DIVISORS ON THE SPACE OF MAPS TO GRASSMANNIANS

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Abstract. In this note we study the divisor theory of the Kontsevich moduli spaces $\overline{\mathcal{M}}_{0,0}(G(k,n),d)$ of genus-zero stable maps to the Grassmannians. We calculate the classes of several geometrically significant divisors. We prove that the cone of effective divisors stabilizes as n increases and we determine the stable effective cone. We also characterize the ample cone.

Contents

1. Introduction

The cones of ample and effective divisors are two of the most fundamental invariants of a variety. They control the birational and projective geometry of the variety. In this paper we determine the ample and stable effective cones of Kontsevich moduli spaces $\overline{\mathcal{M}}_{0,0}(G(k,n),d)$ of genus-zero stable maps to Grassmannians.

The papers [CHS1] and [CHS2] introduce techniques for studying the ample and effective cones of Kontsevich moduli spaces and determine these cones when the target is projective space. The techniques developed in [CHS1] and [CHS2] apply to more general targets, as we illustrate here with the example of Grassmannians. Our first main theorem is:

Theorem 1.1. *Let k,n and d be positive integers such that $2 \leq k < k+2 \leq n$. Let m be a non-negative integer such that $m+d \geq 3$. There is an injective linear map*

$$v : Pic_{\mathbb{Q}}(\overline{M}_{0,m+d}/\mathfrak{S}_d) \to Pic_{\mathbb{Q}}(\overline{M}_{0,m}(G(k,n),d))$$

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that maps NEF (resp. base-point-free) divisors to NEF (resp. base-point-free) divisors. The NEF (resp. base-point-free) cone of $\overline{M}_{0,m}(G(k,n),d)$ is the product of the cones spanned by $H_{\sigma_2}, H_{\sigma_{1,1}}, T, L_1, \ldots, L_m$ and the image under v of the NEF (resp. base-point-free) cone of $\overline{M}_{0,m+d}/\mathfrak{S}_d$.

Here the action of the symmetric group $\mathfrak{S}_d$ permutes the last d marked points. The divisors $H_{\sigma_{1,1}}$ and H_{σ_2} are the divisors of curves whose image intersects fixed Schubert varieties $\Sigma_{1,1}$ and Σ_2, respectively. The divisor T is the tangency divisor and L_i are the pull-backs of the ample generator of $Pic(G(k,n))$ by the i-th evaluation morphism.

There is a natural way of generating NEF divisors on the Kontsevich moduli spaces to arbitrary targets X from NEF codimension one and two classes on X. For a large class of target varieties X, these natural NEF divisors together with the pull-back of NEF divisors from the Deligne-Mumford moduli space generate the NEF cone of $\overline{\mathcal{M}}_{0,m}(X,\beta)$. Our proof of Theorem 1.1 will remain valid in a more general setting—proving, in particular, the corresponding statement for flag varieties.

In contrast, the effective cones of Kontsevich moduli spaces are harder to understand and closely depend on the target variety. We first prove that the effective cones of $\overline{\mathcal{M}}_{0,0}(G(k,n),d)$ stabilize as long as $n \geq k+d$. We then give a complete description of the effective cone of $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$. Our methods bound the effective cone of $\overline{\mathcal{M}}_{0,0}(G(k,n),d)$ when $n < k+d$; however, we do not have a complete description of the cone in this case.

In order to describe the effective cone we have to construct several effective divisors. Surprisingly the construction depends on whether k divides d. When k divides d, let D_{unb} be the class of the divisor of maps for which the pull-back of the tautological bundle of $G(k,k+d)$ has non-balanced splitting. If k does not divide d, set $d = kq + r$ for $0 < r < k$. A rational curve in a Grassmannian induces a rational scroll in projective space. D_{unb} is the class of the closure of maps where the linear spans of the directrices of corresponding scrolls intersect a fixed codimension $(k-r)(q+1)$ linear space. D_{deg} is the class of the divisor of maps whose image lies in a subgrassmannian $G(k,k+d-1)$. With this notation, our second main theorem is:

Theorem 1.2. *The effective cone of $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$ is a simplicial cone generated by D_{deg}, D_{unb} and the boundary divisors.*

The organization of the paper is as follows. In §2 we collect the facts that we use about Kontsevich moduli spaces and state our results more precisely. In §3 we prove that the effective cones of $\overline{\mathcal{M}}_{0,0}(G(k,n),d)$ stabilize when $n \geq k+d$. In §4 we express the divisor classes D_{deg} and D_{unb} in terms of the standard basis of the Picard group of $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$. In §5 we prove a generalization of Theorem 1.1. In §6 we prove Theorem 1.2.

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2. Preliminaries

In this section we recall the necessary facts about Kontsevich moduli spaces.

Let X be a smooth, complex projective homogeneous variety. Let $\beta \in H_2(X, \mathbb{Z})$ be the homology class of a curve. The Kontsevich moduli space of m-pointed, genus-zero stable maps $\overline{\mathcal{M}}_{0,m}(X, \beta)$ provides a useful compactification of the space of rational curves on X whose homology class is β. $\overline{\mathcal{M}}_{0,m}(X, \beta)$ is the smooth, proper Deligne-Mumford stack parameterizing the data of

(i) C, a proper, connected, at-worst-nodal curve of arithmetic genus 0,
(ii) $p_1, \ldots, p_m$, an ordered sequence of distinct, smooth points of C,
(iii) and $f : C \to X$, a morphism with $f_*[C] = \beta$ satisfying the following stability condition: every irreducible component of C mapped to a point under f contains at least 3 special points, i.e., marked points p_i and nodes of C.

By [KP] $\overline{\mathcal{M}}_{0,m}(X, \beta)$ is irreducible.

Oprea in [Opr] proves that the rational complex codimension one classes on $\overline{\mathcal{M}}_{0,m}(X, \beta)$ are tautological. When X is the Grassmannian $G(k, n)$, one can very explicitly describe the Picard group $Pic_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(G(k, n), d)$ as follows. The cohomology ring of the Grassmannian $G(k, n)$ is generated by the Poincaré duals of the classes of Schubert varieties. We use the convention in [GH] to denote Schubert varieties. There are two divisor classes $H_{\sigma_{1,1}}$ and H_{σ_2} on $\overline{\mathcal{M}}_{0,0}(G(k, n), d)$ corresponding to the two codimension-two classes $\sigma_{1,1}$ and σ_2 on $G(k, n)$. Informally, these are the classes of the divisors that parameterize the maps whose images intersect fixed Schubert varieties $\Sigma_{1,1}$ and Σ_2, respectively. More precisely, let

$$ev_i : \overline{\mathcal{M}}_{0,m}(G(k, n), d) \to G(k, n)$$

denote the i-th evaluation morphism and let

$$\pi : \overline{\mathcal{M}}_{0,1}(G(k, n), d) \to \overline{\mathcal{M}}_{0,0}(G(k, n), d)$$

denote the forgetful morphism. $H_{\sigma_{1,1}}$ and H_{σ_2} are defined as follows:

$$H_{\sigma_{1,1}} := \pi_*\ ev_1^*(\sigma_{1,1}), \quad H_{\sigma_2} := \pi_*\ ev_1^*(\sigma_2).$$

In addition to these, for every $1 \leq i \leq \lfloor d/2 \rfloor$, there is a boundary divisor on $\overline{\mathcal{M}}_{0,0}(G(k, n), d)$ whose general point parameterizes a map with a reducible domain $C_1 \cup C_2$ where the map has Plücker degree i on C_1 and degree $d - i$ on C_2, respectively. We will denote the class of the boundary divisor by $\Delta_{i,d-i}$. The theorem of Oprea reduces to the following in the special case of $G(k, n)$.

