which is isomorphic to $[Z/G_k]$ where Z is a local Artin k-scheme and G is a finite group. Then there is a $GL_{n,\mathbb{Z}}$ representation V which is a finite free $\mathbb{Z}$-module and such that $\mathcal{E}_V$ generates every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$.*

Proof. The reduced stack of $\mathcal{X}$ is a stack which satisfies the hypotheses of the last lemma. Thus we know that we may find V such that $\mathcal{E}_V$ generates $\mathcal{F}$ for each quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module which is isomorphic to the push-forward of a quasi-coherent sheaf on $\mathcal{X}^{\text{red}}$. Denoting by n the length of Z, every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ admits a finite decreasing filtration by $\mathcal{O}_{\mathcal{X}}$-modules

$$\mathcal{F} = \mathcal{F}^0 \supset \cdots \supset \mathcal{F}^n = 0 \tag{40}$$

such that each associated graded $\mathcal{F}^m/\mathcal{F}^{m+1}$ is isomorphic to the push-forward of a quasi-coherent sheaf on $\mathcal{X}^{\text{red}}$. We will prove by induction on m that each quasi-coherent sheaf $\mathcal{F}^{n-m}$ is generated by $\mathcal{E}_V$. For $m = 0$ this is obvious. Suppose we have proved the result for $m < n$ and consider the case $m + 1$. We have a short exact sequence of quasi-coherent sheaves:

$$0 \longrightarrow \mathcal{F}^{n-m} \longrightarrow \mathcal{F}^{n-m-1} \longrightarrow \mathcal{F}^{n-m-1}/\mathcal{F}^{n-m} \longrightarrow 0. \tag{41}$$

By assumption, $\mathcal{F}^{n-m}$ is generated by $\mathcal{E}_V$. And by construction of V and lemma 5.4, $\mathcal{E}_V$ generates $\mathcal{F}^{n-m-1}/\mathcal{F}^{n-m}$. Since $G_{\mathcal{E}_V}$ is right-exact, we have a commutative diagram of exact sequences:

$$\begin{tikzcd}
 & G_{\mathcal{E}_V}(\mathcal{F}^{n-m}) \arrow[r] \arrow[d, "\theta_1"'] & G_{\mathcal{E}_V}(\mathcal{F}^{n-m-1}) \arrow[r] \arrow[d, "\theta_2"'] & G_{\mathcal{E}_V}(\mathcal{F}^{n-m-1}/\mathcal{F}^{n-m}) \arrow[r] \arrow[d, "\theta_3"'] & 0 \\
0 \arrow[r] & \mathcal{F}^{n-m} \arrow[r] & \mathcal{F}^{n-m-1} \arrow[r] & \mathcal{F}^{n-m-1}/\mathcal{F}^{n-m} \arrow[r] & 0
\end{tikzcd} \tag{42}$$

Since θ_1 and θ_3 are surjective, it follows from the snake lemma that also θ_2 is surjective. Thus $\mathcal{E}_V$ generates $\mathcal{F}^{n-m-1}$ and we conclude by induction that $\mathcal{E}_V$ generates $\mathcal{F}$. ☐

Now we come to the main lemma which shows that we can find a generator for a single locally finitely-presented sheaf.

Lemma 5.6. *Suppose S is an affine scheme, $f : \mathcal{X} \to S$ is a finitely-presented, separated 1-morphism of a tame Deligne-Mumford stack to S such that $\mathcal{X}$ is a global quotient over S, say $\mathcal{X}$ is equivalent to $[Y/GL_{n,S}]$. If $\mathcal{F}$ is a locally finitely-presented quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module, then there exists a GL_n representation V which is a finite free $\mathbb{Z}$-module and such that $\mathcal{E}_V$ is a generator for $\mathcal{F}$.*

Proof. By standard limit arguments, we may find a finite-type, affine $\mathbb{Z}$-scheme, S', a 1-morphism of stacks $f' : \mathcal{X}' \to S'$, a locally finitely-presented quasi-coherent $\mathcal{O}_{\mathcal{X}'}$-module $\mathcal{F}'$, and a morphism of schemes $S \to S'$ such that S', f', $\mathcal{F}'$ satisfy the same hypotheses as the lemma, and such that $\mathcal{X}$, $\mathcal{F}$ is isomorphic to the base-change of $\mathcal{X}'$, $\mathcal{F}'$ by $S \to S'$. Clearly it suffices to prove the result for $\mathcal{F}'$. Thus, without loss of generality, we may suppose that S is a finite-type, affine $\mathbb{Z}$-scheme.

