CUBIC FOURFOLDS AND SPACES OF RATIONAL CURVES

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Abstract. For a general nonsingular cubic fourfold $X \subset \mathbb{P}^5$ and $e \geq 5$ an odd integer, we show that the space M_e parametrizing rational curves of degree e on X is non-uniruled. For $e \geq 6$ an even integer, we prove that the generic fiber dimension of the maximally rationally connected fibration of M_e is at most one, i.e. passing through a very general point of M_e there is at most one rational curve. For $e < 5$ the spaces M_e are fairly well understood and we review what is known.

1. Introduction

Let k be an algebraically closed field of characteristic 0. Let X be a nonsingular cubic hypersurface in $\mathbb{P}^5_k$. For each integer $e \geq 1$ denote by M_e the variety parametrizing smooth, geometrically connected curves in X of degree e and arithmetic genus 0, i.e., M_e is the scheme of rational curves of degree e in X. The scheme M_e is an irreducible variety of dimension $3e+1$. This is nontrivial and is discussed in Section 2. In this paper we consider the birational geometry of M_e, specifically the following questions:

1. What is the Kodaira dimension of M_e?
2. In case the Kodaira dimension is negative, what is the dimension of the general fiber of the maximally rationally connected fibration of M_e (cf. [16])?

These questions were originally raised by Joe Harris with regard to the well-known problem of rationality/irrationality of cubic fourfolds (we do not solve this problem). It is a pleasure to acknowledge useful conversations with Joe Harris.

In Section 2 we discuss different compactifications of M_e and how they are related. Let $\overline{M}_e$ be a desingularization of a compactification of M_e. Question 2 can be rephrased: For a very general point $p \in \overline{M}_e$, what is the maximal dimension of a closed, rationally connected subvariety $Z \subset \overline{M}_e$ containing p? Equivalently, denoting by $\overline{M}_e \to Q$ the MRC fibration (in the sense of [16, Def. IV.5.3]), what is the difference $\dim(M_e) - \dim(Q)$? If this is 0, then for a very general point $p \in \overline{M}_e$ there is no non-constant morphism $\mathbb{P}^1 \to \overline{M}_e$ whose image contains p, i.e., $\overline{M}_e$ is not uniruled. We note that the invariant $\dim Z$ is a birational invariant of M_e (so it is independent of the choice of desingularized compactification).

Discussions with Joe Harris have led to the list of maximal dimensions for small values of e:

e	1	2	3	4
$\dim M_e$	4	7	10	13
$\dim Z$	0	3	2	3

Date: February 5, 2004.

We pause to explain this table. The case of lines is well known, namely M_1 is a 4-dimensional hyperKähler manifold [3, Prop. 1]. In the case of conics, the set of all conics residual to a fixed line is a 3 dimensional rationally connected variety Z. In the case of cubic rational curves, note that a general cubic curve lies on a unique cubic surface and moves in a 2-dimensional linear system on it. So Z has dimension at least 2. A general quartic rational curve lies on a unique cubic threefold, and moves in a 3-dimensional rationally connected family on it (c.f. [10, Theorem 8.2]), so Z has dimension at least 3. This gives a lower bound for the numbers in the bottom row of the diagram, which is easily seen to be the actual dimension of Z when $e = 1$ or 2. For $e = 3$ and $e = 4$, we have not verified these numbers give the actual dimensions, but we would be surprised if they turn out to be larger. Ana-Maria Castravet conjectured that for $e = 4$, the actual dimension of Z is 3 and the target of the MRC fibration of $\overline{M}_4$ is birational to the *relative intermediate Jacobian* of the family of hyperplane sections of X – in other words, the relative intermediate Jacobian of the family of hyperplane sections of X is not uniruled.

Theorem 1.1. *Let $X \subset \mathbb{P}^5$ be a very general cubic fourfold. For every odd degree $e \geq 5$, the variety M_e is not uniruled. For every even degree $e \geq 6$ the variety M_e has dim$(Z) \leq 1$.*

Actually our method gives something a little better than Theorem 1.1.

Theorem 1.2. *Let $X \subset \mathbb{P}^5$ a smooth cubic hypersurface, and let $\overline{M_e}$ be a nonsingular projective model of M_e. There is a canonical section $\omega_e \in H^0(\overline{M}_e, \Omega^2_{\overline{M}_e})$ with the following property:*
(a) If e is odd, $e \geq 5$, if X is general, and if p is a general point of $\overline{M}_e$, then ω_e induces a nondegenerate pairing on $T_p(\overline{M}_e)$.
(b) If e is even, $e \geq 6$, if X is general, and if $p \in \overline{M}_e$ a general point, then the linear map $T_p(\overline{M}_e) \to T_p^\vee(\overline{M}_e)$ induced by ω_e has a 1-dimensional kernel.

Corollary 1.3. *If e is odd, $e \geq 5$, and if X is general, then the Kodaira dimension $\kappa(M_e) \geq 0$.*

The corollary follows as the form $\omega_e^{(3e+1)/2}$ is a nonzero section of the canonical line bundle.

In Section 2 we recall three different moduli spaces and how they are related. In Section 4 we give a general method to produce ω_e on the Kontsevich moduli stack $\mathcal{M}_e$ of stable maps for any $e \geq 1$. This is different than producing the form ω_e on $\overline{M}_e$. In a preliminary section, Section 3, we prove that every p-form on any tame, finite type Deligne-Mumford stack over a field k (not necessarily algebraically closed, nor of characteristic 0) gives rise to a p-form on every desingularization of the coarse moduli space, cf. Lemma 3.6. Thus producing the 2-form on $\mathcal{M}_e$ is *stronger* than producing the 2-form on $\overline{M}_e$. In Section 5 we describe how to compute the associated alternating pairing on Zariski tangent spaces of $\mathcal{M}_e$. In Section 6 we show that this pairing is nondegenerate for a general point of $\mathcal{M}_5$. The case $e = 5$ is particularly simple: almost no explicit calculation is necessary. In Section 7 we prove that ω_e is generically non-degenerate for every odd degree $e \geq 5$, and the kernel of the pairing is generically 1-dimensional for every even degree $e \geq 6$. In

Section 8 we sketch a proof that M_6 is not uniruled and pose some questions about the spaces M_e.

Finally, Theorem 1.2 implies Theorem 1.1 thanks to the following lemma.

Lemma 1.4. *Let M be a smooth, projective scheme, let ω be a 2-form on M, and suppose that at a general point $p \in \overline{M}_e$ the rank of the 2-form ω is r. Then $dim(Z) \leq dim(M) - r$, i.e., the codimension of the maximal rationally connected subvariety Z passing through a very general point of M is at least r.*

Proof. Denote $d = \dim(Z)$. If $d = 0$, there is nothing to prove. Suppose that d is positive. Then, by [16, Theorem IV.5.8], for a very general point $p \in M$ there is a morphism $g : \mathbb{P}^1 \to M$ whose image contains p and such that g^*T_M contains a locally free subsheaf $\mathcal{E} \subset g^*T_M$ with $\mathcal{E}$ an ample locally free sheaf of rank d and whose cokernel is a trivial locally free sheaf of rank $n-d$ (this is in the proof of [16, Theorem IV.5.8], not in the statement). Consider the sheaf map induced by ω, i.e., $g^*T_M \to g^*\Omega_M$. Since g^*T_M is semipositive, the sheaf $g^*\Omega_M$ is seminegative. There is no nonzero map from an ample locally free sheaf to a seminegative locally free sheaf. So $\mathcal{E}$ is contained in the kernel of the sheaf map. Therefore $d \leq \dim(M) - r$. $\square$

2. Discussion of moduli spaces

In this section we discuss three related functors, each of which gives a compactification of the space of smooth rational curves. The spaces representing these functors are birational. Since we are studying birational properties of these spaces the distinction between them is not crucial to the rest of the paper. We find it useful to pause, compare these three spaces, and point out what is and is not known about them.

Let k be a field, not necessarily algebraically closed, nor of characteristic 0. Let $X \subset \mathbb{P}^N$ be a quasi-projective scheme over k. Denote by M_e the scheme parametrizing families of smooth, proper, geometrically connected curves $C \subset X$ of arithmetic genus 0 and degree e. Even before compactifying M_e, there are several versions of M_e and we concentrate on two of these: M_e^h and M_e^c. The scheme M_e^h is the open subscheme of the Hilbert scheme $\mathrm{Hilb}^{et+1}(X)$ (cf. [9]) parametrizing smooth curves. And M_e^c is the open subvariety of the Chow variety $\mathrm{Chow}_{1,e}(X)$ (cf. [16, Def. I.3.20]) parametrizing cycles of smooth curves. There is not universal acceptance of the definition of the Chow variety (e.g. there is also the definition in [2]), but Kollár's definition is best suited to our needs. In particular, there is the *fundamental class morphism*, also called the *Hilbert-Chow morphism*, from M_e^h to M_e^c.

Lemma 2.1. [16, Thm. 6.3] *There exists a fundamental class morphism $FC : \left(M_e^h\right)^{sn} \to M_e^c$, where $\left(M_e^h\right)^{sn}$ is the* semi-normalization *of M_e^h (cf.* [16, Def. I.7.2.1]*). The morphism FC is an isomorphism. Therefore there is a morphism $(FC)^{-1} : M_e^c \to M_e^h$ that is equivalent to the semi-normalization of M_e^h. In particular it is bijective on points.*

Proof. This follows from [16, Thm. 6.3] and the semi-normal analogue of Zariski's Main Theorem. $\square$

It does happen that M_e^h is not semi-normal so that M_e^c and M_e^h are not isomorphic, e.g. whenever M_e^h is not reduced. A simple example of this is given by any pair (X, L) where $L \subset \mathbb{P}^3$ is a line and $X \subset \mathbb{P}^3$ is a smooth hypersurface of degree $d \geq 4$ containing L. In this case there is a unique connected component of M_1^h whose reduced scheme consists just of the point $[L] \in M_1^h$, but M_1^h is non-reduced.

For the special case that $X \subset \mathbb{P}^n$ is a smooth cubic hypersurface – the case of interest in this paper – we expect that M_e^h is always semi-normal.

Question 2.2. If $X \subset \mathbb{P}^n$ is a smooth cubic hypersurface, and if $\text{char}(k) = 0$, is M_e^h semi-normal? Is M_e^h normal?

There are some partial answers. For n arbitrary and $e = 1$, M_1^h is smooth by [5, Thm. 7.8]. For $n = 3$ and e arbitrary, M_e^h is an open subset of a projective space and so it is smooth. For $n = 4$ and $e = 2, 3$, M_e^h is smooth by [11, Lemma 3.2, Lemma 4.6]. For $n = 4$ and e arbitrary, M_e^h is an irreducible, reduced, local complete intersection scheme by [12]. So, by Serre's criterion, to prove that M_e^h is normal it suffices to prove that M_e^h is nonsingular in codimension one. We do not know whether this is true.

Let $X \subset \mathbb{P}^N$ be a projective scheme over a field k. Denote by $\overline{M}_e^h$ the closure of M_e^h in $\text{Hilb}^{et+1}(X)$ and denote by $\overline{M}_e^c$ the closure of M_e^c in $\text{Chow}_{1,e}(X)$. These are the first two compactifications of M_e which we consider.

Many results about the Hilbert scheme and the Chow variety are known. For instance, by [16, Thm. I.6.3], the morphism FC extends to a morphism $\text{FC} : (\overline{M}_e^h)^{sn} \to \overline{M}_e^c$. Both $\overline{M}_e^c$ and $\overline{M}_e^h$ have certain drawbacks. For example the morphism $(\text{FC})^{-1}$ does not extend to a regular morphism $\overline{M}_e^c \to \overline{M}_e^h$ (this fails even in the case $X = \mathbb{P}^N$). Moreover, the closed subsets $\overline{M}_e^h \subset \text{Hilb}^{et+1}(X)$ and $\overline{M}_e^c \subset \text{Chow}_{1,e}(X)$ are typically not open (i.e., they are typically not a union of connected components of the full Hilbert scheme, resp. Chow variety). Because of this, it is difficult to carry out an infinitesimal analysis of $\overline{M}_e^h$ and $\overline{M}_e^c$ as in [16, Section I.2].

If $\text{char}(k) = 0$, there is a third compactification of M_e that is very useful: the Kontsevich moduli space of stable maps (this compactification also exists in positive characteristic, but it is not as well-behaved). A *prestable map from an r-pointed curve of genus g to X of degree e* defined over a field L/k is a triple $(C, (p_1, \ldots, p_r), f)$ where C is a geometrically connected, reduced, at-worst-nodal curve of arithmetic genus g defined over L, where $p_1, \ldots, p_r$ is an ordered set of distinct L-rational points in the nonsingular locus of C, and where $f : C \to X$ is a morphism of k-schemes such that the degree of $f^*\mathcal{O}(1)$ is e. The triple is called a *stable map* if there are no infinitesimal automorphisms of the triple. There is a notion of families of stable maps and morphisms between stable maps. There is a *Deligne-Mumford stack* that is proper over k, $\overline{\mathcal{M}}_{g,n}(X, e)$, parametrizing flat families of stable maps from r-pointed curves of genus g to X of genus g of degree e. The *coarse moduli space* $\overline{M}_{g,n}(X, e)$ of the stack $\overline{\mathcal{M}}_{g,n}(X, e)$ is a projective k-scheme. The Deligne-Mumford stack and its coarse moduli space are described in detail in [4, 7].

In the special case that $X \subset \mathbb{P}^5$ is a smooth cubic hypersurface, we denote by $\overline{\mathcal{M}}_e$ the Kontsevich moduli space of stable maps to X of genus 0 with no marked points and degree e.

Lemma 2.3. *The scheme M_e^h is isomorphic to an open substack of $\overline{\mathcal{M}}_e$.*

Proof. This follows from the definitions of $\overline{\mathcal{M}}_e$ and M_e^h. □

There is an analogue of the morphism FC, i.e., a 1-morphism FC : $\left(\overline{\mathcal{M}}_e\right)^{sn} \to \overline{M}_e^c$. One drawback of $\overline{\mathcal{M}}_e$ as compared to $\overline{M}_e^h$ and $\overline{M}_e^c$ is that it is a stack rather than a scheme, which makes some arguments more technical. On the other hand, the deformation and obstruction theory of $\overline{\mathcal{M}}_e$ and the "boundary" are understood quite well. These are the key components in the proof of the following proposition.

Proposition 2.4. [13, Prop. 7.4] *Let $n \geq 5$ be an integer and let $X \subset \mathbb{P}^n$ a cubic hypersurface. If X is general, the stack $\overline{\mathcal{M}}_e$ is irreducible and reduced of the expected dimension $(n-2)e + (n-4)$ and has only local complete intersection singularities.*

Proof. Every case except $n = 5$ follows from [13, Prop. 7.4]. Thus suppose that $n = 5$.

By [13, Cor. 7.3], to prove the proposition it suffices to check that for $e = 1$ and $e = 2$, the following three conditions hold:

(i) the evaluation morphism ev : $\overline{\mathcal{M}}_{0,0}(X, e) \to X$ is surjective and has constant fiber dimension,
(ii) a general fiber of ev is irreducible, and
(iii) there exists a *free* stable map of degree e, i.e., a stable map $[C, f]$ such that f^*T_X is generated by global sections.