Theorem 2.1. *([Opr]) Let $2 \leq k < k + 2 \leq n$ and $d \geq 1$ be integers. The divisor classes $H_{\sigma_{1,1}}, H_{\sigma_2}$ and $\Delta_{i,d-i}$ for $1 \leq i \leq \lfloor d/2 \rfloor$ generate $Pic_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(G(k, n), d))$.*

Let $P_{k,n,d}$ be the $\mathbb{Q}$-vector space spanned by basis elements labelled by $H_{\sigma_{1,1}}, H_{\sigma_2}$ and $\Delta_{i,d-i}$ for $1 \leq i \leq \lfloor d/2 \rfloor$. $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(G(k, n), d))$ can be naturally identified with $P_{k,n,d}$. Let Eff (k, n, d) denote the image of the effective cone under this identification. We can view the effective cones Eff (k, n, d) for different n as cones in a fixed vector space since $P_{k,n,d}$ does not depend on n. The first basic result about the cones Eff (k, n, d) is that if we fix k and d, they stabilize for large n.

Theorem 2.2. *For integers $k \geq 2$ and $r \geq 2$, the cone Eff $(k, k+r, d)$ is contained in the cone Eff $(k, k+r+1, d)$. Furthermore, Eff $(k, k+r, d) =$ Eff $(k, k+d, d)$ for every $r \geq d$.*

In view of Theorem 2.2 it is especially interesting to describe Eff $(k, k+d, d)$. For the rest of the paper we will concentrate on this case. On $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ there are two natural effective $\mathbb{Q}$-Cartier divisors D_{deg} and D_{unb}. Their classes will play an essential role in the description of the effective cone of $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$.

Let D_{deg} be the class of the divisor of maps whose image lies in a subgrassmannian $G(k, k+d-1)$ induced from a $k+d-1$-dimensional linear subspace. This divisor is the analogue of the divisor of degenerate maps D_{deg} on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ (see §2.2 of [CHS2]). We will calculate the class D_{deg} in §4.1.

The definition of D_{unb} depends on whether k divides d. If k divides d, then a general point in $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ parameterizes a map $f : \mathbb{P}^1 \to G(k, k+d)$ such that the pull-back of the tautological bundle from $G(k, k+d)$ has balanced splitting. In codimension one the pull-back of the tautological bundle does not have balanced splitting. Informally the class of the closure of the subvariety where the splitting is not balanced is D_{unb}. More precisely, let $F(k-1, k, k+d)$ denote the two-step flag variety parameterizing $k-1$-dimensional linear subspaces contained in k-dimensional linear subspaces of a fixed $k+d$-dimensional linear space. The flag variety $F(k-1, k, k+d)$ admits two projections π_1 and π_2 to the Grassmannians $G(k-1, k+d)$ and $G(k, k+d)$, respectively. A curve class in $F(k-1, k, k+d)$ is determined by specifying two integers d_1, d_2, where d_i is the intersection of the curve class with the pull-back of the ample generator by π_i.

Consider the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(F(k-1, k, k+d), (d-k-1, d))$. The second projection π_2 induces a 1-morphism

$$\overline{\mathcal{M}}_{0,0}(\pi_2) : \overline{\mathcal{M}}_{0,0}(F(k-1, k, k+d), (d-k-1, d)) \to \overline{\mathcal{M}}_{0,0}(G(k, k+d), d).$$

The morphism $\overline{\mathcal{M}}_{0,0}(\pi_2)$ is birational onto its image. Since $\overline{\mathcal{M}}_{0,0}(F(k-1, k, k+d), (d-k-1, d))$ is irreducible, the image is irreducible. Moreover, an easy dimension count shows that the image has codimension one in $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$. It follows that the image is an irreducible divisor. We denote its class by D_{unb}.

If k does not divide d, then the locus of maps where the splitting type of the tautological bundle is not balanced has codimension two, hence does not form a divisor. In that case, we can write $d = kq + r$ where $0 < r < k$. Let $F(k-r, k, k+d)$ be the two-step flag variety of $(k-r)$-dimensional subspaces contained in k-dimensional subspaces of a fixed $(k+d)$-dimensional linear space. The second projection

$$\pi_2 : F(k-r, k, k+d) \to G(k, k+d)$$

induces a birational morphism

$$\overline{\mathcal{M}}_{0,0}(\pi_2) : \overline{\mathcal{M}}_{0,0}(F(k-r, k, k+d), ((k-r)q, d)) \to \overline{\mathcal{M}}_{0,0}(G(k, k+d), d).$$

The first projection

$$\pi_1 : F(k-r, k, k+d) \to G(k-r, k+d)$$

induces a 1-morphism

$$\overline{\mathcal{M}}_{0,0}(\pi_1) : \overline{\mathcal{M}}_{0,0}(F(k-r, k, k+d), ((k-r)q, d)) \to \overline{\mathcal{M}}_{0,0}(G(k-r, k+d), (k-r)q).$$

Let S denote the tautological bundle of $G(k-r, k+d)$. $\mathbb{P}S$ has natural projections ϕ_1 and ϕ_2 to $G(k-r, k+d)$ and $\mathbb{P}^{k+d-1}$, respectively. By the universal property of $\mathbb{P}S$, the evaluation morphism

$$ev : \overline{\mathcal{M}}_{0,1}(G(k-r, k+d), (k-r)q) \to G(k-r, k+d)$$

induces a 1-morphism to $\mathbb{P}S$ and hence by composing with ϕ_2 to $\mathbb{P}^{k+d-1}$. Asking for the linear span of the image to intersect a fixed $k+d-1-(k-r)(q+1)$-dimensional projective linear subspace of $\mathbb{P}^{k+d-1}$ defines a class in $\overline{\mathcal{M}}_{0,1}(G(k-r, k+d), (k-r)q)$ whose push-forward by the forgetful morphism is a divisor D on $\overline{\mathcal{M}}_{0,0}(G(k-r, k+d), (k-r)q)$. We define D_{unb} as

$$D_{\text{unb}} = \overline{\mathcal{M}}_{0,0}(\pi_2)_* \, \overline{\mathcal{M}}_{0,0}(\pi_1)^* D.$$

With this notation, we can rephrase Theorem 1.2 as follows:

Theorem 2.3. *A divisor D on $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ lies on the effective cone if and only if D is a non-negative linear combination*

$$D = a \; D_{deg} + b \; D_{unb} + \sum_{i=1}^{\lfloor d/2 \rfloor} c_i \Delta_{i,d-i}, \qquad a, b, c_i \geq 0$$

of the divisors D_{deg}, D_{unb} and the boundary divisors.

The proof is similar to the proof of the main theorem in [CHS2]. Recall that a moving curve is a reduced, irreducible curve whose deformations cover a Zariski open subset of $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$. Every moving curve in $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ gives rise to an inequality on the effective cone. If B is a moving curve and D is an effective Cartier divisor, then $B \cdot D \geq 0$. Since the divisors D_{deg}, D_{unb} and the boundary divisors are effective, any non-negative linear combination is also effective. The proof is completed by constructing enough moving curves such that the inequalities they give rise to cut out precisely the cone spanned by D_{deg}, D_{unb} and the boundary divisors.

The following lemma gives an easy criterion for deciding whether a reduced, irreducible curve $B \subset \overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ is a moving curve.

Lemma 2.4. *A reduced, irreducible curve $B \subset \overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ is a moving curve if B contains a point parameterizing a map*

$$f : C \to G(k, k+d),$$

where the domain of C is irreducible, the image of f is non-degenerate and the pull-back of the tautological bundle of $G(k, k+d)$ to C has balanced splitting.

Proof. Let $f : C \to G(k, k+d)$ be a map satisfying the assumptions of the lemma. The orbit of this map under the action of $\mathbb{P}GL(k+d)$ forms a dense subset of $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$. Consequently, the $\mathbb{P}GL(k+d)$ orbit of B covers a Zariski open subset of $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$. Therefore, B is a moving curve. □

3. The stability of the effective cone

In this section we prove that the effective cones of $\overline{\mathcal{M}}_{0,0}(G(k, n), d)$ stabilize when $n \geq k+d$. This proves Theorem 2.2 stated in §2.