For each $GL_{n,\mathbb{Z}}$ representation V which is a finite free $\mathbb{Z}$-module, the cokernel of $\theta_{\mathcal{E}_V}(\mathcal{F})$ is a coherent sheaf, and so has finite-dimensional support. Define $d(V)$ to be the dimension of the support (the maximum of the dimensions of the irreducible components), and define $n(V)$ to be the number of irreducible components of the support which have dimension equal to $d(V)$. We will prove that for some V $d(V) = -\infty$, i.e. $\mathcal{E}_V$ generates $\mathcal{F}$. By way of contradiction, suppose this is false. Then there is a V such that $d(V)$ is minimum. And among all V for which $d(V)$ is minimum, there is a V such that $n(V)$ is minimum.

We have an exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules

$$G_{\mathcal{E}_V}(\mathcal{F}) \xrightarrow[\theta]{} \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{43}$$

where $\mathcal{G}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module. Let $Z \subset \mathcal{X}$ be an irreducible component of the support of $\mathcal{G}$. Let $p : \mathcal{X} \to X$ denote the coarse moduli space of $\mathcal{X}$, and let $z \in p(Z)$ be a closed point. Let k denote the algebraic closure of $\kappa(z)$ and consider $\mathcal{X}_k = \mathcal{X} \times_X \operatorname{Spec} k$. Denote by $\mathcal{G}_k$ the pullback of $\mathcal{G}$ to $\mathcal{X}_k$. Now $\mathcal{X}_k$ is a stack which satisfies the hypotheses of corollary 5.5. Thus we can find W such that $\mathcal{E}_W$ generates $\mathcal{G}_k$. By flat base-change, we conclude that $\mathcal{E}_W$ generates $\mathcal{G}_z$. Since $G_{\mathcal{E}_W}$ is right-exact, this means that z is not in the support of the cokernel of $\theta_{\mathcal{E}_W}(\mathcal{G})$. So the support of $\theta_{\mathcal{E}_{V\oplus W}}(\mathcal{G})$ is contained in the support of $\mathcal{G}$ and does not contain Z. Since $G_{\mathcal{E}_{V\oplus W}}$ is right-exact, we conclude that the cokernel of $\theta_{\mathcal{E}_{V\oplus W}}(\mathcal{F})$ is contained in the support of $\mathcal{G}$ and does not contain Z. But then either $d(V \oplus W) < d(V)$ or else $d(V \oplus W) = d(V)$ and $n(V \oplus W) < n(V)$. This contradicts the choice of V, so we conclude by contradiction that there exists a V such that $\theta_{\mathcal{E}_V}(\mathcal{F})$ is surjective. □

Now we come to the main theorem of this section:

Theorem 5.7. *Suppose S is a quasi-compact algebraic space, $f : \mathcal{X} \to S$ is a separated, finitely-presented 1-morphism of a tame Deligne-Mumford stack to S such that $\mathcal{X}$ is a global quotient over S, say $\mathcal{X} = [Y/GL_{n,S}]$. Let $p : \mathcal{X} \to X$ denote the coarse moduli space of $\mathcal{X}$. There exists a $GL_{n,\mathbb{Z}}$ representation V which is a finite free $\mathbb{Z}$-module such that for every morphism of algebraic spaces $X' \to X$ and for each quasi-coherent sheaf $\mathcal{F}$ on $X' \times_X \mathcal{X}$, $\mathcal{E}_V$ is a generator for $\mathcal{F}$.*

Proof. Since S is quasi-compact, we may find a finite étale covering $\{S_i \to S\}$ such that each S_i is an affine scheme. For each i, $\mathcal{X}_i$ is a separated, tame, Deligne-Mumford stack. Let $p_i : \mathcal{X}_i \to X_i$ denote the coarse moduli space of $\mathcal{X}_i$. By [1, lemma 2.2.3], there is a finite étale covering $\{X_{i,\alpha} \to X_i\}$ such that each $\mathcal{X}_i \times_{X_i} X_{i,\alpha}$ is of the form $[Y_{i,\alpha}/G_{i,\alpha}]$ where $Y_{i,\alpha}$ is a finitely-presented S_i-scheme and $G_{i,\alpha}$ is a finite group acting on $Y_{i,\alpha}$ by S_i-morphisms.