Case I, $e = 1$: First consider (iii). For every every smooth cubic hypersurface $X \subset \mathbb{P}^5$, and for every point $p \in X$, there exists a line $L \subset X$ containing p. By [6, Prop. 4.14], for every smooth variety X and for a very general point $p \in X$, every rational curve in X containing p is free. So for a very general point p in X, and for any line L containing p, (iii) holds.

By [16, Cor. II.3.5.4.2], for a very general point $p \in X$, the evaluation morphism ev : $\overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is smooth over p. The fiber F is canonically a complete intersection of hypersurfaces in $\mathbb{P}^{n-1}$ of dimension $n-4$. Whenever $n \geq 5$, this complete intersection is connected. Since F is smooth and connected, it is irreducible. Thus (ii) holds.

Finally, if X is a *general* hypersurface, then by [11, Thm. 2.1], (i) holds.

Case II, $e = 2$: Let $p \in X$ be a general point and let $L \subset X$ be any line containing p. Then L is free. Thus any degree 2 cover of L by a rational curve is a stable map that is free. Thus (iii) holds.

There is an *a priori* lower bound on the dimension of every irreducible component of every fiber of $\mathrm{ev}_f : \overline{\mathcal{M}}_{0,1}(X, 2) \to X$, namely the difference of the expected dimension of $\overline{\mathcal{M}}_{0,1}(X, 2)$ and $\dim(X)$, which is 4. Condition (i) is the condition that every fiber of ev_f has dimension exactly 4. Condition (ii) is the condition that at least one fiber is irreducible and reduced of dimension 4.

Suppose that X contains no linear $\mathbb{P}^2$ – this holds for a general cubic hypersurface in $\mathbb{P}^5$. Then every stable map $f : C \to X$ of degree 2 that is not a double cover of a line is an embedded plane conic. The span of the conic C, say $\Lambda \subset \mathbb{P}^n$, intersects X in a plane cubic curve $C' \subset \Lambda$. Of course $C \subset C'$, and the residual curve is a line $L \subset X$.

Conversely, for a general pair of a line $L \subset X$ and a linear $\mathbb{P}^2$ Λ containing L, the residual to L in $\Lambda \cap X$ is a plane conic. Thus the set of embedded plane conics in X passing through a general point p, is isomorphic to an open subset of the space of lines $\overline{\mathcal{M}}_1$. This space is smooth of dimension 4. So to finish the proof of (i) and (ii), it suffices to show that this set is Zariski dense in $\mathrm{ev}^{-1}(p)$ for every $p \in X$. In other words, for every $p \in X$, the subset of $\mathrm{ev}^{-1}(p)$ consisting of double covers of lines is not dense in any irreducible component of $\mathrm{ev}^{-1}(p)$.

Since X is general, the morphism $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is flat. Thus the variety parametrizing lines in X containing p has dimension 1. Thus the variety parametrizing double covers of lines containing p has dimension 3 (there is a 2-dimensional family of double covers of a given line by rational curves). Combined with the lower bound of 4, it follows that the variety parametrizing double covers of lines containing p is not dense in any irreducible component of $\mathrm{ev}^{-1}(p)$. Therefore (i) and (ii) hold. □

Remark 2.5. (1) The proposition is false for $n = 3$ and $n = 4$. For $n = 3$, M_e is disconnected (the Picard number of X is not 1). For $n = 4$ and $e \geq 2$, there is an irreducible component $Y_e \subset \overline{\mathcal{M}}_e$ parametrizing degree e covers of lines in X. The open subset $\overline{\mathcal{M}}_e - Y_e$ is irreducible, reduced of the expected dimension and has only local complete intersections (cf. [12]).
(2) Even though the proof above works only for a *general* hypersurface X, we suspect the proposition holds for *every* smooth cubic hypersurface $X \subset \mathbb{P}^n$.
(3) In fact the argument above proves much more than the proposition, namely for every stable genus 0 A-graph τ and every flag f of τ, a certain condition $\mathcal{B}(X, \tau, f)$ holds (cf. [13, Cor. 7.3]). In particular, $\overline{\mathcal{M}}(X, \tau)$ is irreducible.

Corollary 2.6. *For $X \subset \mathbb{P}^5$ a general cubic hypersurface, the schemes M_e^c and M_e^h are irreducible and reduced of dimension $3e + 1$. They are birational to each other and to $\overline{\mathcal{M}}_e$.*

3. Trace maps and descent for p-forms

Let X be a quasi-projective variety over a field k with $\mathrm{char}(k) = 0$. If X is smooth and projective and if $k = \mathbb{C}$, Hodge theory gives linear maps from $H^{p+1,q+1}(X)$ to $H^{p,q}(\widetilde{M}_e)$, where $\widetilde{M}_e$ is a desingularization of $\overline{\mathcal{M}}_{0,0}(X, e)$. The map pulls back forms to the universal curve over $\overline{\mathcal{M}}_{0,0}(X, e)$, and then uses "integration along fibers". In the proof of the main theorem, we need a version of this that holds when X is neither smooth nor projective. In the next two sections we prove the following version (the proof is algebraic, not Hodge-theoretic): Let X be a quasi-projective variety. Let g, r, p and q be nonnegative integers. There are linear maps,

$$H^{q+1}\left(X, \Omega^{p+1}\right) \to H^{q}\left(\overline{\mathcal{M}}_{g,r}(X,e), \Omega^{p}\right).$$

When $q = 0$ this map gives p-forms on the Kontsevich moduli stack. This, in turn, gives p-forms on a desingularization of the coarse moduli space of the stack. This follows by a more general result, Proposition 3.6, which is the main result of this section: Let k be a field (not necessarily algebraically closed, nor of characteristic 0), and let $\mathcal{M}$ be a finite type Deligne-Mumford stack over k. If $\mathcal{M}$ is *tame* and if the coarse moduli space M is smooth, then every p-form on $\mathcal{M}$ is the pullback of a unique p-form on M (up to torsion).

The proof uses *trace maps* for proper morphisms, $f : Y \to Z$, where Z is normal and f is étale on a dense open subset of Y,

$$\mathrm{Tr}_f^p : f_*(\Omega_Y^p) \to (\Omega_Z^p)^{\vee\vee}.$$

3.1. **If f is finite étale.** Let $f : Y \to Z$ be a morphism of schemes and let $\mathcal{F}$ be a coherent sheaf on Z. There is a morphism of $\mathcal{O}_Z$-modules, $f_*\mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{F} \to f_*f^*\mathcal{F}$. If f is finite and $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_Z$-module, this morphism is an isomorphism. Also, there is a trace map $\mathrm{Tr}_f : f_*\mathcal{O}_Y \to \mathcal{O}_Z$ defined in the usual way. Therefore, there is a trace map $\mathrm{Tr}_f : f_*f^*\mathcal{F} \to \mathcal{F}$.

Let $f : Y \to Z$ be a finite étale k-morphism of finite type k-schemes. For each integer $p \geq 0$, the pullback map $(df)^\dagger : f^*\Omega^p_{Z/k} \to \Omega^p_{Y/k}$ is an isomorphism. Combined with the trace map from the last paragraph, we get a map satisfying the following properties.

Lemma 3.1. *Let $f : Y \to Z$ be a finite étale k-morphism of finite type k-schemes. Denote by n the degree of f. For each integer $p \geq 0$ there exists a unique morphism of $\mathcal{O}_Z$-modules, $Tr_f^p : f_*\Omega^p_{Y/k} \to \Omega^p_{Z/k}$ satisfying the following properties.*

- (i) *For $p = 0$, $Tr_f^0 : f_*\mathcal{O}_Y \to \mathcal{O}_Z$ is the usual trace map.*
- (ii) *For every open subset $U \subset Z$, for every pair of integers, $p, p' \geq 0$, for every section $\sigma \in H^0(U, \Omega^p_{Z/k})$, and for every section $\tau \in H^0(f^{-1}(U), \Omega^{p'}_{Y/k})$,*

$$Tr_f^{p+p'} f_*(f^*\sigma \wedge \tau) = \sigma \wedge Tr_f^{p'}(f_*\tau).$$

Moreover, the following properties hold.

- (iii) *For every integer p and every section $\tau \in H^0(Y, \Omega^p_{Y/k})$,*

$$Tr_f^{p+1}(f_* d\tau) = d(Tr_f^p(f_*\tau)).$$

- (iv) *Let Z' be a finite type k-scheme and let $g : Z' \to Z$ be a morphism of k-schemes. Denote by Y' the fiber product $Z' \times_Z Y$, and denote by $f' : Y' \to Z'$ and $g' : Y' \to Y$ the projection morphisms. For each integer $p \geq 0$, there is a commutative diagram of $\mathcal{O}_{Z'}$-modules.*

$$\begin{array}{ccccc}
g^* f_* \Omega^p_{Y/k} & \xrightarrow{g^* Tr_f^p} & g^*\Omega^p_{Z/k} & \xrightarrow{(dg)^\dagger} & \Omega^p_{Z'/k} \\
\cong \downarrow & & & & \downarrow = \\
(f')_*(g')^*\Omega^p_{Y/k} & \xrightarrow{(f')_*(dg')^\dagger} & (f')_*\Omega^p_{Y'/k} & \xrightarrow{Tr^p_{f'}} & \Omega^p_{Z'/k}
\end{array}$$

(v) *Let $h : X \to Y$ be a finite étale morphism. Then for each integer $p \geq 0$, $Tr^p_{f\circ h} = Tr^p_f \circ f_* Tr^p_h$.*

3.2. **If f is proper and generically étale.** Let $f : Y \to Z$ be a proper k-morphism of finite type k-schemes. Suppose that Z is connected and geometrically normal, and suppose that there is a dense open subset of Y on which f is étale. Denote by $Z_{\text{smooth}} \subset Z$ the maximal open subscheme that is smooth over k. Then there exists an open subset $U \subset Z_{\text{smooth}}$, dense in Z, such that $V = f^{-1}(U)$ is dense in Y and such that $f|_V : V \to U$ is finite étale. By Lemma 3.1, for each $p \geq 0$, there is a trace map $\mathrm{Tr}^p_{f|_V} : (f|_V)_* \Omega^p_{V/k} \to \Omega^p_{U/k}$.

Definition 3.2. A *trace map* is a morphism of $\mathcal{O}_Z$-modules, $\mathrm{Tr}^p_f : f_*\Omega^p_Y \to (\Omega^p_Z)^{\vee\vee}$, whose restriction to U equals $\mathrm{Tr}^p_{f|_V}$.

It is straightforward to check that, if a trace map exists, it is unique and it is independent of the choice of U.

Proposition 3.3. *Let Z be a finite type k-scheme that is connected and geometrically normal. Let $f : Y \to Z$ be a proper morphism that is étale on a dense open subset of Y. For each integer $p \geq 0$, there exists a trace map $Tr^p_f : f_*\Omega^p_{Y/k} \to (\Omega^p_{Z/k})^{\vee\vee}$.*

Proof. Denote by $i : U \to Z$ the open immersion. The morphism $\mathrm{Tr}_{f|_V}$ determines a morphism of $\mathcal{O}_Z$-modules, $i_*\mathrm{Tr}^p_{f|_V} : f_*\Omega^p_{Y/k} \to i_*i^*\Omega^p_{Z/k}$. Because $U \subset Z_{\text{smooth}}$, $i^*\Omega^p_{Z/k} = i^*(\Omega^p_{Z/k})^{\vee\vee}$. Therefore there is a injective morphism of $\mathcal{O}_Z$-modules, $(\Omega^p_{Z/k})^{\vee\vee} \to i_*i^*\Omega^p_{Z/k}$. The proposition exactly says that the image of $i_*\mathrm{Tr}^p_{f|_V}$ is contained in $(\Omega^p_{Z/k})^{\vee\vee}$. By hypothesis, Z is normal. And $(\Omega^p_{Z/k})^{\vee\vee}$ is reflexive. Therefore, $(\Omega^p_{Z/k})^{\vee\vee}$ is the intersection (in $i_*i^*\Omega^p_{Z/k}$) of its localization at every codimension 1 point. Thus it suffices to check that for every codimension 1 point $\eta \in Z - U$, the image of $i_*\mathrm{Tr}^p_{f|_V}$ is contained in the localization of $(\Omega^p_{Z/k})^{\vee\vee}$ at η.

The image of $i_*\mathrm{Tr}^p_{f|_V}$ does not change if we replace Y by the disjoint union of the irreducible components of Y, with the induced reduced scheme structure. Therefore assume that Y is reduced and every connected component of Y is irreducible. The morphism f is finite on an open subset of Z whose complement has codimension ≥ 2. Thus there exists an open affine $W \subset Z_{\text{smooth}}$ such that $\eta \in W$ and such that $f|_W : f^{-1}(W) \to W$ is finite. Denote $A = H^0(W, \mathcal{O}_Z)$ and denote $B = H^0(f^{-1}(W), \mathcal{O}_Y)$. By [19], the image of $\Omega^p_{B/k}$ under $i_*\mathrm{Tr}^p_{f|_V}$ is contained in $\Omega^p_{A/k}$ (Zannier only proves this when Y is connected, but the generalization here follows trivially). In particular, the image of $i_*\mathrm{Tr}^p_{f|_V}$ is contained in the localization of $(\Omega^p_{Z/k})^{\vee\vee}$ at η.

□

Remark 3.4. For each integer $p \geq 0$, there is a *generic trace map*

$$(\mathrm{Tr}^{\otimes p}_f)_{\text{gen}} : f_*(\Omega^1_{Y/k})^{\otimes p} \to (\Omega^1_{Z/k})^{\otimes p} \otimes_{\mathcal{O}_Z} K(Z).$$

By the proposition, when restricted to the submodule corresponding to an exterior power, the generic trace map factors through $(\Omega^p_Z)^{\vee\vee}$. One might hope that the entire generic trace map factors through the reflexive hull of $(\Omega^1_Z)^{\otimes p}$. This is true, for instance, if $f : Y \to Z$ is étale away from codimension 2. But typically this is not the case: Consider the morphism $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ which pulls back a coordinate t on the target to u^2, the square of a coordinate u on the domain. Then the generic trace of $du \otimes du$ is $\frac{1}{4t} dt \otimes dt$.