Proof of Theorem 2.2. For the proof it is more convenient to think of $G(k, k+r) = \mathbb{G}(k-1, k+r-1)$ as the parameter space for $k-1$-dimensional projective linear spaces in $\mathbb{P}^{k+r-1}$. Let $p \in \mathbb{P}^{k+r}$ be a point. Let $U = \mathbb{P}^{k+r} - \{p\}$ be the complement of the point. Let

$$\tilde{\pi} : U \to \mathbb{P}^{k+r-1}$$

denote linear projection from the point p. The morphism $\tilde{\pi}$ induces a corresponding morphism

$$\pi : V \to \mathbb{G}(k-1, k+r-1),$$

where V denotes the open subscheme of $\mathbb{G}(k-1, k+r)$ parameterizing $\mathbb{P}^{k-1}$ not containing the point p. This in turn induces a smooth 1-morphism

$$\overline{\mathcal{M}}_{0,0}(\pi, k+r) : \overline{\mathcal{M}}_{0,0}(V, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{G}(k-1, k+r-1), d).$$

Let $i : V \to \mathbb{G}(k-1, k+r)$ be the open immersion of V into $\mathbb{G}(k-1, k+r)$. The morphism i also induces a 1-morphism between Kontsevich moduli spaces

$$\overline{\mathcal{M}}_{0,0}(i, d) : \overline{\mathcal{M}}_{0,0}(V, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{G}(k-1, k+r), d).$$

The complement of the image has codimension r. Since $r \geq 2$, the pull-back morphism on the Picard groups

$$\overline{\mathcal{M}}_{0,0}(i, d)^* : \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(\mathbb{G}(k-1, k+r), d)) \to \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(V, d))$$

is an isomorphism. There is a unique homomorphism

$$h : \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(\mathbb{G}(k-1, k+r-1), d)) \to \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(\mathbb{G}(k-1, k+r), d))$$

such that

$$\overline{\mathcal{M}}_{0,0}(\pi, k+r)^* = \overline{\mathcal{M}}_{0,0}(i, d)^* \circ h.$$

Since $\overline{\mathcal{M}}_{0,0}(\pi, k+r)$ is smooth it pulls-back effective divisors to effective divisors. It follows that Eff $(k, k+r, d) \subset$ Eff $(k, k+r+1, d)$.

Next suppose that $r \geq d$. Let D be any effective divisor on $\overline{\mathcal{M}}_{0,0}(G(k, k+r, d)$. A general point in the complement of D parameterizes a stable map $f : C \to G(k, k+r)$ such that the image lies in a subgrassmannian $G(k, k+d)$. The linear embedding of the corresponding $\mathbb{P}^{k+d-1}$ into $\mathbb{P}^{k+r-1}$ induces a 1-morphism

$$\overline{\mathcal{M}}_{0,0}(k+d, k+r) : \overline{\mathcal{M}}_{0,0}(G(k, k+d, d) \to \overline{\mathcal{M}}_{0,0}(G(k, k+r, d).$$

By construction $\overline{\mathcal{M}}_{0,0}(k+d, k+r)^*([D])$ is the class of the effective divisor $\overline{\mathcal{M}}_{0,0}(k+d, k+r)^{-1}(D)$. Therefore, Eff $(k, k+d, d)$ contains Eff $(k, k+r, d)$. Hence, by the previous paragraph the two are equal. This concludes the proof of Theorem 2.2. ∎

4. The divisor class calculations

4.1. The divisor class $D_{\mathbf{deg}}$. In this section we express the divisor class D_{deg} in terms of the standard generators of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,0}(G(k, k+d), d))$. We recall that D_{deg} is the class of the divisor that parameterizes stable maps whose image lies in a proper subgrassmannian $G(k, k+d-1)$.

Lemma 4.1. *The class of the divisor D_{deg} in $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ can be expressed as*

$$D_{deg} = \frac{1}{2d}\left((1-d)\ H_{\sigma_{1,1}} + (d+1)\ H_{\sigma_2} - \sum_{i=1}^{\lfloor d/2 \rfloor} i(d-i)\ \Delta_{i,d-i}\right).$$

Proof. We determine the class of the divisor D_{deg} by intersecting it with test curves. Fix a general rational normal scroll of degree i and a rational normal curve C of degree $d-i-1$ intersecting the scroll in one point p. Consider the one-parameter family of degree d rational curves consisting of C union a pencil (with p as a base point) of rational curves of degree $i+1$ on the scroll joined to C along p. Take a general linear $\mathbb{P}^{k-2}$ and form the cone over this one-parameter family of curves. Each cone gives a one-parameter family of $\mathbb{P}^{k-1}$, hence by the universal property of the Grassmannians induces a curve of degree d in $G(k,k+d)$. Denote the resulting one-parameter family of degree d curves in $G(k,k+d)$ by C_i. The following intersection numbers are easy to see:

$$C_i \cdot H_{\sigma_{1,1}} = 0, \quad C_i \cdot H_{\sigma_2} = i, \quad C_i \cdot D_{\text{deg}} = 0.$$

The curve C_i is contained in the boundary $\Delta_{i+1,d-i-1}$. The intersection numbers with the boundary divisors are as follows:

$$C_i \cdot \Delta_{i+1,d-i-1} = -1, \quad C_i \cdot \Delta_{i,d-i} = 1, \quad C_i \cdot \Delta_{1,d-1} = i+1,$$

provided that $i \geq 1$. The intersection of C_i with the other boundary divisors is clearly zero.

Next consider the one-parameter family of degree d rational curves containing $d+2$ general points and intersecting a general line in a fixed $\mathbb{P}^d$. Taking a cone over these curves with a fixed general $\mathbb{P}^{k-2}$ induces a curve C in $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$ with the following intersection numbers (see Lemma 2.1 in [CHS2]):

$$C \cdot H_{\sigma_{1,1}} = 0, \quad C \cdot H_{\sigma_2} = \frac{d^2+d-2}{2}, \quad C \cdot \Delta_{1,d-1} = \frac{(d+2)(d+1)}{2}.$$

The intersection of C with the remaining boundary divisors is zero.

Next consider a fixed degree $d-1$ rational normal curve R and a general pencil of lines based at a point of R. Take a general $\mathbb{P}^{k-2}$ and take the cone of the curves with vertex this $\mathbb{P}^{k-2}$. This one-parameter family of cones induces a one-parameter family of rational curves in $G(k,k+d)$. Denote by B the resulting family in $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$. We have the intersection numbers

$$B \cdot H_{\sigma_{1,1}} = 0, \quad B \cdot H_{\sigma_2} = 1, \quad B \cdot D_{\text{deg}} = 1.$$

B is disjoint from all the boundary divisors $\Delta_{i,d-i}$ for $i > 1$. It is contained in $\Delta_{1,d-1}$ with intersection number $B \cdot \Delta_{1,d-1} = -1$. These test curves suffice to determine the class of D_{deg} except for the coefficient of $H_{\sigma_{1,1}}$.

In order to determine the coefficient of $H_{\sigma_{1,1}}$, we take the following one-parameter family. Fix a rational normal scroll S of dimension $k-1$ and degree d. Fix a general line l. Consider the one-parameter family of cones obtained by taking the cone over S with vertex a point of l. This family of scrolls induces a curve A in $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$. A has the following intersection numbers:

$$A \cdot H_{\sigma_{1,1}} = d, \quad A \cdot H_{\sigma_2} = d, \quad A \cdot D_{\text{deg}} = 1.$$

A is disjoint from all the boundary divisors. The class of D_{deg} follows form these intersection numbers. $\square$

4.2. **The divisor class D_{unb} when k does not divide d.** In this subsection we express the class of the divisor D_{unb} in $\overline{\mathrm{M}}_{0,0}(G(k,k+d),d)$ when k does not divide d in terms of the standard generators. Express $d = qk + r$ where q and r are the unique integers satisfying $0 < r < k$. The projective $k-1$-planes in the image of a general point of $\overline{\mathrm{M}}_{0,0}(G(k,k+d),d)$ sweep out a k-dimensional scroll of degree d in $\mathbb{P}^{k+d-1}$. Such a scroll has a unique $k-r$-dimensional minimal subscroll of degree $(k-r)q$. Recall from §2 that the divisor D_{unb} is the closure of the locus where the span of this minimal subscroll intersects a fixed codimension $(k-r)(q+1)$ linear space. We have already described the closure explicitly as the image of a divisor from the space of stable maps to the two-step flag variety under the natural map

$$\overline{\mathcal{M}}_{0,0}(\pi_2) : \overline{\mathcal{M}}_{0,0}(F(k-r,k,k+d),((k-r)q,d)) \to \overline{\mathcal{M}}_{0,0}(G(k,k+d),d).$$

Note that D_{unb} may be defined in $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ provided $n > (k-r)(q+1)$. The expression of the class in terms of the standard generators is clearly independent of n.