Let $\mathcal{E}_{i,\alpha}$ denote the locally free sheaf on $\mathcal{X}_{i,\alpha}$ which corresponds to the (left) regular representation of $G_{i,\alpha}$ as in proposition 5.2. By lemma 5.6, there is $GL_{n,\mathbb{Z}}$ representation $V_{i,\alpha}$ which is a finite free $\mathbb{Z}$-module and such that $\mathcal{E}_{V_{i,\alpha}}$ generates $\mathcal{E}_{i,\alpha}$. By corollary 5.3, we conclude that $\mathcal{E}_{V_{i,\alpha}}$ generates all quasi-coherent $\mathcal{O}_{\mathcal{X}_{i,\alpha}}$-modules. Define V to be the direct sum of the finitely many $GL_{n,\mathbb{Z}}$-representation $V_{i,\alpha}$. Then for each i, each α, and each quasi-coherent $\mathcal{O}_{\mathcal{X}_{i,\alpha}}$-module $\mathcal{F}$, we see that $\mathcal{E}_V$ generates $\mathcal{F}$.

Now suppose that $X' \to X$ is a morphism of algebraic spaces and $\mathcal{F}$ is a quasi-coherent sheaf on $X' \times_X \mathcal{X}$. The claim is that $\mathcal{E}_V$ generates $\mathcal{F}$. This may be checked étale locally on the coarse moduli space of $X' \times_X \mathcal{X}$. Combining [1, lemma 2.3.3] with the arguments in the last paragraph, we see that this is true étale locally on the coarse moduli space of $\mathcal{X}_U$: Each of the $X' \times_X \mathcal{X}_{i,\alpha}$ is of the form $[X' \times_X Y_{i,\alpha}/G_{i,\alpha}]$. By the same argument in the last paragraph, $\mathcal{E}_V$ generates all quasi-coherent sheaves on $X' \times_X \mathcal{X}_{i,\alpha}$. Therefore $\mathcal{E}_V$ generates $\mathcal{F}$. □

6. Natural Transformation of Quot Functors

Suppose that S is a quasi-compact algebraic space, $f : \mathcal{X} \to S$ is a separated, finitely-presented 1-morphism of a tame Deligne-Mumford stack to S such that $\mathcal{X}$ is a global quotient over S, say $\mathcal{X} = [Y/GL_{n,X}]$. By theorem 5.7, there is a $GL_{n,\mathbb{Z}}$ representation V which is a finite free $\mathbb{Z}$-module and such that for every morphism of algebraic spaces $T \to S$ and every quasi-coherent sheaf $\mathcal{G}$ on $T \times_S \mathcal{X}$, $\mathcal{E}_V$ generates $\mathcal{G}$.

Suppose that $\mathcal{F}$ is a locally finitely-presented $\mathcal{O}_\mathcal{X}$-module. Suppose that $P : K^0(\mathcal{X}) \to \mathbb{Z}$ is a given Hilbert polynomial. Let $p : \mathcal{X} \to X$ denote the coarse moduli scheme of $\mathcal{X}$ and let $P_V : K^0(X) \to \mathbb{Z}$ denote the map

$$P_V([E]) = P([Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{E}_V, p^*E)]). \tag{44}$$

Suppose $T \to S$ is a morphism of a scheme to S and suppose that $\mathcal{F}_T \to \mathcal{G}$ is an element of $Q^P(\mathcal{F}/\mathcal{X}/S)(T)$. Consider $F_{\mathcal{E}_V}(\mathcal{F}_T) \to F_{\mathcal{E}_V}(\mathcal{G})$. Since $F_{\mathcal{E}_V}$ is exact, this is still surjective. By lemma 4.2, $F_{\mathcal{E}_V}(\mathcal{G})$ is locally finitely-presented, has proper support over T and is flat over T. Moreover every geometric fiber has Hilbert polynomial P_V, i.e. $F_{\mathcal{E}_V}(\mathcal{F}_T) \to F_{\mathcal{E}_V}(\mathcal{G})$ is an element of $Q^{P_V}(F_{\mathcal{E}_V}/X/S)(T)$. This defines a natural transformation

$$F_{\mathcal{E}_V} : Q^P(\mathcal{F}/\mathcal{X}/S) \Rightarrow Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S). \tag{45}$$

Lemma 6.1. *The natural transformation $F_{\mathcal{E}_V}$ is a monomorphism, i.e. for each $T \to S$ the morphism of sets*

$$F_{\mathcal{E}_V}(T) : Q^P(\mathcal{F}/\mathcal{X}/S)(T) \to Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S)(T), \tag{46}$$

is an injection of sets.