Lemma 3.5. *Let Z be a finite type k-scheme that is connected and geometrically normal. Let $f : Y \to Z$ be a proper morphism that is étale on a dense open subset of Y. The trace maps Tr^p_f satisfy the following properties.*

(i) *For $p = 0$, $Tr^0_f : f_*\mathcal{O}_Y \to \mathcal{O}_Z$ is the usual trace map.*

(ii) *For every pair of integers, $p, p' \geq 0$, for every section $\sigma \in H^0(Z, \Omega^p_Z)$, and for every section $\tau \in H^0(Y, \Omega^q_Y)$,*

$$Tr^{p+p'}_f f_*(f^*\sigma \wedge \tau) = \sigma \wedge Tr^{p'}_f(f_*\tau).$$

(iii) *For every integer p and every section $\tau \in H^0(Y, \Omega^p_{Y/k})$,*

$$Tr^{p+1}_f(f_* d\tau) = d(Tr^p_f(f_*\tau)).$$

(iv) *Let Z' be a finite type k-scheme that is connected and geometrically normal. Let $g : Z' \to Z$ be a morphism of k-schemes such that $g^{-1}(U)$ is dense in Z'. Denote by Y' the closure in $Z' \times_Z Y$ of the inverse image of $g^{-1}(U)$. And denote by $f' : Y' \to Z'$ and $g' : Y' \to Y$ the projection morphisms. For each integer $p \geq 0$, there is a commutative diagram of $\mathcal{O}_{Z'}$-modules.*

$$\begin{array}{ccccc} g^*f_*\Omega^p_{Y/k} & \xrightarrow{g^*Tr^p_f} & g^*\Omega^p_{Z/k} & \xrightarrow{(dg)^\dagger} & \Omega^p_{Z'/k} \\ \downarrow & & & & \downarrow = \\ (f')_*(g')^*\Omega^p_{Y/k} & \xrightarrow{(f')_*(dg')^\dagger} & (f')_*\Omega^p_{Y'/k} & \xrightarrow{Tr^p_{f'}} & \Omega^p_{Z'/k} \end{array}$$

(v) *Let $h : X \to Y$ be a proper morphism. Suppose that Y is normal and h is étale on a dense open subset of X. Then $f \circ h$ is étale on a dense open subset of X, and for each integer $p \geq 0$, $Tr^p_{f\circ h} = Tr^p_f \circ f_* Tr^p_h$.*

3.3. **Descent for p-forms on a stack.** Let k be a field (not necessarily algebraically closed nor of characteristic 0). Let $\mathcal{M}$ be a finite type Deligne-Mumford stack over k. Recall from [1], that $\mathcal{M}$ is *tame* if the stabilizer group of every geometric point of $\mathcal{M}$ has order prime to char(k). Recall from [15] that there exists a coarse moduli space, M, for $\mathcal{M}$, and M is an algebraic space of finite type over k. Suppose that $\mathcal{M}$ is tame, irreducible, and generically reduced and that M is a geometrically normal k-scheme.

Denote by $c : \mathcal{M} \to M$ the morphism of $\mathcal{M}$ to the coarse moduli space. Let $U \subset M$ be the maximal open subscheme over which c is smooth. Then for every p, the pullback map $H^0(U, \Omega^p_{M/k}) \to H^0(c^{-1}(U), \Omega^p_{\mathcal{M}/k})$ is an isomorphism. It is not true that the pullback map over all of M is an isomorphism, even modulo torsion. For instance, let char$(k) \neq 2$, consider $\mathbb{A}^2_k$ with coordinates x, y, let Γ be the cyclic group of order 2, and let Γ act on $\mathbb{A}^2_k$ by $x \mapsto -x, y \mapsto -y$. Let $\mathcal{M}$ be the quotient stack $[\mathbb{A}^2_k/\Gamma]$. Then the 2-form $dx \wedge dy$ is Γ-invariant and thus gives rise to a global section of $\Omega^2_{\mathcal{M}/k}$. But this 2-form is not the pullback of any global section of $\Omega^2_{M/k}$.

A slightly weaker result is true, and will be proved in Proposition 3.6. This result is good enough for our application. First we explain the result, then we prove it. If $\mathcal{M}$ is tame, irreducible and generically reduced and if M is a geometrically normal k-scheme, then for each integer $p \geq 0$ there is a k-linear map,

$$c_* : H^0(\mathcal{M}, (\Omega^p_{\mathcal{M}/k})^{\text{free}}) \to H^0(M, (\Omega^p_{M/k})^{\vee\vee}).$$

Here, for a coherent sheaf $\mathcal{F}$, the notation $(\mathcal{F})^{\text{free}}$ denotes the maximal torsion-free quotient, i.e., the image of the sheaf map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_M} K(M)$. There is also a *generic pullback map*,

$$c^*_{\text{gen}} : H^0(M, (\Omega^p_{M/k})^{\vee\vee}) \to H^0(\mathcal{M}, (\Omega^p_{\mathcal{M}/k})^{\text{free}} \otimes_{\mathcal{O}_M} K(M)).$$

And the composition $c^*_{\text{gen}} \circ c_*$ equals the obvious inclusion map.

Let N be a finite type k-scheme that is geometrically normal. Let $g : N \to M$ be a k-morphism such that $g^{-1}(U)$ is dense. There is a generic pullback map,

$$g^*_{\text{gen}} : H^0(M, (\Omega^p_{M/k})^{\vee\vee}) \to H^0(N, (\Omega^p_{N/k})^{\vee\vee} \otimes_{\mathcal{O}_N} K(N)).$$

In fact the image of $g^*_{\text{gen}} c_*$ is contained in the image of $H^0(N, (\Omega^p_{N/k})^{\vee\vee})$.

Proposition 3.6. *Let k be a field (not necessarily algebraically closed nor of characteristic 0), and let $\mathcal{M}$ be a finite type Deligne-Mumford stack. Suppose that $\mathcal{M}$ is tame, irreducible, and generically reduced, and that the coarse moduli space M is a k-scheme that is geometrically normal. For each integer $p \geq 0$, there is a k-linear map,*

$$c_* : H^0(\mathcal{M}, (\Omega^p_{\mathcal{M}/k})^{free}) \to H^0(M, (\Omega^p_{M/k})^{\vee\vee}),$$

whose composition with the generic pullback map, c^*_{gen}, is the obvious inclusion map. Moreover, for every finite type k-scheme N, and for every k-morphism $g : N \to M$, if N is geometrically normal and if $g^{-1}(U) \subset N$ is dense, then the image of $g^*_{gen} c_*$ is contained in the image of $H^0(N, (\Omega^p_{N/k})^{\vee\vee})$.

Proof. If there exists a map c_* such that $c^*_{\text{gen}} c_*$ is the inclusion map, then c_* is unique. Thus we may prove that c_* exists after étale base change of M: the uniqueness of c_* guarantees the cocycle condition for étale descent.

By [1, Lem. 2.2.3], there exists an étale covering $\{M_i \to M\}$ such that,

(i) each base change $M_i \times_M \mathcal{M}$ is a finite quotient stack $[U_i/\Gamma_i]$,
(ii) each U_i is a scheme finite over M_i,
(iii) each Γ_i is a finite group whose order is prime to $\text{char}(k)$,
(iv) Γ_i acts on U_i by M_i-morphisms, and
(v) the quotient M_i-scheme $U_i//\Gamma_i$ equals M_i.

Thus, without loss of generality, assume that $\mathcal{M} = [U/\Gamma]$ where U is a scheme finite over M and where Γ is a group whose order is prime to $\text{char}(k)$ acting on U by M-morphisms.

Denote by $h : U \to M$ the morphism above. By Proposition 3.3, there is a morphism

$$\text{Tr}^p_h : H^0(U, (\Omega^p_{U/k})^{\text{free}}) \to H^0(M, (\Omega^p_{M/k})^{\vee\vee}).$$

The global sections of $(\Omega^p_{\mathcal{M}/k})^{\text{free}}$ are precisely the Γ-invariant global sections of $(\Omega^p_{U/k})^{\text{free}}$. So there is an induced morphism

$$(\mathrm{Tr}^p_h)^\Gamma : H^0(\mathcal{M}, (\Omega^p_{\mathcal{M}/k})^{\text{free}}) \to H^0(M, (\Omega^p_{M/k})^{\vee\vee}).$$

It is straightforward to check that $\frac{1}{|\Gamma|}(\mathrm{Tr}^p_h)^\Gamma$ satisfies the condition for c_*.

Consider $g^*_{\text{gen}} c_*$. Denote by $\mathcal{N}$ the fiber product $N \times_M \mathcal{M}$. Denote by $c' : \mathcal{N} \to N$ and $g' : \mathcal{N} \to \mathcal{M}$ the projection morphisms. Then $(c')^*_{\text{gen}} g^*_{\text{gen}} c_*$ equals $(g')^* c^*_{\text{gen}} c_*$. And this equals $(c')^*_{\text{gen}}(c')_*(g')^*$. Since $(c')^*_{\text{gen}}$ is injective, $g^*_{\text{gen}} c_* = (c')_*(g')^*$. In particular, the image is contained in $H^0(N, (\Omega^p_{N/k})^{\vee\vee})$.

□

4. Construction of the 2-form

Let k be a field (not necessarily algebraically closed, nor of characteristic 0). Let $\mathcal{M}$ be a finite type Deligne-Mumford stack over k and let $p : \mathcal{C} \to \mathcal{M}$ be a representable 1-morphism of Deligne-Mumford stacks such that

(i) p is proper and flat of relative dimension 1, and
(ii) every geometric fiber of p is a reduced, at-worst-nodal curve,

i.e., $p : \mathcal{C} \to \mathcal{M}$ is a semi-stable family of curves. There is a canonical morphism from the sheaf of relative Kähler differentials to the dualizing sheaf $\Omega^1_p \to \omega_p$. This is an isomorphism on the open substack $U \subset \mathcal{C}$ which is the smooth locus of p. For each integer $i \geq 0$, this isomorphism induces a morphism of $\mathcal{O}_U$-modules,

$$\phi_{U,i} \colon \Omega^{i+1}_{\mathcal{C}/k}|_U \to (\Omega^{i+1}_{\mathcal{C}/k}/p^*\Omega^{i+1}_{\mathcal{M}/k})|_U \cong p^*\Omega^i_{\mathcal{M}/k} \otimes \omega_p|_U.$$

This morphism has the property that for every section $\alpha \in \Omega^i_{\mathcal{M}/k}$ and $\beta \in \Omega^j_{\mathcal{C}/k}$, $\phi_{U,i+j}(p^*\alpha \wedge \beta) = p^*\alpha \wedge \phi_{U,j}(\beta)$.

Lemma 4.1. *For each integer $i \geq 0$ there exists a unique morphism of $\mathcal{O}_{\mathcal{C}}$-modules, $\phi_i : \Omega^{i+1}_{\mathcal{C}/k} \to p^*\Omega^i_{\mathcal{M}/k} \otimes_{\mathcal{O}_{\mathcal{C}}} \omega_p$, such that $\phi_i|_U = \phi_{U,i}$ and such that for every section $\alpha \in \Omega^i_{\mathcal{M}/k}$ and $\beta \in \Omega^j_{\mathcal{C}/k}$, $\phi_{i+j}(p^*\alpha \wedge \beta) = p^*\alpha \wedge \phi_j(\beta)$.*

Proof. If ϕ_i exists, then by construction it annihilates $p^*\Omega^{i+1}_{\mathcal{M}}$, i.e., it factors through the quotient. The quotient has a canonical subsheaf isomorphic to $p^*\Omega^i_{\mathcal{M}} \otimes \Omega^1_p$ with an obvious map to $p^*\Omega^i_{\mathcal{M}} \otimes \omega_p$. The lemma claims this map extends to the entire quotient. It also claims the extension is unique. Uniqueness is straightforward: the cokernel of $p^*\Omega^i_{\mathcal{M}} \otimes \Omega^1_p$ is a sheaf that is torsion on fibers, whereas the sheaf $p^*\Omega^i_{\mathcal{M}} \otimes \omega_p$ is torsion-free on fibers. So there is no nonzero map from the cokernel to $p^*\Omega^i_{\mathcal{M}} \otimes \omega_p$, i.e., if the map extends, then the extension is unique. The extension problem is equivalent to the vanishing of a section of a sheaf Ext. This vanishing can be checked after passing to the completion of the local ring at each geometric closed point of $\mathcal{C}$, i.e., it suffices to check that the sheaf map extends formally locally at each geometric closed point of $\mathcal{C}$.

Since the property can be checked formally locally, without loss of generality assume that $\mathcal{M}$ is a scheme. Let $z \in \mathcal{C}$ be a closed point. Denoting $A = \widehat{\mathcal{O}_{\mathcal{M},p(z)}}$ and $B = \widehat{\mathcal{O}_{\mathcal{C},z}}$, there is an isomorphism

$$B \cong A[[x,y]]/\langle xy - a \rangle.$$

for some element $a \in A$. By Remark 4.4, the base change of $\phi_{U,i}$ does extend to a map $\phi_i \otimes_{\mathcal{O}_\mathcal{C}} B$ as required. $\square$

Corollary 4.2. *For each pair of integers, $i, j \geq 0$, there is a k-linear map,*

$$H^{j+1}(\mathcal{C}, \Omega^{i+1}_{\mathcal{C}/k}) \to H^j(\mathcal{M}, \Omega^i_{\mathcal{M}/k}).$$

Proof. The morphism ϕ_i induces a k-linear map,

$$H^{j+1}(\mathcal{C}, \Omega^{i+1}_{\mathcal{C}/k}) \to H^{j+1}(\mathcal{C}, p^*\Omega^i_{\mathcal{M}/k} \otimes \omega_p).$$

Associated to the morphism p, there is a Leray spectral sequence for the target vector space. Because $R^l p_*(p^*\Omega^i_{\mathcal{M}/k} \otimes \omega_p) = \Omega^i_{\mathcal{M}/k} \otimes R^l p_* \omega_p$ is zero for $l \geq 2$, there is an abutment map,

$$H^{j+1}(\mathcal{C}, p^*\Omega^i_{\mathcal{M}/k} \otimes \omega_p) \to H^j(\mathcal{M}, \Omega^i_{\mathcal{M}/k} \otimes R^1 p_* \omega_p).$$

And there is a trace isomorphism $R^1 p_* \omega_p \xrightarrow{\cong} \mathcal{O}_\mathcal{M}$. Composing these maps gives the k-linear map,

$$H^{j+1}(\mathcal{C}, \Omega^{i+1}_{\mathcal{C}/k}) \to H^j(\mathcal{M}, \Omega^i_{\mathcal{M}/k}).$$

$\square$

Assume that $\text{char}(k) = 0$. Let X be a quasi-projective k-scheme and let $\overline{\mathcal{M}}_{g,r}(X, e)$ be the Kontsevich moduli space of stable maps from r-pointed curves of arithmetic genus g to X of degree e. There is a universal curve $p : \mathcal{C} \to \overline{\mathcal{M}}_{g,r}(X, e)$ satisfying the hypotheses above. And there is an evaluation morphism $\text{ev} : \mathcal{C} \to X$. For each pair of integers, $i, j \geq 0$, there is a pullback morphism

$$\text{ev}^* : H^{j+1}(X, \Omega^{i+1}_{X/k}) \to H^{j+1}(\mathcal{C}, \Omega^{i+1}_{\mathcal{C}/k}).$$

Composing with the k-linear map from Corollary 4.2 gives the following.

Corollary 4.3. *For each pair of integers $i, j \geq 0$, there is an "integration along fibers" morphism,*

$$H^{j+1}(X, \Omega^{i+1}_{X/k}) \to H^j(\overline{\mathcal{M}}_{g,r}(X, e), \Omega^i_{\overline{\mathcal{M}}_{g,r}(X,e)/k}).$$

In particular, suppose X is the smooth locus of a cubic hypersurface in $\mathbb{P}^4$. As will be recalled in the next section, the Griffiths residue calculus gives a canonical map from a 1-dimensional k-vector space to $H^1(X, \Omega^3_{X/k})$. If X is projective, this map is an isomorphism of k-vector spaces. Using the map above, for each integer $e > 0$, this gives a global section ω_e of Ω^2 on the stack $\overline{\mathcal{M}}_e$ parametrizing stable maps from curves of arithmetic genus 0 to X of degree e. The section ω_e is well-defined up to non-zero scalar. This is the object of study in the rest of the article.