Lemma 4.2. *Let $d = qk + r$ with $0 < r < k$. Assume $n > (k-r)(q+1)$. Let $i = q_i k + r_i$ where q_i and r_i are the unique integers satisfying $0 \le r_i < k$. Then the divisor class D_{unb} in $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ is given by*

$$D_{unb} = -\frac{(q+1)(d-q-1)}{2d} H_{\sigma_2} + \frac{(q+1)(d+q+1)}{2d} + \sum_{i=1}^{\lfloor d/2 \rfloor} \gamma_i \Delta_i$$

where

$$\gamma_i = \begin{cases} \binom{q_i+1}{2}(r-k) + \frac{q+1}{2d}\left(i^2(q+1) - idq_i - dr_i(q_i+1)\right) & \text{if } r_i \le r \\ \binom{q_i+1}{2}(r-k) - (q_i+1)(r_i-r) + \frac{q+1}{2d}\left(i^2(q+1) - idq_i - dr_i(q_i+1)\right) \end{cases}$$

if $r < r_i$.

Proof. We will prove the lemma by intersecting both sides of the equation by one-parameter families. Since the class of the divisor is independent of n we may construct the families in $G(k,n)$ for any $n > (k-r)(q+1)$. Fix a balanced rational normal scroll of degree $d+q$ and dimension $k+1$. Let C'_q be the one-parameter family of subscrolls of degree d and dimension k obtained by taking the joins of a fixed degree $d-q$ balanced rational subscroll of dimension $k-1$ with rational normal curves of degree q varying in a 2-dimensional subscroll of degree $2q$. Denote by C_q the one-parameter family in $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ induced by the family of scrolls. The following intersection numbers are easy to compute

$$C_q \cdot D_{\text{unb}} = q+1, \quad C_q \cdot H_{\sigma_2} = d+q, \quad C \cdot H_{\sigma_{1,1}} = d-q.$$

C_q clearly has intersection number zero with all the boundary components.

Similarly, fix a rational normal scroll of degree $d+q+1$ and dimension $k+1$. Let C'_{q+1} be the one-parameter family of subscrolls of degree d and dimension k obtained by taking the joins of a fixed degree $d-q-1$ balanced rational subscroll of dimension $k-1$ with rational normal curves of degree $q+1$ varying in a 2-dimensional subscroll of degree $2q+2$. Denote by C_{q+1} the one-parameter family in $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$ induced by the family of scrolls. The following intersection numbers are easy to compute:

$$C_{q+1} \cdot D_{\text{unb}} = 0, \quad C_q \cdot H_{\sigma_2} = d+q+1, \quad C \cdot H_{\sigma_{1,1}} = d-q-1.$$

The intersection of C_{q+1} with the boundary divisors is zero. The coefficients of H_{σ_2} and $H_{\sigma_{1,1}}$ follow from these two families.

Let X be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. Denote by F_1 and F_2 the pull-backs of the two fiber classes from $\mathbb{P}^1 \times \mathbb{P}^1$. Let E be the exceptional divisor. Let V be the following vector bundle of rank k on X given by

$$V = \begin{cases} \oplus_{j=1}^{k-r} \mathcal{O}_X(-qF_1 - mF_2 + q_iE) \oplus \oplus_{j=1}^{r-r_i} \mathcal{O}_X(-(q+1)F_1 - mF_2 + q_iE) \\ \oplus \oplus_{j=1}^{r_i} \mathcal{O}_X(-(q+1)F_1 - mF_2 + (q_i+1)E) \ \text{ if } \ r \geq r_i \\ \oplus_{j=1}^{k-r_i+r} \mathcal{O}_X(-qF_1 - mF_2 + q_iE) \oplus \oplus_{j=1}^{r_i-r} \mathcal{O}_X(-qF_1 - mF_2 + (q_i+1)E) \\ \oplus \oplus_{j=1}^{r} \mathcal{O}_X(-(q+1)F_1 - mF_2 + (q_i+1)E) \ \text{ if } \ r < r_i \end{cases}$$

Let $\pi : X \to \mathbb{P}^1$ denote the natural map to $\mathbb{P}^1 \times \mathbb{P}^1$ followed by the first projection. Mapping $\mathbb{P}(V)$ to projective space under $\mathcal{O}_{\mathbb{P}(V)}(1)$ gives a one-parameter family of scrolls of degree d parameterized by $\mathbb{P}^1$ which in turn induces a one-parameter family B_i in $\overline{\mathrm{M}}_{0,0}(G(k,n),d)$. By construction the pull-back of the tautological bundle of $G(k,n)$ to X is V. Hence we have the following intersection numbers

$$B_i \cdot H_{\sigma_2} = c_1^2(V) - c_2(V), \quad B_i \cdot H_{\sigma_{1,1}} = c_2(V).$$

The Chern classes of V are determined by the Whitney sum formula to be

$$c_1(V) = -dF_1 - kmF_2 + iE$$

$$c_1(V)^2 = 2kmd - i^2$$

$$\begin{aligned} c_2(V) \quad &= qm(k-r)(k-r-1) + (q+1)mr(r-1) + (2q+1)mr(k-r) \\ &\quad -q_i^2\tbinom{k-r_i}{2} - q_i(q_i+1)r_i(k-r_i) - (q_i+1)^2\tbinom{r_i}{2} \\ &= md(k-1) - \tfrac{1}{2}(q_i i(k-1) + r_i(i-q_i-1)). \end{aligned}$$

Hence, we conclude that

$$B_i \cdot H_{\sigma_2} = md(k+1) - i^2 + \frac{1}{2}(q_i i(k-1) + r_i(i-q_i-1))$$

and

$$B_i \cdot H_{\sigma_{1,1}} = md(k-1) - \frac{1}{2}(q_i i(k-1) + r_i(i-q_i-1)).$$

The intersection number $B_i \cdot \Delta_j = \delta_i^j$, where δ_i^j is the Krönecker delta function. Finally let W be the subbundles of V giving rise to the minimal subscrolls of degree $(k-r)q$, i.e., the direct sum of the line bundles where the coefficient of F_1 is $-q$. The intersection number $B_i \cdot D_{\text{unb}}$ is given by $c_1(\pi_*(W^*))$ The latter may be computed by Riemann-Roch

$$B_i \cdot D_{\text{unb}} = \begin{cases} ((q+1)m - \binom{q_i+1}{2})(k-r) \text{ if } r \geq r_i \\ ((q+1)m - \binom{q_i+1}{2})(k-r) - (q_i+1)(r_i - r) \text{ if } r < r_i \end{cases}$$

The lemma follows by elementary algebra. $\square$

4.3. **The divisor class of D_{unb} when k divides d.** In this section we express the class of D_{unb} in terms of the standard generators of the Picard group of $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$ when k divides d.

Lemma 4.3. *Suppose k divides d. For each integer i, let r_i be the unique integer $0 \leq r_i < k$ such that k divides $i - r_i$. The class of the divisor D_{unb} satisfies the equation*

$$D_{unb} = \frac{1}{2k}\left((k+1)H_{\sigma_{1,1}} + (1-k)H_{\sigma_2} - \sum_{i=1}^{\lfloor d/2 \rfloor} r_i(k-r_i)\Delta_{i,d-i}\right).$$

As it occurs, it is simpler to prove a more general result.