Proof. Given any morphism $T \to S$ of a scheme to S and given any element $\alpha : F_{\mathcal{E}_V}(\mathcal{F})_T \to G$ in $Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S)(T)$, define $\beta : K \to F_{\mathcal{E}_V}(\mathcal{F})_T$ to be the kernel of α and define $\eta_T(\alpha) : \mathcal{F} \to \mathcal{G}$ to be the cokernel of the composition:

$$p^*K \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E}_V \xrightarrow{p^*\beta\otimes 1} G_{\mathcal{E}_V}(\mathcal{F}_T) \xrightarrow{\theta_{\mathcal{E}_V}(\mathcal{F}_T)} \mathcal{F}_T. \tag{47}$$

Now suppose given $\nu : \mathcal{F}_T \to \mathcal{G}$ in $Q^P(\mathcal{F}/\mathcal{X}/S)(T)$ and let $\mu : \mathcal{K} \to \mathcal{F}_T$ denote the kernel of ν. Then we have a short exact sequence:

$$0 \longrightarrow F_{\mathcal{E}_V}(\mathcal{K}) \xrightarrow{F_{\mathcal{E}_V}(\mu)} F_{\mathcal{E}_V}(\mathcal{F})_T \xrightarrow{F_{\mathcal{E}_V}(\nu)} F_{\mathcal{E}_V}(\mathcal{G}) \longrightarrow 0. \tag{48}$$

Notice that if $\alpha = F_{\mathcal{E}_V}(\nu)$, then $\beta = F_{\mathcal{E}_F}(\mu)$. Since p^* is right-exact, we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & G_{\mathcal{E}_V}(\mathcal{K}) & \xrightarrow{p^*\beta\otimes 1} & G_{\mathcal{E}_V}(\mathcal{F}_T) & \xrightarrow{p^*\alpha\otimes 1} & G_{\mathcal{E}_V}(\mathcal{G}) & \longrightarrow & 0 \\ & & {\scriptstyle\theta_{\mathcal{E}_V}(\mathcal{K})}\downarrow & & {\scriptstyle\theta_{\mathcal{E}_V}(\mathcal{F}_T)}\downarrow & & {\scriptstyle\theta_{\mathcal{E}_V}(\mathcal{G})}\downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \xrightarrow{\mu} & \mathcal{E}_T & \xrightarrow{\nu} & \mathcal{K} & \longrightarrow & 0 \end{array} \tag{49}$$

It follows by the snake lemma that we have the formula:

$$\eta_T(F_{\mathcal{E}_V}(\mu)) = \mu. \tag{50}$$

Therefore $F_{\mathcal{E}_V}$ is injective. □

Remark: Notice that the association $T \mapsto \eta_T$ is functorial for arbitrary S-morphisms $T_1 \to T_2$. This might seem odd since the formation of the kernel K is not compatible arbitrary base-change. But one could define η_T equivalently to be the coequalizer of the diagram:

$$\begin{tikzcd} G_{\mathcal{E}_V}(\mathcal{F}_T) \arrow[r, "p^*\alpha\otimes 1"] \arrow[d, "\theta_{\mathcal{E}_V}(\mathcal{F}_T)"'] & p^*G \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}_V \\ \mathcal{F}_T & \end{tikzcd} \tag{51}$$

Since coequalizers are compatible with arbitrary base-change, we see that η is compatible with arbitrary S-morphisms $T_1 \to T_2$.

Proposition 6.2. *The monomorphism $F_{\mathcal{E}_V}$ of functors is relatively representable by schemes. In fact $F_{\mathcal{E}_V}$ is a finitely-presented, quasi-finite, monomorphism. If $\mathcal{F}$ has proper support over S, then $F_{\mathcal{E}_V}$ is a finitely-presented, finite, monomorphism, i.e. a finitely-presented closed immersion.*

Proof. Suppose $T \to S$ is a morphism of a scheme to S and suppose that $\alpha : F_{\mathcal{E}_V}(\mathcal{F}_T) \to G$ is an element of $Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S)$. Now form $\eta_T(\alpha) : \mathcal{F}_T \to \mathcal{G}$. By theorem 3.2, there is a flattening stratification $g : \Sigma \to T$ for $\mathcal{G}$ and g is a surjective, finitely-presented, quasi-finite monomorphism. By lemma 4.3, for each connected component Σ_i of Σ, the restriction of $\mathcal{G}$ to Σ_i has constant Hilbert polynomial P_i. In particular, there is a connected component (possibly empty) Σ_i of Σ on which the restriction of $\mathcal{G}$ has Hilbert polynomial P. Of course we have an induced natural transformation of functors

$$\Sigma_i \to T \times_{\alpha, Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S)} Q^P(\mathcal{F}/\mathcal{X}/S). \tag{52}$$

By equation 50, we have an inverse natural transformation. Thus we conclude that the fiber functor $T \times_{\alpha, Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S)} Q^P(\mathcal{F}/\mathcal{X}/S)$ is represented by the morphism $g_i : \Sigma_i \to T$. Notice that this is a finitely-presented, quasi-finite monomorphism of schemes.