Remark 4.4. Let A be a ring and let $B = A[x, y]/(xy - a)$ for some $a \in A$. Consider the canonical exact sequence

$$0 \to \Omega^1_A \otimes B \to \Omega^1_B \to \Omega^1_{B/A} \to 0.$$

Exactness on the left follows as B is a complete intersection flat over A whose cotangent complex $L_{B/A}$ is quasi-isomorphic to $\Omega^1_{B/A}$. Moreover, the relative dualizing sheaf is the determinant of $L_{B/A}$ (which is perfect of amplitude $[-1,0]$). So, the relative dualizing module $\omega_{B/A}$ is free with generator

$$\theta = \frac{\mathrm{d}x \wedge \mathrm{d}y}{xy - a}.$$

and there is a canonical B-module homomorphism

$$\Omega^1_{B/A} \longrightarrow \omega_{B/A}$$

which is determined by the rules $\mathrm{d}x \mapsto x\theta$ and $\mathrm{d}y \mapsto -y\theta$. From this we will define maps

$$\Omega^i_B \to \Omega^{i-1}_A \otimes_A \omega_{B/A}.$$

Namely, any element in Ω^i_B can be written as a B-linear combination of forms of the type η, $\eta \wedge \mathrm{d}x$, $\eta \wedge \mathrm{d}y$ and $\eta \wedge \mathrm{d}x \wedge \mathrm{d}y$, where η is in Ω^j_A, with $j = i$, $i-1$, or $i-2$. We claim there exists a map as above such that

$$\eta \mapsto 0, \quad \eta \wedge \mathrm{d}x \mapsto \eta \otimes x\theta, \quad \eta \wedge \mathrm{d}y \mapsto -\eta \otimes y\theta, \quad \eta \wedge \mathrm{d}x \wedge \mathrm{d}y \mapsto -\eta \wedge \mathrm{d}a \otimes \theta.$$

The reader easily verifies that this is well defined (the main concern being that forms of the type $\eta \wedge (y\mathrm{d}x + x\mathrm{d}y - \mathrm{d}a)$ and $\eta \wedge (y\mathrm{d}x + x\mathrm{d}y - \mathrm{d}a) \wedge \mathrm{d}x$ get mapped to zero).

5. Explicit description of the 2-form

Let k be a field of characteristic 0 and let X be a quasi-projective k-scheme. As in the last section, for each pair of integers, $i, j \geq 0$, there is k-linear map $H^{j+1}(X, \Omega^{i+1}_{X/k}) \to H^j(\overline{\mathcal{M}}_{g,r}(X,e), \Omega^i_{\overline{\mathcal{M}}_{g,r}(X,e)/k})$. When $j = 0$, this gives global sections of Ω^i. Let $z \in \mathcal{M}$ be a geometric closed point, and consider the fiber of this section at z. The goal of this section is to describe the fiber of the section in terms of the local geometry of the associated stable map $f : C \to X$, i.e., in terms of the pullback of the tangent bundle of X, etc. In the special case that X is the smooth locus of a cubic hypersurface in $\mathbb{P}^4$ and $g = 0$, we give an explicit description of the fiber of this section.

5.1. **Explicit description of $H^1(X, \Omega^3_X)$.** First we recall a small part of the Griffiths residue calculus [8, Section 8]. This is also discussed very briefly in [18, Section 0]. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d, and let $U \subset X$ be the smooth locus. The cotangent sequence is,

$$0 \longrightarrow \mathcal{O}_U(-d) \longrightarrow \Omega^1_{\mathbb{P}^n}|_U \longrightarrow \Omega^1_U \longrightarrow 0$$

Taking the exterior power of this sequence, and twisting by $\mathcal{O}_X(d)|_U$, gives an exact sequence,

$$0 \longrightarrow \Omega^{n-2}_U \longrightarrow \Omega^{n-1}_{\mathbb{P}^n}|_U \otimes \mathcal{O}_U(d) \longrightarrow \Omega^n_{\mathbb{P}^n}|_U \otimes \mathcal{O}_U(2d) \longrightarrow 0$$

(This also follows by taking the dual of the first exact sequence and twisting by $\Omega^n_{\mathbb{P}^n}|_U \otimes \mathcal{O}_U(d)$.) The connecting homomorphism in cohomology gives a map,

$$H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(2d)) \to H^1(U, \Omega^{n-2}_U).$$

In the special case of a cubic fourfold, there is an exact sequence,

$$0 \longrightarrow \Omega^3_U \longrightarrow \Omega^4_{\mathbb{P}^5}|_U \otimes \mathcal{O}_U(3) \longrightarrow \Omega^5_{\mathbb{P}^5}|_U \otimes \mathcal{O}_U(6) \longrightarrow 0. \qquad (1)$$

Of course $\Omega^5_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5}(6) \cong \mathcal{O}_{\mathbb{P}^5}$. Thus the connecting homomorphism,

$$H^0(\mathbb{P}^5, \Omega^5_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5}(6)) \to H^1(U, \Omega^3_U),$$

is a map from a 1-dimensional vector space to $H^1(U, \Omega^3_U)$. If $U = X$, this map is an isomorphism. Choose a nonzero element in $H^0(\mathbb{P}^5, \Omega^5_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5}(6))$, and define ω_e^{pre} to be the image of this element in $H^1(U, \Omega^3_U)$. Define $\omega_e \in H^0(\overline{\mathcal{M}}_e, \Omega^2_{\overline{\mathcal{M}}_e/k})$ to be the global section associated to ω_e^{pre}.

5.2. **The explicit description.** Let $f : C \to X$ be a point of $M_e \subset \overline{\mathcal{M}}_e$. Assume that $C \cong \mathbb{P}^1$ is smooth and that f is a regular embedding into the smooth locus $U \subset X$. Consider the sequence of vector bundles over C given by the normal bundle $N_{C/X}$ of C in X mapping to the normal bundle $N_{C/\mathbb{P}^5}$ of C in $\mathbb{P}^5$,

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^5} \longrightarrow f^* N_{X/\mathbb{P}^5} \longrightarrow 0. \qquad (2)$$

Of course $N_{X/\mathbb{P}^5} \cong \mathcal{O}_X(3)$, so that $f^* N_{X/\mathbb{P}^5} \cong \mathcal{O}_C(3e)$, where the notation $\mathcal{O}_C(a)$ indicates any invertible sheaf of degree a on C. In particular, observe that $\bigwedge^3 N_{C/X} = \mathcal{O}_C(3e-2)$ and that $\bigwedge^4 N_{C/\mathbb{P}^5} = \mathcal{O}_C(6e-2)$. The Zariski tangent space $T_{[f]}(M_e)$, which is the same thing as the dual vector space of the fiber $\Omega^1_{M_e}|_{[f]}$, is given by the space of global sections $H^0(C, N_{C/X})$ (c.f. [16, Theorem I.2.8]). So the fiber $\Omega^2_{M_e}|_{[f]}$ is just the vector space dual of $\bigwedge^2 H^0(C, N_{C/X})$. And the 2-form ω_e gives a procedure to associate to any two sections of $N_{C/X}$ a complex number.

Next consider the exact sequence

$$0 \longrightarrow \bigwedge^3 N_{C/X} \otimes \mathcal{O}_C(-3e) \longrightarrow \bigwedge^3 N_{C/\mathbb{P}^5} \otimes \mathcal{O}_C(-3e) \longrightarrow \bigwedge^2 N_{C/X} \longrightarrow 0. \quad (3)$$

This sequence is obtained from Equation 2 by taking exterior powers and twisting by $\mathcal{O}_C(-3e)$. In any case, the sheaf on the left is $\mathcal{O}_C(-2)$ by what was said above. Choose an isomorphism $H^1(C, \mathcal{O}_C(-2)) = \mathbb{C}$, and let

$$\delta : H^0(C, \bigwedge^2 N_{C/X}) \to H^1(C, \bigwedge^3 N_{C/X} \otimes \mathcal{O}_C(-3)) = H^1(C, \mathcal{O}(-2)) = \mathbb{C},$$

be the boundary map on cohomology coming from the exact sequence above. This is another procedure which associates to any two sections of $N_{C/X}$ a complex number. In the following theorem we prove that the two procedures agree. The best argument for this is the usual one: What else could it be? The actual proof is even more annoying.

Theorem 5.1. *Up to a nonzero scalar factor the pairing associated to ω_e on $T_{[f]}(\mathcal{M}_e) = H^0\left(C, N_{C/X}\right)$ is equal to the pairing $(s_1, s_2) \mapsto \delta(s_1 \wedge s_2)$.*

Proof. Observe that the construction of Section 4 is compatible with arbitrary base change of the stack $\mathcal{M}$. To prove the theorem, perform a base change to the Artin local ring $Z = \operatorname{Spec} A$ which is the base of the universal first order deformation of $C \subset X$, say $\mathcal{C} \subset Z \times X$. The construction of Section 4 restricts the exact sequence from Equation 1 to $\mathcal{C}$ and then pushes-out the sequence by the map

$$f^*\left(\Omega^3_X\right) \to \Omega^3_{\mathcal{C}} \to p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}.$$

Then the construction takes the cohomology of the resulting sequence to obtain the 2-form ω_e. By a diagram chase, the resulting sequence is simply the "Serre dual" of the sequence from Equation 3 from which the theorem follows.

First consider the universal first order deformation of $C \subset X$. By Serre duality the vector space $V = H^1\left(C, I/I^2 \otimes \omega_C\right)$ is dual to $H^0\left(C, N_{C/X}\right)$. Here I is the ideal sheaf of C in X. Consider the local Artin k-algebra, $A = k \oplus V$, where $V \subset A$ is an ideal of square zero. Set $Z = \operatorname{Spec} A$. Denote by $\mathcal{C} \to Z$ the universal first order deformation of C. Let $s_1, \dots, s_A$ be an ordered basis for $H^0\left(C, N_{C/X}\right)$ and let $t_1, \dots, t_A$ in V be the dual ordered basis. The elements $s_1, \dots, s_A$ are canonically identified with $\mathcal{O}_C$-linear maps $I/I^2 \to \mathcal{O}_C$. Let $p \in C$ be a point, let $U \subset X$ be an open affine subset containing p, and let g_1, g_2, g_3 be generators for $H^0(U, I)$ as an $H^0(U, \mathcal{O}_X)$-module. Then the ideal of $\mathcal{C}$ is locally generated by the equations

$$\widetilde{g}_j := g_j + \sum_{i=1}^{A} t_i \cdot s_i(f_j), \quad j = 1, 2, 3$$
$$\widetilde{g}_j \in \mathcal{O}_X[t_1, \dots, t_A]/\langle t_i t_{i'}, t_i g_j, g_j g_{j'} | i, i' = 1, \dots, A, j, j' = 1, 2, 3\rangle.$$

Denote by $p\colon \mathcal{C} \to Z$ and $\widetilde{f}\colon \mathcal{C} \to X$ the two projections.

To prove the theorem, we compute the 2-form on Z obtained from the construction of Section 4 applied to $(p\colon \mathcal{C} \to Z, \widetilde{f}\colon \mathcal{C} \to X)$. This is not as crazy as it sounds; namely $\Omega^2_{A/k} \otimes_A k = \wedge^2 V$, so this computation will provide the necessary information.

To compute $\widetilde{f}^*\eta$, form the pullback by $\widetilde{f}^*$ of the exact sequence from Equation 1. Considered as an element of the Yoneda-Ext group $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \Omega^3_{\mathcal{C}})$, the element $\widetilde{f}^*\eta$ is simply the push-out of this exact sequence by the canonical map $\widetilde{f}^*\left(\Omega^3_X\right) \to \Omega^3_{\mathcal{C}}$. According to Section 4, take the image of $f^*\eta$ under the map

$$\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \Omega^3_{\mathcal{C}}) \to \operatorname{Ext}^1_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z})$$

In terms of Yoneda-Ext, take an additional push-out of the exact sequence by $\Omega^3_{\mathcal{C}} \to p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}$. So, in terms of Yoneda-Ext, the exact sequence is obtained as the push-out of the pullback of Equation 1 by the map $\widetilde{f}^*\Omega^3_X \to p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}$.

Of course it is only necessary to compute the restriction of this exact sequence to the closed fiber, so restrict the push-out exact sequence to the closed fiber. In particular, the restriction to the closed fiber of $p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}$ is $\bigwedge^2 V \otimes_k \Omega^1_C$. The next step is an explicit local description of the map

$$\psi\colon \Omega^3_X|_C \to \bigwedge^2 V \otimes_k \Omega^1_C.$$

Let t be a regular function on X restricting to a local coordinate on C. Any local 3-form on X is an $\mathcal{O}_X$-linear combination of the forms $\epsilon_{jj'} = \mathrm{d}f_j \wedge \mathrm{d}f_{j'} \wedge \mathrm{d}t, 1 \leq j < j' \leq 3$ and the form $\mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3$. So it suffices to evaluate ψ on these 3-forms. The result is

$$\psi(\eta_{jj'}) = \sum_{i,i'=1}^{A} s_i(f_j) s_{i'}(f_{j'}) t_i \wedge t_{i'} \otimes dt, \ 1 \leq j < j' \leq 3, \tag{4}$$

$$\psi(df_1 \wedge df_2 \wedge df_3) = 0. \tag{5}$$

Of course there is a more "global" way of thinking about ψ. The exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_X|_C \longrightarrow \Omega^1_C \longrightarrow 0, \tag{6}$$

determines a canonical map $\alpha : \Omega^3_X|_C \to \bigwedge^2 I/I^2 \otimes_{\mathcal{O}_C} \Omega^1_C$. And there is a map of $\mathcal{O}_C$-modules $\beta : I/I^2 \to V \otimes_k \mathcal{O}_C$ defined as the transpose of the map $H^0\left(C, N_{C/X}\right) \otimes_k \mathcal{O}_C \to N_{C/X}$. The global description of ψ is as the composition of α with $\bigwedge^2 \beta \otimes \mathrm{Id}_{\Omega^1_C}$.

Just as the exact sequence in Equation 6 induces the map α, also the exact sequence

$$0 \longrightarrow \tilde{I}/\tilde{I}^2 \longrightarrow \Omega^1_{\mathbb{P}^5}|_C \longrightarrow \Omega^1_C \longrightarrow 0. \tag{7}$$

induces a map $\alpha' : \Omega^4_{\mathbb{P}^5}|_C \to \bigwedge^3 \widetilde{I}/\widetilde{I}^2 \otimes \Omega^1_C$ where $\widetilde{I}$ is the ideal sheaf of C in $\mathbb{P}^5$. By adjunction, there are isomorphisms $\Omega^5_{\mathbb{P}^5}|_C \otimes \mathcal{O}_C(3e) \cong \Omega^4_X|_C$ and $\Omega^4_X|_C \cong \bigwedge^3 I/I^2 \otimes \Omega^1_C$. Combining these adjunction isomorphisms gives an isomorphism,

$$\alpha'' \colon \Omega^5_{\mathbb{P}^5}|_C \otimes \mathcal{O}_C(6e) \to \bigwedge^3 I/I^2 \otimes \mathcal{O}_C(3e) \otimes \Omega^1_C.$$

Both terms in this map are isomorphic to $\mathcal{O}_C$. Choosing such isomorphisms, α'' is just an isomorphism of $\mathcal{O}_C$ to itself.