Definition 4.4. A *versal triple* is a triple (S, C, E) of

(i) a Deligne-Mumford stack S with quasi-projective coarse moduli space,
(ii) a family of prestable, genus 0 curves $\pi : C \to S$,
(iii) and a locally free sheaf E on C of rank k and degree d

which is *versal* at every geometric point s of S: the induced map from the formal neighborhood of s to the versal deformation space of $(C_s, E|_{C_s})$ is (formally) smooth.

Lemma 4.5. *If the triple (S, C, E) is versal then S is smooth. Conversely, assuming S is smooth at s, (S, C, E) is versal at s if the map from the Zariski tangent space T_sS to the first-order deformation space of (C_s, E_s) is surjective.*

Proof. Since the natural obstruction group to deformations of $(C_s, E|_{C_s})$ is of the form $H^2(C_s, -)$, the versal deformation space is smooth. Thus S is smooth if (S, C, E) is versal.

Conversely, assume S is smooth. A k-algebra homomorphism between power series rings is formally smooth if and only if it is surjective on Zariski tangent spaces. □

For every versal triple (S, C, E), S contains a collection of reduced, effective Cartier divisors D_{unb} and $(\Delta_{i,d-i})_{i\in\mathbb{Z}}$ with the following properties.

(i) A codimension 1 point s is in D_{unb} if and only if

$$C_s \cong \mathbb{P}^1, \quad E|_{C_s} \cong \mathcal{O}((d/k)-1) \oplus \mathcal{O}(d/k)^{\oplus(k-2)} \oplus \mathcal{O}((d/k)+1).$$

(ii) A codimension 1 point s is in $\Delta_{i,d-i}$ if and only if

$$C_s = C'_s \cup C''_s, \quad \deg(E|_{C'_s}) = i, \quad \deg(E|_{C''_s}) = d - i.$$

Denote by U the complement of the divisor D_{unb} and all the divisors $\Delta_{i,d-i}$. For every geometric point s of U

$$C_s \cong \mathbb{P}^1, \quad E|_{C_s} \cong \mathcal{O}(d/k)^{\oplus k}.$$

Lemma 4.6. *For every versal triple (S, C, E) there is a linear equivalence of $\mathbb{Q}$-Cartier divisor classes on S,*

$$2k[D_{unb}] \sim \pi_* C_2(End(E)) - \sum_{i \leq \lfloor d/2 \rfloor} r_i(k-r_i)[\Delta_{i,d-i}].$$

First we deduce Lemma 4.3 assuming Lemma 4.6.

Proof of Lemma 4.3. Form the triple with $S = \overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$, with $\pi : C \to \overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ the universal family of curves, and with $E = f^*S_k^\vee$. Here S_k is the tautological rank k subbundle on $G(k+d, d)$ and $f : C \to G(k, k+d)$ is the universal map.

The stack S is smooth, in fact smooth over the stack $\mathfrak{M}_{0,0}$ of prestable, genus 0 curves, cf. [Beh]. Using the relative smoothness, Lemma 4.5 says (S, C, E) is versal if each relative Zariski tangent space of S over $\mathfrak{M}_{0,0}$ surjects to the first-order deformation space of E_s.

The Euler sequence on $G(k, n)$ is,

$$0 \longrightarrow \mathrm{Hom}(S_k, S_k) \longrightarrow \mathrm{Hom}(S_k, \mathcal{O}_{G(k,n)}^{\oplus n}) \longrightarrow T_{G(k,n)} \longrightarrow 0.$$

Pulling back by f and pushing forward by π gives an exact sequence

$$\pi_* f^* T_{G(k,n)} \longrightarrow R^1\pi_*\mathrm{Hom}(E, E) \longrightarrow R^1\pi_*(f^*S_k^\vee)^{\oplus n}.$$

The first term is the relative tangent bundle of $\overline{\mathcal{M}}_{0,0}(G(k, k+d), d)$ over $\mathfrak{M}_{0,0}$. The second term is the bundle of first-order deformation spaces of E. Since $f^*S_k^\vee$ is globally generated, $R^1\pi_*(f^*S_k^\vee) = (0)$ by [HS, Lemma 2.3(i)]. Therefore (S, C, E) is versal.

Clearly the divisors D_{unb} and $(\Delta_{i,d-i})_{i\in\mathbb{Z}}$ associated to (S, C, E) are the same as the divisors defined earlier. Finally, by straightforward computation

$$c_2(\mathrm{End}(S_k)) = (k + 1)\sigma_{1,1} + (1 - k)\sigma_2.$$

□

Proof of Lemma 4.6. Let

$$0 \longrightarrow \omega_\pi \longrightarrow \mathcal{E}_\pi \longrightarrow \mathcal{O}_C \longrightarrow 0$$

be the extension whose class in $H^1(C, \omega_\pi)$ defines the trace. Form the locally free sheaf $\mathcal{F} = \mathrm{End}(E) \otimes \mathcal{E}_\pi$. For every geometric point s of U

$$\mathcal{F}|_{C_s} = (\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \otimes \mathrm{End}(\mathcal{O}(d/k)^{\oplus k}) \cong \mathcal{O}(-1)^{\oplus 2k^2}.$$

Since $\mathcal{O}(-1)$ has no cohomology

$$h^0(C_s, \mathcal{F}|_{C_s}) = h^1(C_s, \mathcal{F}|_{C_s}) = 0.$$

Since U is a dense Zariski open, $\pi_*\mathcal{F} = (0)$ and $R^1\pi_*\mathcal{F}$ is a torsion sheaf supported on D_{unb} and the divisors $\Delta_{i,d-i}$. The determinant of this torsion sheaf is an effective Cartier divisor R supported on D_{unb} and the divisors $\Delta_{i,d-i}$, cf. [KM]. Thus

$$R = r_{\text{unb}} D_{\text{unb}} + \sum_{i \le \lfloor d/2 \rfloor} r_{i,d-i}\Delta_{i,d-i}.$$

If D_{unb} is nonempty r_{unb} is the length of the stalk $(R^1\pi_*\mathcal{F})_s$ at a generic point s of D_{unb}. Similarly for $r_{i,d-i}$.

Because the formal neighborhood of s in S maps smoothly to the versal deformation space of (C_s, E_s), the length of $(R^1\pi_*\mathcal{F})_s$ can be computed using the versal deformation space and thus using any other versal triple. In other words, there exist integers r_{unb} and $r_{i,d-i}$ such that the equation above holds for *every* versal triple (S, C, E). This allows us to compute the coefficients r_{unb} and $r_{i,d-i}$ using special versal triples, i.e., using the method of *test families*.

Also Riemann-Roch gives

$$R \sim 2\pi_* c_2(\mathrm{End}(E)),$$

cf. [dJS]. This gives a linear equivalence of $\mathbb{Q}$-Cartier classes

$$2\pi_* c_2(\mathrm{End}(E)) \sim r_{\mathrm{unb}} D_{\mathrm{unb}} + \sum_{i \leq \lfloor d/2 \rfloor} r_{i,d-i} \Delta_{i,d-i}.$$

Let V and W be 2-dimensional vector spaces. Set $S = \mathbb{P}V \cong \mathbb{P}^1$. Let $C = \mathbb{P}V \times \mathbb{P}W \cong \mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi : C \to S$ be the projection pr_V.