In case $\mathcal{F}$ has proper support over S, we know from theorem 1.1 that $Q^P \to S$ satisfies the valuative criterion of properness. Therefore $g_i : \Sigma_i \to T$ satisfies the valuative criterion of properness, i.e. g_i is finite. But a finite monomorphism is precisely a closed immersion, therefore g_i is a finitely-presented closed immersion. $\square$

Now we are in a position to prove theorem 4.4. By proposition 6.2, we know that

$$Q^P(\mathcal{F}/\mathcal{X}/S) \to Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S) \tag{53}$$

is relatively representable by a finitely-presented, quasi-finite monomorphism (resp. finitely-presented closed immersion).

Now the proof that $Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S) \to S$ is represented by a scheme which is quasi-projective over S is essetially [6, théorème 3.2, part IV]. In his proof, Grothendieck makes some Noetherian hypotheses which are eliminated in a standard way, cf. [5, section IV.8.9]. For the sake of completeness, we include the proof here.

We have a locally closed immersion $X \hookrightarrow \mathbb{P}^N_S$ for some N. By [5, proposition IV.8.9.1], we can find a finite-type affine $\mathbb{Z}$-scheme, S', a quasi-projective (resp. projective) S'-scheme, X', and a coherent sheaf F' on X' along with a morphism $S \to S'$ such that X is isomorphic to $S \times_{S'} X'$, and under this isomorphism $F_{\mathcal{E}_V}(\mathcal{F})$ is isomorphic to the pullback of F'. Let

$p : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$ be the polynomial $p(t^n) = P_V([\mathcal{O}_X(n)])$ where $\mathcal{O}_X(n)$ is the pullback to X of the invertible sheaf $\mathcal{O}_{\mathbb{P}^N_S}(n)$ on $\mathbb{P}^N_S$. polynomial, and consider the Quot functor $Q^p(F'/X'/S')$ which is the connected component of $Q(F'/X'/S')$ parametrizing families of quotients $F'_T \to G$ such that for each closed point $t \in T$, we have $\chi(G_t \otimes \mathcal{O}(n)) = p(t^n)$. By lemma 4.3, it follows that $Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S)$ is isomorphic to a connected component of the fiber product $S \times_{S'} Q^p(F'/X'/S')$. So to prove that $Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S) \to S$ is quasi-projective, it suffices to show that $Q^p(F'/X'/S') \to S'$ is quasi-projective.

By [6, théorème 3.2, part IV], we know that the restriction of $Q^p(F'/X'/S')$ to the category of locally Noetherian S'-schemes is represented by a quasi-projective S'-scheme. Using [5, proposition IV.8.9.1] and [5, théorème IV.11.2.6], it follows that in fact this quasi-projective S'-scheme represents $Q^p(F'/X'/S')$ on the category of all S'-schemes. Thus $Q^{P_V}(F_{\mathcal{E}_V}(\mathcal{F})/X/S) \to S$ is represented by a quasi-projective scheme. This completes the proof of theorem 4.4.

References

[1] D. Abramovich and A. Vistoli. Compactifying the space of stable maps. *Journal of the American Mathematical Society*, 15:27–75, 2002. arXiv:math.AG/9908167.

[2] M. Artin. Algebraization of formal moduli. I. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 21–71. Univ. Tokyo Press, Tokyo, 1969.

[3] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1990.

[4] A. K. Dan Edidin, Brendan Hassett and A. Vistoli. Brauer groups and quotient stacks. *American Journal of Mathematics*, 123:761–777, 2001. arXiv:math.AG/9905049.

[5] J. Dieudonné and A. Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1961-1967.

[6] A. Grothendieck. *Fondements de la géométrie algébrique*. Secrétariat mathématique, 1962.

[7] S. Keel and S. Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145:193–213, 1997.

[8] D. Knutson. *Algebraic spaces*, volume 203 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[9] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, 2000.

[10] D. Mumford. *Lectures on curves on an algebraic surface*, volume 59 of *Annals of Mathematics Studies*. Princeton University Press, 1966.

[11] M. Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139

E-mail address: `molsson@math.mit.edu`

Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139

E-mail address: `jstarr@math.mit.edu`