We leave it to the reader to verify that the following diagram commutes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^3_X|_C & \longrightarrow & \Omega^4_{\mathbb{P}^5}|_C(3e) & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha'} & & \Big\downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & \bigwedge^2 I/I^2 \otimes \Omega^1_C & \longrightarrow & \bigwedge^3 \tilde{I}/\tilde{I}^2(3e) \otimes \Omega^1_C & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0. \end{array} \tag{8}$$

The top exact sequence is the restriction to C of Equation 1, and the bottom exact sequence is the dual of Equation 3 tensored with Ω^1_C. More canonically, the last term in the top sequence is $\Omega^5_{\mathbb{P}^5}|_C(6e)$ and the last term in the bottom sequence is $\bigwedge^3 I/I^2(3e) \otimes \Omega^1_C$. The diagram follows using the isomorphisms of these sheaves with $\mathcal{O}_C$ from the last paragraph.

The conclusion is that the extension of $\mathcal{O}_C$ by $\wedge^2 V \otimes_k \Omega^1_C$ obtained from $\widetilde{f}^*\eta$ is precisely the Serre dual exact sequence of Equation 3 used to define the coboundary map δ. Hence the coboundary map on cohomology $H^0(C, \mathcal{O}_C) \to H^1(C, \bigwedge^2 I/I^2 \otimes \Omega^1_C)$ is the dual of δ. $\square$

6. Proof of Theorem 1.2: degree five case

The strategy of the proof of Theorem 1.2 is the following. Form the $\mathbb{P}^{55}$ parametrizing all cubic hypersurfaces in $\mathbb{P}^5$. Let $U_e \to \mathbb{P}^{55}$ be the Deligne-Mumford stack over $\mathbb{P}^{55}$ parametrizing pairs $([X], [C])$ of a cubic hypersurface $X \subset \mathbb{P}^5$ and a smooth rational curve $C \subset X$ of degree e such that X is smooth along C and such that $H^1(C, N_{C/X})$ is zero (i.e. $C \subset X$ is *unobstructed*). The last condition guarantees that $U_e \to \mathbb{P}^{55}$ is a smooth morphism. Also, by Proposition 2.4, a general fiber of $U_e \to \mathbb{P}^{55}$ is irreducible. In particular, U_e is also irreducible.

There is a straightforward generalization of the construction of Section 4 to the relative setting. This produces (locally over $\mathbb{P}^{55}$) a 2-form ω_e that is a global section of $\Omega^2_{U_e/\mathbb{P}^{55}}$ such that the restriction of ω_e to any fiber is the 2-form of the fiber constructed in Section 4. The rank of ω_e on fibers is a lower semicontinuous function on U_e, so to prove that the rank of ω_e is the maximum possible for a general pair $([X], [C])$, it suffices to find a single pair $([X], [C]) \in U_e$ where the rank of ω_e is the maximum possible.

Let

$$0 \longrightarrow \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(a_3) \longrightarrow N_{C/\mathbb{P}^5} \longrightarrow \mathcal{O}(3e) \longrightarrow 0$$

be the usual exact sequence, where $a_1 + a_2 + a_3 = 3e - 2$. In other words $N_{C/X} = \bigoplus \mathcal{O}(a_i)$. The extension class of this sequence is an element ψ of $H^1(\mathbb{P}^1, \mathcal{O}(a_1 - 3e) \oplus \mathcal{O}(a_2 - 3e) \oplus \mathcal{O}(a_3 - 3e))$. Write $\mathbb{P}^1 = \mathrm{Proj}\,(S)$, where $S = \mathbb{C}[X_0, X_1]$. Then, using Serre duality, ψ equals $\psi_1 \oplus \psi_2 \oplus \psi_3$ for $\psi_i \in \mathrm{Hom}\,(S_{3e-a_i-2}, \mathbb{C})$. Writing elements of $H^0(C, N_{C/X})$ in the form (g_1, g_2, g_3) for $g_i \in H^0(C, \mathcal{O}(a_i))$, then the pairing takes the following form

$$\left\langle \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\rangle = \psi_3(g_1h_2 - g_2h_1) + \psi_2(g_1h_3 - g_3h_1) + \psi_1(g_2h_3 - g_3h_2).$$

To compute the pairing for a given curve, we have to find the linear functionals ψ_1, ψ_2, ψ_3 above. For large e this reduces to a rather involved computation. We will present this computation later, but first we show that in the special case $e = 5$ there is a short solution (which will hopefully motivate the reader to brave the computations of the next two sections).

Theorem 6.1. *Let $f : C \to X$ be a general quintic rational curve on a general cubic fourfold X. Then $N_{C/X} = \mathcal{O}(4) \oplus \mathcal{O}(4) \oplus \mathcal{O}(5)$ and the extension class ψ of the sequence $0 \to N_{C/X} \to N_{C/\mathbb{P}^5} \to \mathcal{O}(15) \to 0$ is a general point of the space* $\mathrm{Hom}\,(S_9 \oplus S_9 \oplus S_8, \mathbb{C})$.

Proof. Fix a rational normal curve $C \subset \mathbb{P}^5$ of degree 5. Its normal bundle $N_{C/\mathbb{P}^5}$ is $\mathcal{O}(7)^{\oplus 4}$. Thus any (not necessarily smooth) cubic fourfold X containing C determines a homomorphism of $\mathcal{O}_C$-modules

$$\varphi_X : \mathcal{O}(7)^{\oplus 4} \to \mathcal{O}(15).$$

Note that $\varphi_X = 0$ if and only if X is singular along C, which happens if and only if the defining equation of X is a section of $I^2(3)$. The following computations are left to the reader,

$$\dim H^0(\mathbb{P}^5, I(3)) = 40, \quad \dim H^0(\mathbb{P}^5, I^2(3)) = 4, \quad \dim \mathrm{Hom}_C(\mathcal{O}(7)^4, \mathcal{O}(15)) = 36.$$

Thus the rule $X \mapsto \varphi_X$ is onto. Hence a general exact sequence of the form $0 \to \mathrm{Ker}\,(\alpha) \to \mathcal{O}(7)^4 - \alpha \to \mathcal{O}(15) \to 0$ occurs as the normal bundle sequence for a general (nonsingular) X. The theorem follows. $\square$

To finish, choose ψ_i as follows,

$$\psi_1(\sum_{i=0}^{9} a_i X_0^{9-i} X_1^i) = \sum_{i=0}^{9} \nu_i a_i, \quad \psi_1(\sum_{i=0}^{9} a_i X_0^{9-i} X_1^i) = \sum_{i=0}^{9} \mu_i a_i,$$

and

$$\psi_1(\sum_{i=0}^{8} a_i X_0^{8-i} X_1^i) = \sum_{i=0}^{8} \lambda_i a_i.$$

Choose ν_i, μ_i and λ_i general. Form the matrix of the pairing with respect to the obvious basis of $H^0(\mathbb{P}^1, \mathcal{O}(4) \oplus \mathcal{O}(4) \oplus \mathcal{O}(5))$. The computation gives,

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \mu_0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\
0 & 0 & 0 & 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 \\
0 & 0 & 0 & 0 & 0 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 \\
0 & 0 & 0 & 0 & 0 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
-\lambda_0 & -\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 & \nu_0 & \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 \\
-\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 & -\lambda_5 & 0 & 0 & 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 \\
-\lambda_2 & -\lambda_3 & -\lambda_4 & -\lambda_5 & -\lambda_6 & 0 & 0 & 0 & 0 & 0 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 \\
-\lambda_3 & -\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & 0 & 0 & 0 & 0 & 0 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \nu_8 \\
-\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & -\lambda_8 & 0 & 0 & 0 & 0 & 0 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \nu_8 & \nu_9 \\
-\mu_0 & -\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 & -\nu_0 & -\nu_1 & -\nu_2 & -\nu_3 & -\nu_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 & -\mu_5 & -\nu_1 & -\nu_2 & -\nu_3 & -\nu_4 & -\nu_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_2 & -\mu_3 & -\mu_4 & -\mu_5 & -\mu_6 & -\nu_2 & -\nu_3 & -\nu_4 & -\nu_5 & -\nu_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_3 & -\mu_4 & -\mu_5 & -\mu_6 & -\mu_7 & -\nu_3 & -\nu_4 & -\nu_5 & -\nu_6 & -\nu_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_4 & -\mu_5 & -\mu_6 & -\mu_7 & -\mu_8 & -\nu_4 & -\nu_5 & -\nu_6 & -\nu_7 & -\nu_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_5 & -\mu_6 & -\mu_7 & -\mu_8 & -\mu_9 & -\nu_5 & -\nu_6 & -\nu_7 & -\nu_8 & -\nu_9 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}.$$

Finally, to complete the proof of Theorem 1.2 for $e = 5$, consider the determinant of this matrix. For $\lambda_0 = 1, \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 1, \lambda_5 = 1, \lambda_6 = -1, \lambda_7 = -4, \lambda_8 = 2, \mu_0 = 1, \mu_1 = 2, \mu_2 = -1, \mu_3 = 2, \mu_4 = 5, \mu_5 = -1, \mu_6 = 13, \mu_7 = -1, \mu_8 = 1, \mu_9 = 1, \nu_0 = 1, \nu_1 = 2, \nu_2 = 3, \nu_3 = 5, \nu_4 = 4, \nu_5 = -5, \nu_6 = -6, \nu_7 = -7, \nu_8 = -5, \nu_9 = 1$ the determinant equals 445717799641. Since this is nonzero, Theorem 1.2 is true for $e = 5$.

7. Proof of Theorem 1.2

By Section 6, to prove Theorem 1.2 it suffices to determine a certain extension class ψ. The proof for $e = 5$ was short because ψ can be chosen to be a general element of the Ext group. Comparing the dimension of the parameter space U_e of pairs $([X], [C])$ (cf. Section 6) and the dimension of the relevant Ext group, the Ext group grows more quickly. So, for large e, the extension class ψ will *not* be a general element of the Ext group.

Instead we work with a specific pair $([X], [C]) \in U_e$ for which we can prove the rank of ω_e is maximal and $h^1(C, N_{C/X}) = 0$. The reader is warned that for this pair, X is not smooth! But X is smooth on an open set containing C, and this is all that matters.

The proof of Theorem 1.2 in the case that e is odd is almost identical to the proof in the case that e is even. For this reason, most of the argument is carried out for both cases simultaneously. For each construction, the even case is specified by a subscript "ϵ" and the odd case is specified by a subscript "o". Arguments that apply verbatim to both cases will not have a subscript (i.e., if there is no subscript, a true statement is obtained by either applying the subscript "o" throughout, or by applying the subscript "ϵ" throughout). In the odd case, the degree is $e_o = 2r_o + 1$ for some integer $r_o \geq 2$. In the even case, the degree is $e_\epsilon = 2r_\epsilon$ for some integer $r_\epsilon \geq 3$.

7.1. **Computation of $N_{C/\mathbb{P}^5}$.** We begin by specifying C and computing $N_{C/\mathbb{P}^5}$. As in the last section, choose homogeneous coordinates X_0, X_1 on $\mathbb{P}^1$. Choose homogeneous coordinates $Y_0, Y_1, Y_2, Y_3, Y_4, Y_5$ on $\mathbb{P}^5$. Consider the maps $f_o : \mathbb{P}^1 \to \mathbb{P}^5$, resp. $f_\epsilon : \mathbb{P}^1 \to \mathbb{P}^5$ given by

$$f_o([X_0 : X_1]) = [X_0^{2r_o+1} : X_0^{2r_o}X_1 : X_0^{r_o+1}X_1^{r_o} : X_0^{r_o}X_1^{r_o+1} : X_0X_1^{2r_o} : X_1^{2r_o+1}],$$

resp.

$$f_\epsilon([X_0 : X_1]) = [X_0^{2r_\epsilon} : X_0^{2r_\epsilon-1}X_1 : X_0^{r_\epsilon+1}X_1^{r_\epsilon-1} : X_0^{r_\epsilon-1}X_1^{r_\epsilon+1} : X_0X_1^{2r_\epsilon-1} : X_1^{2r_\epsilon}].$$

This is a closed immersion, and local inverses are given by $[Y_0 : \cdots : Y_5] \mapsto [Y_0 : Y_1]$ and $[Y_0 : \cdots : Y_5] \mapsto [Y_4 : Y_5]$ on $\mathbb{P}^5 - \mathbb{V}(Y_0, Y_1)$ and $\mathbb{P}^5 - \mathbb{V}(Y_4, Y_5)$ respectively (the image of C does not intersect $\mathbb{V}(Y_0, Y_1, Y_4, Y_5)$). To compute the normal bundle of C in $\mathbb{P}^5$, we use the Euler sequence for $T_{\mathbb{P}^1}$ and for $T_{\mathbb{P}^5}$. There is a map between these Euler sequences induced by f_o, resp. f_ϵ, and the important term is

$$\widetilde{df_o} : \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to f_o^*\left(\mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6}\right) = \mathcal{O}_{\mathbb{P}^1}(2r_o+1)^{\oplus 6},$$

resp.

$$\widetilde{df_\epsilon} : \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to f_\epsilon^*\left(\mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6}\right) = \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon)^{\oplus 6}.$$

These maps are given by the matrices

$$\widetilde{\mathrm{d}f_o} = \begin{bmatrix} (2r_o+1)X_0^{2r_o} & 0 \\ 2r_oX_0^{2r_o-1}X_1 & X_0^{2r_o} \\ (r_o+1)X_0^{r_o}X_1^{r_o} & r_oX_0^{r_o+1}X_1^{r_o-1} \\ r_oX_0^{r_o-1}X_1^{r_o+1} & (r_o+1)X_0^{r_o}X_1^{r_o} \\ X_1^{2r_o} & 2r_oX_0X_1^{2r_o-1} \\ 0 & (2r_o+1)X_1^{2r_o} \end{bmatrix},$$

resp.

$$\widetilde{\mathrm{d}f_\epsilon} = \begin{bmatrix} 2r_\epsilon X_0^{2r_\epsilon-1} & 0 \\ (2r_\epsilon-1)X_0^{2r_\epsilon-2}X_1 & X_0^{2r_\epsilon-1} \\ (r_\epsilon+1)X_0^{r_\epsilon}X_1^{r_\epsilon-1} & (r_\epsilon-1)X_0^{r_\epsilon+1}X_1^{r_\epsilon-2} \\ (r_\epsilon-1)X_0^{r_\epsilon-2}X_1^{r_\epsilon+1} & (r_\epsilon+1)X_0^{r_\epsilon-1}X_1^{r_\epsilon} \\ X_1^{2r_\epsilon-1} & (2r_\epsilon-1)X_0X_1^{2r_\epsilon-2} \\ 0 & 2r_\epsilon X_1^{2r_\epsilon-1} \end{bmatrix}.$$