The relative dualizing sheaf of π is $\mathrm{pr}_W^* \omega_{\mathbb{P}(W)}$. And the sheaf $\mathcal{E}_\pi$ is,

$$\mathcal{E}_\pi = \mathrm{pr}_W^*(W^\vee \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(W)}(-1)).$$

1. Computation of $r_{\mathbf{unb}}$. There is a natural isomorphism of vector spaces,

$$H^1(\mathbb{P}V \times \mathbb{P}W, \mathrm{pr}_V^* \mathcal{O}(1) \otimes \mathrm{pr}_W^* \omega_{\mathbb{P}W}) \cong$$
$$H^0(\mathbb{P}V, \mathcal{O}(1)) \otimes H^1(\mathbb{P}W, \omega_{\mathbb{P}W}) \cong V^\vee.$$

Let x be a nonzero element in $V^\vee$. Denote by p the unique point of $\mathbb{P}V$ where x vanishes. Using the isomorphism

$$\mathrm{Ext}^1(\mathrm{pr}_W^*(\mathcal{O}(1)), \mathrm{pr}_V^*(\mathcal{O}(1)) \otimes \mathrm{pr}_W^*(\omega_{\mathbb{P}W}(1))) \cong$$
$$H^1(\mathbb{P}V \times \mathbb{P}W, \mathrm{pr}_V^* \mathcal{O}(1) \otimes \mathrm{pr}_W^* \omega_{\mathbb{P}W}),$$

the element x determines an extension class

$$0 \longrightarrow \mathrm{pr}_V^* \mathcal{O}(1) \otimes \mathrm{pr}_W^*(\omega_{\mathbb{P}W}(1)) \longrightarrow G_x \longrightarrow \mathrm{pr}_W^* \mathcal{O}(1) \longrightarrow 0.$$

Because the section x of $\mathcal{O}(1)$ is nonzero on $D_+(x)$, the restriction of G_x to $D_+(x)$ is balanced. However, the restriction of G_x to the fiber over p is unbalanced, $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

Define locally free sheaves E' and E to be

$$E' := G_x \oplus \mathcal{O}_C^{\oplus(k-2)}, \quad E := E' \otimes \mathrm{pr}_W^* \mathcal{O}(d/k).$$

The datum (S, C, E) is versal on $D_+(x)$. We claim it is also versal at p, i.e., the induced morphism ζ from the formal neighborhood of p to the versal deformation space is smooth. The versal deformation space is $k[\![t]\!]$, and the Cartier divisor D_{unb} over the deformation space is just (t). The pullback $\zeta^{-1}(D_{\mathrm{unb}})$ equals $m\{p\}$ for some positive integer m. By Lemma 4.5, ζ is smooth if $m = 1$. Clearly m is the same for both the triple (S, C, E) and the triple (S, C, E').

Form the sheaf $\mathcal{T}_S := R^1\pi_*(E' \otimes \mathcal{E}_\pi)$, and form the analogous sheaf $\mathcal{T}_{\mathrm{versal}}$ on $k[\![t]\!]$. As above, $\pi_*(E' \otimes \mathcal{E}_\pi) = (0)$ and $\mathcal{T}_S$ is a torsion sheaf supported on p. Since ζ is flat and since $R\pi_*(E' \otimes \mathcal{E}_\pi)$ is of formation compatible with flat base-change,

$$\mathrm{length}(\mathcal{T}_S) = m \cdot \mathrm{length}(\mathcal{T}_{\mathrm{versal}}) \geq m \cdot \dim(\mathcal{T}_{\mathrm{versal}}/t\mathcal{T}_{\mathrm{versal}}) = m \cdot \dim(\mathcal{T}_S \otimes \kappa(p)).$$

By cohomology and base change

$$\mathcal{T}_S \otimes \kappa(p) \cong H^1(\mathbb{P}W, [\omega_{\mathbb{P}W}(1) \oplus \mathcal{O}_{\mathbb{P}W}^{\oplus(k-2)} \oplus \mathcal{O}_{\mathbb{P}W}(+1)] \otimes [W^\vee(-1)])$$

which is canonically $W^\vee$. Since this is 2-dimensional

$$\mathrm{length}(\mathcal{T}_S) \geq 2m.$$

Since $\mathcal{T}_S$ is a torsion sheaf, $\mathrm{length}(\mathcal{T}_S) = c_1(\mathcal{T}_S)$. By [dJS, Lemma 4.3]

$$c_1(\mathcal{T}_S) = -c_1(R\pi_*(E' \otimes \mathcal{E}_\pi)) = -\pi_*(c_1(E')^2 - 2c_2(E')).$$

By the Whitney sum formula

$$c_t(E') = c_t(G_x) = 1 + \mathrm{pr}_V^* h_V + (\mathrm{pr}_V^* h_V) \cap (\mathrm{pr}_W^* h_W),$$

where $h_V = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ and $h_W = c_1(\mathcal{O}_{\mathbb{P}(W)}(1))$. Hence

$$\mathrm{length}(\mathcal{T}_S) = 2\pi_*((\mathrm{pr}_V^* h_V) \cap (\mathrm{pr}_W^* h_W)) = 2.$$

Since $2 = \mathrm{length}(\mathcal{T}_S) \geq 2m$, $m = 1$, i.e., (S, C, E) is versal.

Because (S, C, E) is versal, because all the divisors $\Delta_{i,d-i}$ are empty, and because D_{unb} is the single reduced point p

$$r_{\text{unb}} = \deg_S(R).$$

By Riemann-Roch, $R \sim 2\pi_* c_2(\mathrm{End}(E))$. Of course $\mathrm{End}(E) \cong \mathrm{End}(E')$. By straightforward computation,

$$2c_2(\mathrm{End}(E')) = -2(\mathrm{rk}(E') - 1)c_1(E')^2 + 4\mathrm{rk}(E')c_2(E').$$

Using the computation above, this is,

$$-2(k-1)\mathrm{pr}_V^*(h_V \cap h_V) + 4k(\mathrm{pr}_V^* h_V) \cap (\mathrm{pr}_W^* h_W) = 4k(\mathrm{pr}_V^* h_V) \cap (\mathrm{pr}_W^* h_W).$$

The pushforward to $\mathbb{P}V$ has degree $4k$;

$$r_{\text{unb}} = 4k.$$

2. Computation of $r_{i,d-i}$. Fix a point p in $\mathbb{P}V$ and a point p' in $\mathbb{P}W$. Let $\nu : \widetilde{C} \to C$ be the blowing-up of $\mathbb{P}V \times \mathbb{P}W$ along $\{(p,p')\}$. Denote by A the exceptional divisor of ν. Denote by $\widetilde{\pi} : \widetilde{C} \to S$ the composition $\mathrm{pr}_V \circ \nu$. Let $i \leq \lfloor d/2 \rfloor$ be an integer. Write,

$$i = kq_i + r_i,$$

where r_i is the unique integer $0 \leq r_i < k$ such that $i - r_i$ is divisible by k. Denote

$$E' := \mathcal{O}_{\widetilde{C}}(-(q_i+1)A)^{\oplus r_i} \oplus \mathcal{O}_{\widetilde{C}}(-q_i A)^{\oplus (k-r_i)}, \quad E := E' \otimes \nu^* \mathrm{pr}_W^* \mathcal{O}(d/k).$$

Consider the triple $(S, \widetilde{C}, E)$.

It is clearly versal on $\mathbb{P}V - \{p\}$. To see that it is versal at p, observe first the versal deformation space of $(\widetilde{C}_p, E|_{\widetilde{C}_p})$ equals the versal deformation space $\widetilde{C}_p$, which is $k\llbracket t \rrbracket$. Let m be the positive integer such that (t) pulls back to $m\{p\}$. As in [DM, p. 81], there is an isomorphism of the formal neighborhood of the node of $\widetilde{C}_p$ in $\widetilde{C}$ with the formal scheme $k\llbracket u, v, s \rrbracket / \langle uv - s^m \rangle$, i.e., an A_{m-1}-singularity. Since $\widetilde{C}$ is smooth, $m = 1$, i.e., (S, C, E) is versal.

As above, $r_{i,d-i}$ equals the degree of the first Chern class of $-R\widetilde{\pi}_*(\mathrm{End}(E') \otimes \mathcal{E}_{\widetilde{\pi}})$,

$$r_{i,d-i} = 2\widetilde{\pi}_* c_2(\mathrm{End}(E')).$$

Of course

$$\mathrm{End}(E') = (E')^\vee \otimes E' = \mathcal{O}_{\widetilde{C}}^{\oplus r_i^2} \oplus \mathcal{O}_{\widetilde{C}}(-A)^{\oplus r_i(k-r_i)} \oplus \mathcal{O}_{\widetilde{C}}(+A)^{\oplus r_i(k-r_i)} \oplus \mathcal{O}_{\widetilde{C}}^{\oplus (k-r_i)^2}.$$

By the Whitney sum formula

$$c_t(\mathrm{End}(E')) = (1 - [A] \cap [A])^{r(i)(k-r(i))}.$$

Therefore

$$r_{i,d-i} = -2\deg_S \widetilde{\pi}_*(r_i(k-r_i)[E] \cap [E]) = 2r_i(k-r_i)\deg_S(\{p\}) = 2r_i(k-r_i).$$

$\square$

5. The ample cone

In this section we prove a generalization of Theorem 1.1 for products of flag varieties. Our proof will be valid in more general contexts; however, for simplicity of exposition we leave generalizations to the reader.

Let X be a product of flag varieties with Picard number ρ. There are ρ morphisms from X to Grassmannians $\phi_i : X \to G_i$. The cone of NEF divisors on X is simplicial and generated by the pull-backs of the ample divisors on G_i by the morphisms ϕ_i. We denote these divisors by $H_1, \dots, H_\rho$. Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class. For our purposes we may and will assume that $\beta \cdot H_i = e_i > 0$ for every $1 \leq i \leq \rho$ (otherwise, we can reduce to the problem of understanding the ample cone when the target has smaller Picard number).

The NEF divisors on X pull-back to NEF divisors on $\overline{\mathcal{M}}_{0,m}(X, \beta)$ via the evaluation morphisms (see [CHS1] Lemma 4.1). We thus obtain the NEF divisors $L_i^j = ev_i^*(H_j)$ on $\overline{\mathcal{M}}_{0,m}(X, \beta)$ for $1 \leq i \leq m$ and $1 \leq j \leq \rho$.

The tangency divisor T plays an important role in the description of the ample cone. T is the class of the locus of maps for which the inverse image of a fixed hyperplane in the Plücker embedding of $G(k,n)$ does not consist of d distinct, smooth points of the domain. For this class to make sense we need that $d > 1$. R. Pandharipande in [Pa] computes the class of the tangency divisor when the target is projective space. Since T on $\overline{\mathcal{M}}_{0,m}(G(k,n), d))$ is the pull-back of this divisor by the Plücker embedding, the following formula easily follows from Pandharipande's calculation in [Pa] Lemma 2.3.1:

$$T = \frac{d-1}{d} H_{\sigma_{1,1}} + \frac{d-1}{d} H_{\sigma_2} + \sum_{i=1}^{\lfloor d/2 \rfloor} \frac{i(d-i)}{d} K_i,$$

where K_i is the sum of the boundary divisors where the map has degree i on one component. The divisor T is NEF on $\overline{\mathcal{M}}_{0,m}(G(k,n), d))$ ([CHS1] Lemma 4.1). Assuming that $e_i > 1$, pulling-back the tangency divisor on $\overline{\mathrm{M}}_{0,m}(G_i, e_i)$ by $\overline{\mathcal{M}}_{0,m}(\phi_i)$ we obtain NEF divisors $T_1, \dots, T_\rho$ on $\overline{\mathcal{M}}_{0,m}(X, \beta)$.

The numerically effective codimension two cycles on X are also simplicial and give rise to NEF divisors on $\overline{\mathcal{M}}_{0,m}(X, \beta)$. Given a cycle $\gamma \in NEF^2(X)$, the divisor

$$H_\gamma := \pi_{m+1*}(ev_{m+1}^*(\gamma))$$

is NEF ([CHS1] Lemma 4.1), where ev_{m+1} is the $m+1$-st evaluation morphism from $\overline{\mathcal{M}}_{0,m+1}(X, \beta)$ to X and π_{m+1} is the forgetful morphism.

Let $\overline{\mathrm{M}}_{0,m+e_1+\cdots+e_\rho}/\mathfrak{S}_{e_1} \times \cdots \times \mathfrak{S}_{e_\rho}$ denote the quotient of the moduli space of genus-zero stable curves with $m + e_1 + \cdots + e_\rho$ marked points by the action of the products of the symmetric groups on $e_1, \dots, e_\rho$ letters acting by the obvious permutation action. There is a map

$$v : \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{M}}_{0,m+e_1+\cdots+e_\rho}/\mathfrak{S}_{e_1} \times \cdots \times \mathfrak{S}_{e_\rho}) \to \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,m}(X, \beta))$$

that maps NEF divisors to NEF divisors. This map is constructed and described in §2 of [CHS1] for the case of $\mathbb{P}^n$. The proofs of that section carry over to the case of X. The map v is induced by the rational map from $\overline{\mathcal{M}}_{0,m}(X, \beta)$ to $\overline{\mathrm{M}}_{0,m+e_1+\cdots+e_\rho}/\mathfrak{S}_{e_1} \times \cdots \times \mathfrak{S}_{e_\rho}$ obtained by intersecting the image of the map with a fixed union of divisors $H_1 \cup \cdots \cup H_\rho$, marking the points of intersection and then forgetting the map and stabilizing.

Theorem 5.1. *The NEF cone of $\overline{\mathcal{M}}_{0,m}(X,\beta)$ is the product of the image under v of the NEF cone of $\overline{\mathrm{M}}_{0,m+e_1+\cdots+e_\rho}/\mathfrak{S}_{e_1} \times \cdots \times \mathfrak{S}_{e_\rho}$ with the cone spanned by non-negative linear combinations of the classes L_i^j, for $1 \leq i \leq m$ and $1 \leq j \leq \rho$, H_γ, for NEF codimension two cycles γ on X, and T_i, for $1 \leq i \leq \rho$ and $e_i > 1$.*

Proof. The proof is almost identical to the proof of the main theorem in [CHS1]. Clearly any divisor D that lies in this cone is NEF. The main theorem of [Opr] and Proposition 2.5 of [CHS1] imply that divisors in this cone span the Picard group. Hence, we can write any NEF divisor as

$$D = v(D_1) + \sum_{\gamma \in NEF^2(X)} a_\gamma H_\gamma + \sum_{i=1}^{\rho} b_i T_i + \sum_{1 \leq i \leq \rho, 1 \leq j \leq m} c_{i,j} L_i^j.$$

We have to show that D_1 is NEF and the constants a_γ, b_i and $c_{i,j}$ are non-negative. We can do this by intersecting with curves that have zero intersection with all the terms in the expression of D but one. Analogues of the necessary curves have appeared in [CHS1].

Briefly take any curve C in $\overline{\mathrm{M}}_{0,m+e_1+\cdots+e_\rho}/\mathfrak{S}_{e_1} \times \cdots \mathfrak{S}_{e_\rho}$. Attach e_i copies of the generator B_i of the curve class in G_i to the e_i marked points of the members of C. Map the curves B_i isomorphically to fixed representatives passing through a point and contract the members of C. This curve has zero intersection with T_i, H_γ and L_i^j. Hence D_1 has to be NEF.

To see that the coefficients of L_i^j are non-negative take a curve in the class β with two components $C_1 \cup C_2$ where the component C_1 has class dual to H_j and C_2 has the residual class. Suppose C_2 contains all the marked points but the i-th marked point. Consider the one-parameter family obtained by varying the i-th marked point p_i. This one-parameter family has positive intersection with L_i^j and zero intersection with all the other divisors occurring in the expression of D. Hence $c_{i,j} \geq 0$.

To prove that the coefficients of T_i are non-negative consider the following one-parameter family. Take a fixed curve C with m marked points in the class $\beta - 2\zeta_i$, where ζ_i is the curve class dual to H_i. At a general point of C attach a copy of a curve in the class ζ_i and attach to that another copy of the same curve at a variable point. This one-parameter family has positive intersection with T_i and does not intersects any of the other divisors occurring in the expression of D.