Observe that both matrices have rank 2 at every point of $\mathbb{P}^1$. The normal bundle of C in $\mathbb{P}^5$ is the cokernel of $\widetilde{df_o}$, resp. $\widetilde{df_\epsilon}$. To compute this, consider the sheaf morphism $T_o : \mathcal{O}_{\mathbb{P}^1}(2r_o+1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(3r_o+1)^{\oplus 4}$, resp. $T_\epsilon : \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon - 1)$ given by the matrices

$$T_o = \left[\begin{array}{ccc|} (r_o-1)X_1^{r_o} & -r_oX_0X_1^{r_o-1} & X_0^{r_o} \\ 0 & X_1^{r_o} & -r_oX_0^{r_o-1}X_1 \\ 0 & 0 & (r_o-1)X_1^{r_o} \\ 0 & 0 & 0 \end{array}\right. \cdots$$

$$\left|\begin{array}{ccc} 0 & 0 & 0 \\ (r_o-1)X_0^{r_o} & 0 & 0 \\ -r_oX_0X_1^{r_o-1} & X_0^{r_o} & 0 \\ X_1^{r_o} & -r_oX_0^{r_o-1}X_1 & (r_o-1)X_0^{r_o} \end{array}\right],$$

resp.

$$T_\epsilon = \left[\begin{array}{ccc|} (r_\epsilon - 2)X_1^{r_\epsilon - 1} & -(r_\epsilon - 1)X_0 X_1^{r_\epsilon - 1} & X_0^{r_\epsilon - 1} \\ 0 & 2X_1^{r_\epsilon} & -r_\epsilon X_0^{r_\epsilon - 2} X_1^2 \\ 0 & 0 & (r_\epsilon - 2)X)1^{r_\epsilon} \\ 0 & 0 & 0 \end{array} \right. \cdots$$

$$\left| \begin{array}{ccc} 0 & 0 & 0 \\ (r_\epsilon - 2)X_0^{r_\epsilon} & 0 & 0 \\ -r_\epsilon X_0^2 X_1^{r_\epsilon - 2} & 2X_0^{r_\epsilon} & 0 \\ X_1^{r_\epsilon - 1} & -(r_\epsilon - 1)X_0^{r_\epsilon - 2} X_1 & (r_\epsilon - 2)X_0^{r_\epsilon - 1} \end{array} \right].$$

It is straightforward to verify that $T_o \circ \widetilde{\mathrm{d}f_o}$ is zero, resp. $T_\epsilon \circ \widetilde{\mathrm{d}f_\epsilon}$ is zero. And T_o, resp. T_ϵ, has rank 4 everywhere. Thus T_o, resp T_ϵ, gives an isomorphism of $N_{C/\mathbb{P}^5}$ with $\mathcal{O}_{\mathbb{P}^1}(3r_o+1)^{\oplus 4}$, resp. $\mathcal{O}_{\mathbb{P}^1}(3r_\epsilon - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon - 1)$. Each of these isomorphisms is taken to be an identification of locally free sheaves.

7.2. **Computation of $N_{C/X}$.** Next we specify X and compute the normal bundle $N_{C/X}$. Observe that the quadric equations $Q_a = Y_1Y_4 - Y_0Y_5$ and $Q_b = Y_2Y_3 - Y_0Y_5$ both vanish on the image of f_o, resp. f_ϵ. Let L_a and L_b be any linear homogeneous polynomials in $Y_0, \dots, Y_5$ which are linearly independent and consider the homogeneous cubic polynomial $F = L_aQ_a + L_bQ_b$ (later we will specialize to the case that L_a and L_b are general linear homogeneous polynomials in Y_0 and Y_5 alone). For our purposes it is convenient to make a "change of variables" and define $M = L_a + L_b$ and $N_o = L_a + r_oL_b$, resp. $N_\epsilon = L_a + (r_\epsilon - 1)L_b$ (here we are using that $r_o \neq 1$, resp. $r_\epsilon \neq 2$, to see that L_a and L_b are uniquely determined by M and N_o, resp. N_ϵ). Consider $X = \{[Y_0 : \dots : Y_5] \in \mathbb{P}^5 | F(Y_0, \dots, Y_5) = 0\}$. Observe that X is singular along the common zero locus of L_a, L_b, Q_a and Q_b – which will typically be a geometrically connected degree 4 curve of arithmetic genus 1.

To determine whether X is smooth along the image of f_o, resp. f_ϵ, we need to compute the pullback of the "gradient vector" $[\frac{\partial F}{\partial Y_i}]_{i=0,\dots,5}$. Define $\widetilde{L}_a = f^*L_a, \widetilde{L_b} = f^*L_b, \widetilde{M} = f^*M$ and $\widetilde{N} = f^*N$, considered as sections of of $H^0(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^5}(1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(5))$. The pullback of the gradient vector of F is the sheaf morphism $U_o : \mathcal{O}_{\mathbb{P}^1}(2r_o+1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(6r_o+3)$, resp. $U_\epsilon : \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(6r_\epsilon)$ given by

$$U_o = \left[\begin{array}{ccc|} -X_1^{2r_o+1}(\widetilde{L}_a + \widetilde{L}_b) & X_0X_1^{2r_o}\widetilde{L}_a & X_0^{r_o}X_1^{r_o+1}\widetilde{L}_b \end{array} \right. \cdots$$
$$\left| \begin{array}{ccc} X_0^{r_o+1}X_1^{r_o}\widetilde{L}_b & X_0^{2r_o}X_1\widetilde{L}_a & -X_0^{2r_o+1}(\widetilde{L}_a + \widetilde{L}_b) \end{array} \right],$$

resp.

$$U_\epsilon = \left[\begin{array}{ccc|} -X_1^{2r_\epsilon}(\widetilde{L}_a + \widetilde{L}_b) & X_0X_1^{2r_\epsilon - 1}\widetilde{L}_a & X_0^{r_\epsilon - 1}X_1^{r_\epsilon+1}\widetilde{L}_b \end{array} \right. \cdots$$
$$\left| \begin{array}{ccc} X_0^{r_\epsilon+1}X_1^{r_\epsilon - 1}\widetilde{L}_b & X_0^{2r_\epsilon - 1}X_1\widetilde{L}_a & -X_0^{2r_\epsilon}(\widetilde{L}_a + \widetilde{L}_b) \end{array} \right].$$

If $\widetilde{L}_a$ and $\widetilde{L}_b$ have no common zeroes and if $\widetilde{L}_a + \widetilde{L}_b$ is nonzero at the points $[1:0]$ and $[0:1]$, then these matrices are everywhere nonzero, i.e., X is smooth along C. From now on, assume this is the case. The matrix U factors as $U = S \circ T$ where $S_o : N_{C/\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^1}(6r_o+3)$, resp. $S_\epsilon : N_{C/\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^1}(6r_\epsilon)$, are given by the matrices

$$S_o = \frac{-1}{r_o - 1} \left[\begin{array}{cccc} X_1^{r_o+1}\widetilde{M} & X_0X_1^{r_o}\widetilde{N}_o & X_0^{r_o}X_1\widetilde{N}_o & X_0^{r_o+1}\widetilde{M} \end{array} \right],$$

resp.

$$S_\epsilon = \frac{-1}{2(r_\epsilon - 2)} \left[\ 2X_1^{r_\epsilon+1}\widetilde{M} \quad X_0X_1^{r_\epsilon-1}\widetilde{N}_\epsilon \quad X_0^{r_\epsilon-1}X_1\widetilde{N}_\epsilon \quad 2X_0^{r_\epsilon+1}\widetilde{M}\ \right].$$

The normal bundle $N_{C/X}$ is the kernel of the sheaf morphism S. To describe this map, we write out

$$\begin{cases} M_o = c_{0,o}Y_0 + c_{1,o}Y_1 + c_{2,o}Y_2 + c_{3,o}Y_3 + c_{4,o}Y_4 + c_{5,o}Y_5, \\ N_o = d_{0,o}Y_0 + d_{1,o}Y_1 + d_{2,o}Y_2 + d_{3,o}Y_3 + d_{4,o}Y_4 + d_{5,o}Y_5 \end{cases},$$

resp.

$$\begin{cases} M_\epsilon = c_{0,\epsilon}Y_0 + c_{1,\epsilon}Y_1 + c_{2,\epsilon}Y_2 + c_{3,\epsilon}Y_3 + c_{4,\epsilon}Y_4 + c_{5,\epsilon}Y_5, \\ N_\epsilon = d_{0,\epsilon}Y_0 + d_{1,\epsilon}Y_1 + d_{2,\epsilon}Y_2 + d_{3,\epsilon}Y_3 + d_{4,\epsilon}Y_4 + d_{5,\epsilon}Y_5 \end{cases}$$

Then we have

$$\begin{cases} \widetilde{M}_o = c_{0,o}X_0^{2r_o+1} + c_{1,o}X_0^{2r_o}X_1 + c_{2,o}X_0^{r_o+1}X_1^{r_o} + c_{3,o}X_0^{r_o}X_1^{r_o+1} + c_{4,o}X_0X_1^{2r_o} + c_{5,o}X_1^{2r_o+1}, \\ \widetilde{N}_o = d_{0,o}X_0^{2r_o+1} + d_{1,o}X_0^{2r_o}X_1 + d_{2,o}X_0^{r_o+1}X_1^{r_o} + d_{3,o}X_0^{r_o}X_1^{r_o+1} + d_{4,o}X_0X_1^{2r_o} + d_{5,o}X_1^{2r_o+1} \end{cases},$$

resp.

$$\begin{cases} \widetilde{M}_\epsilon = c_{0,\epsilon}X_0^{2r_\epsilon} + c_{1,\epsilon}X_0^{2r_\epsilon-1}X_1 + c_{2,\epsilon}X_0^{r_\epsilon+1}X_1^{r_\epsilon-1} + c_{3,\epsilon}X_0^{r_\epsilon-1}X_1^{r_\epsilon+1} + c_{4,\epsilon}X_0X_1^{2r_\epsilon-1} + c_{5,\epsilon}X_1^{2r_\epsilon}, \\ \widetilde{N}_\epsilon = d_{0,\epsilon}X_0^{2r_\epsilon} + d_{1,\epsilon}X_0^{2r_\epsilon-1}X_1 + d_{2,\epsilon}X_0^{r_\epsilon+1}X_1^{r_\epsilon-2} + d_{3,\epsilon}X_0^{r_\epsilon-1}X_1^{r_\epsilon+1} + d_{4,\epsilon}X_0X_1^{2r_\epsilon-1} + d_{5,\epsilon}X_1^{2r_\epsilon} \end{cases}$$

Denote by n_o, n'_o, m_o, m'_o the following expressions,

$$\begin{cases} n_o = d_{4,o}X_0^2X_1^{r_o} + d_{5,o}X_0X_1^{r_o+1}, \\ n'_o = d_{0,o}X_0^{r_o+1}X_1 + d_{1,o}X_0^{r_o}X_1^2 + d_{2,o}X_0X_1^{r_o+1} + d_{3,o}X_1^{r_o+2}, \\ m_o = c_{4,o}X_0X_1^{r_o+1} + c_{5,o}X_1^{r_o+2}, \\ m'_o = c_{0,o}X_0^{r_o+2} + c_{1,o}X_0^{r_o+1}X_1 + c_{2,o}X_0^2X_1^{r_o} + c_{3,o}X_0X_1^{r_o+1} \end{cases}$$

Denote by $n_\epsilon, n'_\epsilon, m_\epsilon, m'_\epsilon$ the following expressions,

$$\begin{cases} n_\epsilon = d_{3,\epsilon}X_0X_1^{r_\epsilon} + d_{4,\epsilon}X_0^2X_1^{r_\epsilon-1} + d_{5,\epsilon}X_0X_1^{r_\epsilon}, \\ n'_\epsilon = d_{0,\epsilon}X_0^{r_\epsilon}X_1 + d_{1,\epsilon}X_0^{r_\epsilon-1}X_1^2 + d_{2,\epsilon}X_0X_1^{r_\epsilon}, \\ m_\epsilon = 2c_{4,\epsilon}X_0X_1^{r_\epsilon+1} + 2c_{5,\epsilon}X_1^{r_\epsilon+2}, \\ m'_\epsilon = 2c_{0,\epsilon}X_0^{r_\epsilon+2} + 2c_{1,\epsilon}X_0^{r_\epsilon+1}X_1 + 2c_{2,\epsilon}X_0^3X_1^{r_\epsilon-1} + 2c_{3,\epsilon}X_0X_1^{r_\epsilon+1} \end{cases}$$

Then $X_0X_1\widetilde{N}_o = X_1^{r_o+1}n_o + X_0^{r_o+1}n'_o$ and $\widetilde{M}_o = X_1^{r_o-1}m_o + X_0^{r_o-1}m'_o$, resp. $X_0X_1\widetilde{N}_\epsilon = X_1^{r_\epsilon+1}n_\epsilon + X_0^{r_\epsilon+1}n'_\epsilon$ and $2\widetilde{M}_\epsilon = X_1^{r_\epsilon-2}m_\epsilon + X_0^{r_\epsilon-2}m'_\epsilon$.

Consider the sheaf morphism $R_o : \mathcal{O}_{\mathbb{P}^1}(2r_o) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_o+2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_o-1) \to N_{C/\mathbb{P}^5}$, resp. $R_\epsilon : \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon+2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2) \to N_{C/\mathbb{P}^5}$ given by the matrices,

$$R_o = \begin{bmatrix} X_0^{r_o+1} & 0 & n_o \\ 0 & X_0^{r_o-1} & -m_o \\ 0 & -X_1^{r_o-1} & -m'_o \\ -X_1^{r_o+1} & 0 & n'_o \end{bmatrix},$$

resp.

$$R_\epsilon = \begin{bmatrix} X_0^{r_\epsilon+1} & 0 & n_\epsilon \\ 0 & X_0^{r_\epsilon-2} & -m_\epsilon \\ 0 & -X_1^{r_\epsilon-2} & -m'_\epsilon \\ -X_1^{r_\epsilon+1} & 0 & n'_\epsilon \end{bmatrix}$$

The composition $S \circ R$ is zero. The matrix R has rank 3 generically (in particular it has rank 3 at $[0:1]$ and $[1:0]$ by the hypothesis that $\widetilde{M}$ is nonzero at those points). By degree considerations, R has rank 3 everywhere and gives an isomorphism of

$\mathcal{O}_{\mathbb{P}^1}(2r_o) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_o+2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_o-1)$, resp. $\mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2)$ with the kernel of S, i.e. with $N_{C/X}$. In particular, $h^1(\mathbb{P}^1, N_{C/X}) = 0$, so $([X],[C])$ is a point of U_e.

7.3. **Initial description of the pairing.** In this subsection we begin the description of the skew-symmetric bilinear pairing on $H^0(C, N_{C/X})$ induced by ω_e. We complete the description in the next subsection. Elements in $H^0(\mathbb{P}^1, N_{C/X})$ are denoted by (g_1, g_2, g_3) or $g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3$ where $\mathbf{e}_i$ is the ith column of the matrix R and where $g_1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_o)), g_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_o+2))$ and $g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_o-1))$, resp. $g_1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2)), g_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon+2))$ and $g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2))$.

By Theorem 5.1, to compute the bilinear pairing ω_e on $H^0(\mathbb{P}^1, N_{C/X})$ it is equivalent (up to a nonzero scalar) to compute the boundary map

$$\delta : H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)).$$

The next term in the long exact sequence of cohomology is $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r_o))^{\oplus 4}$, resp. $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-1))^{\oplus 2} \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-2))^{\oplus 2}$, both of which are zero. Therefore the connecting homomorphism is the cokernel of the map on global sections

$$R_o^\dagger : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r_o))^{\oplus 4} \to H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X})$$

resp.

$$\begin{aligned} R_\epsilon^\dagger : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-2) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-2)) \oplus \mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-1)) \\ \longrightarrow H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X}), \end{aligned}$$

determined by the sheaf morphism $R^\dagger : N^\vee_{C/\mathbb{P}^5} \otimes \bigwedge^3 N_{C/X} \to \bigwedge^2 N_{C/X}$ that is adjoint to R. (The adjoint $R^\dagger = \mathrm{diag}(1,-1,1) \circ R^t$, where R^t is the transpose of R.) If we use as "ordered basis" for $\bigwedge^2 N_{C/X}$ the elements $\mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_1 \wedge \mathbf{e}_3$ and $\mathbf{e}_1 \wedge \mathbf{e}_2$, then the matrix of $R^\dagger$ is

$$R_o^\dagger = \begin{bmatrix} X_0^{r_o+1} & 0 & 0 & -X_1^{r_o+1} \\ 0 & -X_0^{r_o-1} & X_1^{r_o-1} & 0 \\ n_o & -m_o & -m_o' & n_o' \end{bmatrix},$$

resp.

$$R_\epsilon^\dagger = \begin{bmatrix} X_0^{r_\epsilon+1} & 0 & 0 & -X_1^{r_\epsilon+1} \\ 0 & -X_0^{r_\epsilon-2} & X_1^{r_\epsilon-2} & 0 \\ n_\epsilon & -m_\epsilon & -m_\epsilon' & n_\epsilon' \end{bmatrix}.$$

In other words, the pairing ω_e is given by

$$\begin{aligned} [(g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3), (h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3)] = (g_1h_2 - g_2h_1)\mathbf{e}_1 \wedge \mathbf{e}_2 \\ + (g_1h_3 - g_3h_1)\mathbf{e}_1 \wedge \mathbf{e}_3 + (g_2h_3 - g_3h_2)\mathbf{e}_2 \wedge \mathbf{e}_3 \bmod \mathrm{Im}(R^\dagger). \end{aligned}$$

7.4. **The image of the map $R^\dagger$.** To compute an explicit formula for the pairing $[\cdot,\cdot]$, we need to find the image of $R^\dagger$. First consider the intersection of $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(4r_o+2))\mathbf{e}_1\wedge\mathbf{e}_2$ with the image of $R_o^\dagger$, resp. $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(4r_\epsilon))\mathbf{e}_1\wedge\mathbf{e}_2$ with the image of $R_\epsilon^\dagger$. A global section of $\mathcal{O}_{\mathbb{P}^1}(3r_o)^{\oplus 4}$, resp. $\mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-1)\oplus\mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-2)\oplus\mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-2)\oplus\mathcal{O}_{\mathbb{P}^1}(3r_\epsilon-1)$ is mapped under $R^\dagger$ into $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(4r_o+2))\mathbf{e}_1\wedge\mathbf{e}_2$, resp. $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(4r_\epsilon))\mathbf{e}_1\wedge\mathbf{e}_2$ iff it is of the form

$$v_o=\begin{bmatrix} X_1^{r_o+1}p_o\\ -X_1^{r_o-1}q_o\\ -X_0^{r_o-1}q_o\\ X_0^{r_o+1}p_o\end{bmatrix}$$

for some $p_o\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2r_o-1))$ and $q_o\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2r_o+1))$, resp. iff it is of the form

$$v_\epsilon=\begin{bmatrix} X_1^{r_\epsilon+1}p_\epsilon\\ -X_1^{r_\epsilon-2}q_\epsilon\\ -X_0^{r_\epsilon-2}q_\epsilon\\ X_0^{r_\epsilon+1}p_\epsilon\end{bmatrix}$$

for some $p_\epsilon\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2))$ and $q_\epsilon\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2r_\epsilon))$. The image of such an element is,

$$R_o^\dagger(v_o)=(X_0X_1\widetilde{N}_op_o+\widetilde{M}_oq_o)\mathbf{e}_1\wedge\mathbf{e}_2,$$

resp.

$$R_\epsilon^\dagger(v_\epsilon)=(X_0X_1\widetilde{N}_\epsilon p_\epsilon+2\widetilde{M}_\epsilon q_\epsilon)\mathbf{e}_1\wedge\mathbf{e}_2.$$

There is one last simplification. Assume that $c_1=c_2=c_3=c_4=0$ and $d_1=d_2=d_3=d_4=0$, in other words L_a and L_b are 2 linearly independent, linear combinations of Y_0 and Y_5 and c_0,c_5,d_0 and d_5 are all nonzero. Consider those q such that $q=X_0X_1q'$ for some $q'\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2r_o-1))$, resp. $q'\in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2))$. Then $R_o^\dagger(v_o)$ equals $X_0X_1(\widetilde{N}_op_o+\widetilde{M}_oq_o')$, resp. $R_\epsilon^\dagger(v_\epsilon)$ equals $X_0X_1(\widetilde{N}_\epsilon p_\epsilon+2\widetilde{M}_\epsilon q_\epsilon)$. Since $\widetilde{M}$ and $\widetilde{N}$ are linearly independent elements in the span of $X_0^{2r_o+1}$ and $X_1^{2r_o+1}$, resp. in the span of $X_0^{2r_\epsilon}$ and $X_1^{2r_\epsilon}$, as p and q' vary the expression $R^\dagger(v)$ varies over the whole linear span of

$$X_0^{4r_o+1}X_1,\dots,X_0^{2r_o+2}X_1^{2r_o},X_0^{2r_o}X_1^{2r_o+2},\dots,X_0X_1^{4r_o+1},$$

resp.

$$X_0^{4r_\epsilon-1}X_1,\dots,X_0^{2r_\epsilon+1}X_1^{2r_\epsilon-1},X_0^{2r_\epsilon-1}X_1^{2r_\epsilon+1},\dots,X_0X_1^{4r_\epsilon-1}.$$

Notice that $X_0^{4r_o+2},X_0^{2r_o+1}X_1^{2r_o+1}$ and $X_1^{4r_o+2}$ are missing, resp. $X_0^{4r_\epsilon},X_0^{2r_\epsilon}X_1^{2r_\epsilon}$ and $X_1^{4r_\epsilon}$ are missing. Taking $q_o=X_0^{2r_o+1}$ and $q_o=X_1^{2r_o+1}$ gives $c_{0,o}X_0^{4r_o+2}+c_{5,o}X_0^{2r_o+1}X_1^{2r_o+1}$ and $c_{0,o}X_0^{2r_o+1}X_1^{2r_o+1}+c_{5,o}X_1^{4r_o+2}$. And taking $q_\epsilon=X_0^{2r_\epsilon}$ and $q_\epsilon=X_1^{2r_\epsilon}$ gives $c_{0,\epsilon}X_0^{4r_\epsilon}+c_{5,\epsilon}X_0^{2r_\epsilon}X_1^{2r_\epsilon}$ and $c_{0,\epsilon}X_0^{2r_\epsilon}X_1^{2r_\epsilon}+c_{5,\epsilon}X_1^{4r_\epsilon}$. Thus the intersection of $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(4r+2))\mathbf{e}_1\wedge\mathbf{e}_2$ with the image of $R^\dagger$ is the subspace with basis

$$c_{0,o}X_0^{4r_o+2}+c_{5,o}X_0^{2r_o+1}X_1^{2r_o+1},X_0^{4r_o+1}X_1,X_0^{4r_o}X_1^2,\dots,X_0^{2r_o+2}X_1^{2r_o},X_0^{2r_o}X_1^{2r_o+2},\dots\\ X_0X_1^{4r_o+1},c_{0,o}X_0^{2r_o+1}X_1^{2r_o+1}+c_{5,o}X_1^{4r_o+2},$$

resp.

$$c_{0,\epsilon}X_0^{4r_\epsilon}+c_{5,\epsilon}X_0^{2r_\epsilon}X_1^{2r_\epsilon},X_0^{4r_\epsilon-1}X_1,X_0^{4r_\epsilon-2}X_1^2,\dots,X_0^{2r_\epsilon+1}X_1^{2r_\epsilon-1},X_0^{2r_\epsilon-1}X_1^{2r_\epsilon+1},\dots\\ X_0X_1^{4r_\epsilon-1},c_{0,\epsilon}X_0^{2r_\epsilon}X_1^{2r_\epsilon}+c_{5,\epsilon}X_1^{4r_\epsilon}.$$

For each pair of nonnegative integers (i,j), denote by $\alpha_{i,j} : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i+j)) \to \mathbb{C}$ the linear functional such that for every homogeneous polynomial g of degree d,

$$g(X_0, X_1) = \sum_{i+j=d} \alpha_{i,j}(g) X_0^i X_1^j,$$

i.e. $\alpha_{i,j}(g)$ is the coefficient of $X_0^i X_1^j$ in g. Then the linear functional $c_{5,o}^2\alpha_{4r_o+2,0} - c_{0,o}c_{5,o}\alpha_{2r_o+1,2r_o+1} + c_{0,o}^2\alpha_{0,4r_o+2}$, resp. $c_{5,\epsilon}^2\alpha_{4r_\epsilon,0} - c_{0,\epsilon}c_{5,\epsilon}\alpha_{2r_\epsilon,2r_\epsilon} + c_{0,\epsilon}^2\alpha_{0,4r_\epsilon}$, is a nonzero linear functional on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r+2))$ whose kernel is precisely the intersection with the image of $R^\dagger$.

Using the first two rows of $R^\dagger$, every element in $H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X})$ is congruent to some element in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r_o+2))\mathbf{e}_1 \wedge \mathbf{e}_2$, resp. $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r_\epsilon))\mathbf{e}_1 \wedge \mathbf{e}_2$ modulo the image of $R^\dagger$. Carrying this out, up to a nonzero scalar, the pairing $[\cdot,\cdot]$ is,

$$\begin{gathered}[(g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3), (h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3)]_o = \\ (c_{5,o}^2\alpha_{4r_o+2,0} - c_{0,o}c_{5,o}\alpha_{2r_o+1,2r_o+2} + c_{0,o}^2\alpha_{0,4r_o+2})(g_1h_2 - g_2h_1) + \\ c_{0,o}c_{5,o}(c_{5,o}\alpha_{3r_o,r_o-1} - c_{0,o}\alpha_{r_o-1,3r_o})(g_1h_3 - g_3h_1) + \\ c_{0,o}c_{5,o}(d_{5,o}\alpha_{3r_o+1,r_o} - d_{0,o}\alpha_{r_o,3r_o+1})(g_2h_3 - g_3h_2),\end{gathered}$$

resp.

$$\begin{gathered}[(g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3), (h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3)]_\epsilon = \\ (c_{5,\epsilon}^2\alpha_{4r_\epsilon,0} - c_{0,\epsilon}c_{5,\epsilon}\alpha_{2r_\epsilon,2r_\epsilon} + c_{0,\epsilon}^2\alpha_{0,4r_\epsilon})(g_1h_2 - g_2h_1) + \\ 2c_{0,\epsilon}c_{5,\epsilon}(c_{5,\epsilon}\alpha_{3r_\epsilon-2,r_\epsilon-2} - c_{0,\epsilon}\alpha_{r_\epsilon-2,3r_\epsilon-2})(g_1h_3 - g_3h_1) + \\ c_{0,\epsilon}c_{5,\epsilon}(d_{5,\epsilon}\alpha_{3r_\epsilon,r_\epsilon} - d_{0,\epsilon}\alpha_{r_\epsilon,3r_\epsilon})(g_2h_3 - g_3h_2).\end{gathered}$$

7.5. **Diagonalizing the pairing.** The antisymmetric bilinear map $[\cdot,\cdot]$ gives a linear transformation $\widetilde{\omega}_e : H^0(\mathbb{P}^1, N_{C/X}) \to H^0(\mathbb{P}^1, N_{C/X})^\vee$ and we want to find the kernel of this linear transformation. This is done by "diagonalizing" the pair $(H^0(\mathbb{P}^1, N_{C/X}), [\cdot,\cdot])$, i.e. by finding a direct sum decomposition

$$H^0(\mathbb{P}^1, N_{C/X})_o = \bigoplus_{i=0}^{r_o-2} E_{i,o} \oplus E_{r_o-1,o} \oplus E_{r_o,o},$$

resp.

$$H^0(\mathbb{P}^1, N_{C/X})_\epsilon = \bigoplus_{i=0}^{r_\epsilon-3} E_{i,\epsilon} \oplus E_{r_\epsilon-2,\epsilon} \oplus E_{r_\epsilon-1,\epsilon} \oplus E_{r,\epsilon}$$

into pairwise orthogonal subspaces with respect to $[\cdot,\cdot]$. In the odd case, to show $[\cdot,\cdot]_o$ has trivial kernel, it suffices to show the restriction to each space $E_{i,o}$ has trivial kernel.

In the even case, there is a vector $\mathbf{w}$ in $E_{r_\epsilon,\epsilon}$ lying in the kernel. On the quotient vector space $H^0(\mathbb{P}^1, N_{C/X})/\mathbb{C}\{\mathbf{w}\}$, there is an induced alternating bilinear form $[\cdot,\cdot]'_\epsilon$ and an induced direct sum decomposition $\bigoplus_{i=0}^{r_\epsilon} E'_{i,\epsilon}$ by pairwise orthogonal subspaces. To show $[\cdot,\cdot]'_\epsilon$ has trivial kernel, it suffices to show the restriction to each space $E'_{i,\epsilon}$ has trivial kernel. In both case, this is done by computing the determinant of the matrix of $[\cdot,\cdot]_o$, resp. $[\cdot,\cdot]'_\epsilon$ with respect to a suitable basis.