$NEF^2(X)$ is simplicial, say generated by classes γ_i. $\mathrm{Eff}_2(X)$ is simplicial, generated by Schubert varieties (or products of Schubert varieties) dual to the previous cone. Each of these surfaces Γ_i^* have a pencil of rational curves with a base point such that for every element ζ in the pencil $\zeta \cdot H_j \leq 1$. To such a pencil attach a fixed curve with m marked points in the class $\beta - \zeta$ at a base point. The resulting one-parameter family has positive intersection with H_{γ_i} and does not intersect any of the other divisors in the expression of D. This concludes the proof. □

Remark 5.2. Since the divisors H_γ, L_i^j and T_i are base-point-free and v maps base-point-free divisors to base-point-free divisors, the theorem remains true if we replace NEF by base-point-free. It is clear that Theorem 5.1 specializes to Theorem 1.1 when X is a Grassmannian.

6. The effective cone

In this section we prove that a divisor lies in the effective cone of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(G(k,k+r),d)$ if and only if it is a non-negative linear combination of the divisors D_{deg}, D_{unb} and the boundary divisors.

Proof of Theorem 2.3. Since D_{deg}, D_{unb} and the boundary divisors are effective, any non-negative linear combination of these divisors lie in the effective cone. We have to show the converse.

Note that the divisors D_{deg}, D_{unb} and the boundary divisors form a $\mathbb{Q}$-basis of $\text{Pic}(\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d) \otimes \mathbb{Q}$. This follows from Theorem 2.1 and Lemmas 4.1, 4.2 and 4.3, but will also follow from the proof we give. Therefore, we can write the class of any effective divisor D as a linear combination

$$D = a\ D_{\text{deg}} + b\ D_{\text{unb}} + \sum_{i=1}^{\lfloor d/2 \rfloor} c_i\ \Delta_{i,d-i}. \tag{1}$$

To conclude the proof we need to show that the coefficients a, b and c_i are non-negative. If for each of the divisors D_{deg}, D_{unb} and $\Delta_{i,d-i}$ we can construct a moving curve in $\overline{\mathcal{M}}_{0,0}(G(k,k+d),d)$ that has positive intersection number with that divisor and zero with the others, the theorem follows since D has non-negative intersection with any moving curve. We will now construct the necessary moving curves in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$.

For each boundary divisor $\Delta_{i,d-i}$, the paper [CHS2] constructs a sequence of curves $C_i(j)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d)$ with the following properties:

(i) The intersection of $C_i(j)$ with D_{deg} is zero.
(ii) The intersection of $C_i(j)$ with $\Delta_{k,d-k}$ is zero if $k \neq 1$ or $k \neq i$.
(iii) The intersection of $C_i(j)$ with $\Delta_{i,d-i}$ is $j(d+1) - n(i,d)$ where $n(i,d)$ is a fixed constant depending only on i and d and not on j.
(iv) The intersection of $C_i(j)$ with $\Delta_{1,d-1}$ is $n(i,d)\frac{i(i+1)}{2}$.

We will use the curves $C_i(j)$ as the basis of our construction, so we recall their construction. Consider the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at

$$j(d+1) + n(i,d)\frac{(i+2)(i-1)}{2}$$

general points p_h. We denote the proper transform of the two fiber classes by F_1 and F_2, respectively, and we denote the exceptional divisor lying over p_h by E_h. The curve $C_i(j)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d)$ is obtained by considering the image of the one-parameter family of fibers in the class F_2 under the linear system

$$d\ F_1 + j\frac{i(i+1)}{2}\ F_2 - \sum_{h=1}^{j(d+1)-n(i,d)} i\ E_h - \sum_{h=j(d+1)-n(i,d)+1}^{j(d+1)+n(i,d)\frac{(i+2)(i-1)}{2}} E_h.$$

The proof in the case k divides d. We assume that $d = kq$. Fix k general $\mathbb{P}^q$s in $\mathbb{P}^{k+d-1}$. In order to show that the coefficient c_i is non-negative, we need a sequence of curves B in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$ that have intersection zero with D_{deg} and D_{unb} and

$$\frac{B \cdot \Delta_k}{B \cdot \Delta_i}$$

tends to zero. Suppose $i = kq_i + r_i$. We then take r_i families of the type $C_{q_i+1}(j)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^q, q)$ and $k - r_i$ families of the type $C_{q_i}(j)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^q, q)$. Taking their join induces a sequence of moving curves in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$ with the desired properties.

Now we show that the coefficient of D_{deg} is non-negative. Take the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,q)$ and take a general projection of the surface to $\mathbb{P}^q$. Fix $k-1$ degree q rational normal curves in $k-1$ general $\mathbb{P}^q$s. Fix an isomorphism between the rational normal curves of degree q and the fibers of degree q of $\mathbb{P}^1 \times \mathbb{P}^1$. Taking the join of the points corresponding under the isomorphism provides a family of balanced scrolls that never become reducible or unbalanced. The scrolls become degenerate finitely many times when the linear span of the fibers of $\mathbb{P}^1 \times \mathbb{P}^1$ is not all of $\mathbb{P}^q$. By the universal property of $G(k, k+d)$ this family of scrolls induces a one-parameter family C of curves in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$. C has intersection number zero with the boundary and D_{unb} and positive with D_{deg}. Moreover, Lemma 2.4 implies that C is a moving curve. Consequently, in the expression (1) the coefficient of D_{deg} has to be non-negative.

Now we show that the coefficient of D_{unb} is non-negative. Recall that $k \geq 2$. By Lemma 2.4 of [C] the dimension of rational scrolls of degree d and dimension k in $\mathbb{P}^{k+d-1}$ is $d(d+2k)-3$. Consider the one-parameter family of scrolls in $\mathbb{P}^{k+d-1}$ which contain $d+2k+1$ general points and intersect $2k-3$ general $(d-2)$-dimensional linear spaces. It is easy to see that there cannot be any reducible scrolls that contain all the points. Moreover, since the points are in general position none of the scrolls containing the points can be degenerate. A general such scroll will be balanced. This family of scrolls induces a curve in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$. By Lemma 2.4 the resulting curve is moving and has the desired intersection numbers with D_{deg}, D_{unb} and the boundary divisors.

The proof in the case k does not divide d. In this case the proof is almost identical. To obtain a moving curve in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$ that has intersection number zero with the boundary divisors and D_{unb}, but positive intersection with D_{deg} take a general projection of the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ by the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,q+1)$ to $\mathbb{P}^{q+1}$. Take the joins of degree $q+1$ fibers with a fixed balanced scroll of dimension $k-1$ and degree $d-q-1$. This induces a moving curve in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$ with the desired intersection numbers. By Lemma 2.4 of [C] the dimension of rational scrolls of degree d and dimension k in $\mathbb{P}^{k+d-1}$ is $d(d+2k)-3$. To obtain a moving curve that has intersection numbers zero with the boundary and D_{deg}, but positive with D_{unb}, consider the one-parameter family of scrolls in $\mathbb{P}^{k+d-1}$ that contain $d+2k+1$ general points and intersects $2k-3$ general $(d-2)$-dimensional linear spaces. It is easy to see that there cannot be any reducible scrolls that contain all the points. Moreover, since the points are in general position none of the scrolls containing the points can be degenerate. This one-parameter family of scrolls induces a moving curve in $\overline{\mathcal{M}}_{0,0}(G(k,k+r,d),d)$ with the desired properties.

Finally to show that the coefficients of the boundary divisors are non-negative we use analogues of the families we used in the case k divides d. Suppose that $d = kq + r$. Either $(k-r)q \geq d/2$ or $r(q+1) \geq d/2$. Let us assume the latter. The argument in the former case is identical replacing $q+1$ by q and r by $k-r$. Let $i = rq_i + r_i$. Consider the join of $k-r$ fixed general rational normal curves of degree q with r_i general curves of type $C_{q_i+1}(j)$ in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{q+1}, q+1)$ and $r - r_i$ general curves of type $C_{q_i}(j)$ in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^{q+1}, q+1)$. The resulting one-parameter family of scrolls induces a curve B in $\overline{\mathrm{M}}_{0,0}(G(k, k+d), d)$. By construction B has intersection zero with D_{unb} and D_{deg}. As j tends to infinity the ratio

$$\frac{B \cdot \Delta_k}{B \cdot \Delta_i}$$

tends to zero for $k \neq i$. This concludes the proof of the theorem.

$\square$

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