For $i = 0, \dots, r_o - 2$, denote by $E_{i,o} \subset H^0(\mathbb{P}^1, N_{C/X})_o$ the subspace generated by

$$\begin{cases} \mathbf{v}_{i,1,o} &= X_0^{r_o+1+i} X_1^{r_o-1-i} \mathbf{e}_1 \\ \mathbf{v}_{i,2,o} &= X_0^{r_o-i} X_1^{r_o+2+i} \mathbf{e}_2 \\ \mathbf{v}_{i,3,o} &= X_0^{2r_o-1-i} X_1^{i} \mathbf{e}_3 \\ \mathbf{v}_{i,4,o} &= X_0^{i} X_1^{2r_o-1-i} \mathbf{e}_3 \\ \mathbf{v}_{i,5,o} &= X_0^{r_o+2+i} X_1^{r_o-i} \mathbf{e}_2 \\ \mathbf{v}_{i,6,o} &= X_0^{r_o-1-i} X_1^{r_o+1+i} \mathbf{e}_1 \end{cases}$$

For $i = 0, \dots, r_\epsilon - 3$, denote by $E_{i,\epsilon} \subset H^0(\mathbb{P}^1, N_{C/X})_\epsilon$ the subspace generated by

$$\begin{cases} \mathbf{v}_{i,1,\epsilon} &= X_0^{r_\epsilon+i} X_1^{r_\epsilon-2-i} \mathbf{e}_1 \\ \mathbf{v}_{i,2,\epsilon} &= X_0^{r_\epsilon-i} X_1^{r_\epsilon+2+i} \mathbf{e}_2 \\ \mathbf{v}_{i,3,\epsilon} &= X_0^{2r_\epsilon-2-i} X_1^{i} \mathbf{e}_3 \\ \mathbf{v}_{i,4,\epsilon} &= X_0^{i} X_1^{2r_\epsilon-2-i} \mathbf{e}_3 \\ \mathbf{v}_{i,5,\epsilon} &= X_0^{r_\epsilon+2+i} X_1^{r_\epsilon-i} \mathbf{e}_2 \\ \mathbf{v}_{i,6,\epsilon} &= X_0^{r_\epsilon-2-i} X_1^{r_\epsilon+i} \mathbf{e}_1 \end{cases}$$

For $i = r_o - 1$ denote by $E_{r_o-1,o} \subset H^0(\mathbb{P}^1, N_{C/X})_o$ the subspace generated by

$$\begin{cases} \mathbf{v}_{r-1,1,o} &= X_0^{2r_o} \mathbf{e}_1 \\ \mathbf{v}_{r-1,2,o} &= X_0^{2r_o+2} \mathbf{e}_2 \\ \mathbf{v}_{r-1,3,o} &= X_0^{2r_o+1} X_1 \mathbf{e}_2 \\ \mathbf{v}_{r-1,4,o} &= X_0^{r_o} X_1^{r_o+1} \mathbf{e}_3 \\ \mathbf{v}_{r-1,5,o} &= X_0^{r_o+1} X_1^{r_o} \mathbf{e}_3 \\ \mathbf{v}_{r-1,6,o} &= X_0 X_1^{2r_o+1} \mathbf{e}_2 \\ \mathbf{v}_{r-1,7,o} &= X_1^{2r_o+2} \mathbf{e}_2 \\ \mathbf{v}_{r-1,8,o} &= X_1^{2r_o} \mathbf{e}_1 \end{cases}$$

For $i = r_\epsilon - 2$, denote by $E_{r_\epsilon-2,\epsilon} \subset H^0(\mathbb{P}^1, N_{C/X})_\epsilon$ the subspace generated by

$$\begin{cases} \mathbf{v}_{r-2,1,\epsilon} &= X_0^{2r_\epsilon-2} \mathbf{e}_1 \\ \mathbf{v}_{r-2,2,\epsilon} &= X_0^{2r_\epsilon+2} \mathbf{e}_2 \\ \mathbf{v}_{r-2,3,\epsilon} &= X_0^{r_\epsilon} X_1^{r_\epsilon-2} \mathbf{e}_3 \\ \mathbf{v}_{r-2,4,\epsilon} &= X_0^{2} X_1^{2r_\epsilon} \mathbf{e}_2 \\ \mathbf{v}_{r-2,5,\epsilon} &= X_0^{2r_\epsilon} X_1^{2} \mathbf{e}_2 \\ \mathbf{v}_{r-2,6,\epsilon} &= X_0^{r_\epsilon-2} X_1^{r_\epsilon} \mathbf{e}_3 \\ \mathbf{v}_{r-2,7,\epsilon} &= X_1^{2r_\epsilon+2} \mathbf{e}_2 \\ \mathbf{v}_{r-2,8,\epsilon} &= X_1^{2r_\epsilon-2} \mathbf{e}_1 \end{cases}$$

For $i = r_o$ denote by $E_{r_o,o} \subset H^0(\mathbb{P}^1, N_{C/X})_o$ the subspace generated by

$$\begin{cases} \mathbf{v}_{r,1,o} &= X_0^{r,o} X_1^{r,o} \mathbf{e}_1 \\ \mathbf{v}_{r,2,o} &= X_0^{r_o+1} X_1^{r_o+1} \mathbf{e}_2 \end{cases}$$

For $i = r_\epsilon - 1$ denote by $E_{r_\epsilon-1,\epsilon} \subset H^0(\mathbb{P}^1, N_{C/X})_\epsilon$ the subspace generated by

$$\begin{cases} \mathbf{v}_{r-1,1,\epsilon} &= X_0^{r_\epsilon-1} X_1^{r_\epsilon-1} \mathbf{e}_1 \\ \mathbf{v}_{r-1,2,\epsilon} &= X_0^{r_\epsilon+1} X_1^{r_\epsilon+1} \mathbf{e}_2 \end{cases}$$

Finally, for $i = r_\epsilon$, denote by $E_{r_\epsilon,\epsilon} \subset H^0(\mathbb{P}^1, N_{C/X})_\epsilon$ the subspace generated by

$$\begin{cases} \mathbf{v}_{r,1,\epsilon} &= X_0^{2r_\epsilon+1}X_1\mathbf{e}_2 \\ \mathbf{v}_{r,2,\epsilon} &= X_0^{r_\epsilon-1}X_1^{r_\epsilon-1}\mathbf{e}_3 \\ \mathbf{v}_{r,3,\epsilon} &= X_0X_1^{2r_\epsilon+1}\mathbf{e}_2 \end{cases}$$

Each of these generating sets is a subbasis of the standard monomial basis of $H^0(\mathbb{P}^1, N_{C/X})$. Visibly, every monomial basis vector is in precisely one of the subspaces E_i, and thus these spaces give a direct sum decomposition of $H^0(\mathbb{P}^1, N_{C/X})$. As a consistency check, observe that for $i = 0, \ldots, r_o - 2$, resp. $i = 0, \ldots, r_\epsilon - 3$, $\dim(E_{i,o}) = 6$, resp. $\dim(E_{i,\epsilon}) = 6$, $\dim(E_{r_o-1,o}) = 8$, resp. $\dim(E_{r_\epsilon-2,\epsilon}) = 8$, $\dim(E_{r_o,o}) = 2$, resp. $\dim(E_{r_\epsilon-1,\epsilon}) = 2$, and $\dim(E_{r_\epsilon,\epsilon}) = 3$. So the sum of the dimensions of the spaces $E_{i,o}$ is

$$6(r_o - 1) + 8 + 2 = 6r_o + 4 = (2r_o + 1) + (2r_o + 3) + 2r_o,$$

i.e.,

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_o))\mathbf{e}_1 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_o+2))\mathbf{e}_2 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_o-1))\mathbf{e}_3.$$

Similarly, the sum of the dimensions of the spaces $E_{i,\epsilon}$ is

$$6(r_\epsilon - 2) + 8 + 2 + 3 = 6r_\epsilon + 1 = (2r_\epsilon - 1) + (2r_\epsilon + 3) + (2r_\epsilon - 1),$$

i.e.,

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2))\mathbf{e}_1 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon+2))\mathbf{e}_2 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r_\epsilon-2))\mathbf{e}_3.$$

Checking the spaces E_i are pairwise orthogonal with respect to $[\cdot,\cdot]$ is straightforward, but tedious. One way to think of it is to consider the graph whose vertices are the standard monomial basis vectors of $H^0(\mathbb{P}^1, N_{C/X})$, and where there is an edge between two such basis vectors iff the pairing is nonzero for this pair. Thus there is never an edge between $g_1\mathbf{e}_1$ and $h_1\mathbf{e}_1$, nor between $g_2\mathbf{e}_2$ and $h_2\mathbf{e}_2$, nor between $g_3\mathbf{e}_3$ and $h_3\mathbf{e}_3$. There is an edge between $g_1\mathbf{e}_1$ and $h_2\mathbf{e}_2$ iff $g_1h_2 = X_0^{4r_o+2}, X_0^{2r_o+1}X_1^{2r_o+1}$ or $X_1^{4r_o+2}$, resp. iff $g_1h_2 = X_0^{4r_\epsilon}, X_0^{2r_\epsilon}X_1^{2r_\epsilon}$ or $X_1^{4r_\epsilon}$. There is an edge between $g_1\mathbf{e}_1$ and $h_3\mathbf{e}_3$ iff $g_1h_3 = X_0^{3r_o}X_1^{r_o-1}$ or $X_0^{r_o-1}X_1^{3r_o}$, resp. iff $g_1h_3 = X_0^{3r_\epsilon-2}X_1^{r_\epsilon-2}$ or $X_0^{r_\epsilon-2}X_1^{3r_\epsilon-2}$. And there is an edge between $g_2\mathbf{e}_2$ and $h_3\mathbf{e}_3$ iff $g_2h_3 = X_0^{3r_o+1}X_1^{r_o}$ or $X_0^{r_o}X_1^{3r_o+1}$, resp. iff $g_2h_3 = X_0^{3r_\epsilon}X_1^{r_\epsilon}$ or $X_0^{r_\epsilon}X_1^{3r_\epsilon}$. Thus, the valences of $X_0^{2r_o}\mathbf{e}_1$, $X_1^{2r_o}\mathbf{e}_1$, $X_0^{r_o}X_1^{r_o-1}\mathbf{e}_3$ and $X_0^{r_o-1}X_1^{r_o}\mathbf{e}_3$ are each 3, resp. the valences of $X_0^{2r_\epsilon-2}\mathbf{e}_1$, $X_1^{2r_\epsilon-2}\mathbf{e}_1$, $X_0^{r_\epsilon}X_1^{r_\epsilon-2}\mathbf{e}_3$ and $X_0^{r_\epsilon-2}X_1^{r_\epsilon}\mathbf{e}_3$ are each 3. Also the valences of $X_0^{r_o}X_1^{r_o}\mathbf{e}_1$ and $X_0^{r_o+1}X_1^{r_o+1}\mathbf{e}_2$ are each 1, resp. the valences of $X_0^{r_\epsilon-1}X_1^{r_\epsilon-1}\mathbf{e}_1, X_0^{r_\epsilon+1}X_1^{r_\epsilon+1}\mathbf{e}_2, X_0^{2r_\epsilon+1}X_1\mathbf{e}_2$ and $X_0X_1^{2r_\epsilon+2}\mathbf{e}_2$ are each 1. Every other vertex has valence two. Moreover, there is an symmetry of the graph by permuting the variables X_0 and X_1. Using this, it is straightforward to compute the maximal connected subgraph containing the vector $\mathbf{v}_{i,1}$ for each i. The vertices of this subgraph are the generators of E_i. Therefore the E_i are pairwise orthogonal.

7.6. **Computing the determinants.** Finally, we will compute the matrix and determinant of the restriction of $\widetilde{\omega}_e$ to each of the subspace E_i. In the odd case, each determinant is nonzero, proving that $\widetilde{\omega}$ has trivial kernel. In the even case, all but one of the determinants is nonzero, and for E_{r_ϵ}, the restriction of $\widetilde{\omega}$ has a 1-dimensional kernel.

For $i = 0, \ldots, r_o - 2$, resp. for $i = 0, \ldots, r_\epsilon - 3$ denote by A_i the matrix of $\widetilde{\omega}_e : E_i \to E_i^\vee$ with respect to the ordered basis $\mathbf{v}_{i,1}, \ldots, \mathbf{v}_{i,6}$ and the dual ordered basis of $E_i^\vee$. The computation gives,

$$A_{i,o} = \begin{bmatrix} 0 & c_0c_5 & -c_0c_5^2 & 0 & 0 & 0 \\ -c_0c_5 & 0 & 0 & c_0c_5d_0 & 0 & 0 \\ c_0c_5^2 & 0 & 0 & 0 & c_0c_5d_5 & 0 \\ 0 & -c_0c_5d_0 & 0 & 0 & 0 & -c_0^2c_5 \\ 0 & 0 & -c_0c_5d_5 & 0 & 0 & -c_0c_5 \\ 0 & 0 & 0 & c_0^2c_5 & c_0c_5 & 0 \end{bmatrix},$$

resp.

$$A_{i,\epsilon} = \begin{bmatrix} 0 & c_0c_5 & -2c_0c_5^2 & 0 & 0 & 0 \\ -c_0c_5 & 0 & 0 & c_0c_5d_0 & 0 & 0 \\ 2c_0c_5^2 & 0 & 0 & 0 & c_0c_5d_5 & 0 \\ 0 & -c_0c_5d_0 & 0 & 0 & 0 & -2c_0^2c_5 \\ 0 & 0 & -c_0c_5d_5 & 0 & 0 & -c_0c_5 \\ 0 & 0 & 0 & 2c_0^2c_5 & c_0c_5 & 0 \end{bmatrix}$$

The Pfaffian of each matrix is $\mathrm{Pfaff}(A_{i,o}) = c_0^3c_5^3(c_0d_5 - c_5d_0)$, resp. $\mathrm{Pfaff}(A_{i,\epsilon}) = 2c_0^3c_5^3(c_0d_5 - c_5d_0)$. Thus the determinant is $\mathrm{Det}(A_{i,o}) = c_0^6c_5^6(c_0d_5 - c_5d_0)^2$, resp. $\mathrm{Det}(A_{i,\epsilon}) = 4c_0^6c_5^6(c_0d_5 - c_5d_0)^2$. By hypothesis, c_0, c_5 are nonzero and (c_0, c_5) is linearly independent from (d_0, d_5). Thus each determinant is nonzero.

For $i = r_o - 1$, resp. $i = r_\epsilon - 2$, denote by A_i the matrix of $\widetilde{\omega}_e : E_i \to E_i^\vee$ with respect to the ordered basis $\mathbf{v}_{i,1}, \ldots, \mathbf{v}_{i,8}$ and the dual ordered basis of $E_i^\vee$. The computation gives,

$$A_{r_o-1,o} = \begin{bmatrix} 0 & -c_5^2 & 0 & -c_0c_5^2 & 0 & c_0c_5 & 0 & 0 \\ c_5^2 & 0 & 0 & 0 & -c_0c_5d_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_0c_5d_5 & 0 & 0 & 0 & -c_0c_5 \\ c_0c_5^2 & 0 & c_0c_5d_5 & 0 & 0 & 0 & -c_0c_5d_0 & 0 \\ 0 & c_0c_5d_5 & 0 & 0 & 0 & -c_0c_5d_0 & 0 & -c_0^2c_5 \\ -c_0c_5 & 0 & 0 & 0 & c_0c_5d_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_0c_5d_0 & 0 & 0 & 0 & c_0^2 \\ 0 & 0 & c_0c_5 & 0 & c_0^2c_5 & 0 & -c_0^2 & 0 \end{bmatrix},$$

resp.

$$A_{r_\epsilon-2,\epsilon} = \begin{bmatrix} 0 & -c_5^2 & -2c_0c_5^2 & c_0c_5 & 0 & 0 & 0 & 0 \\ c_5^2 & 0 & 0 & 0 & 0 & -c_0c_5d_5 & 0 & 0 \\ 2c_0c_5^2 & 0 & 0 & 0 & c_0c_5d_5 & 0 & -c_0c_5d_0 & 0 \\ -c_0c_5 & 0 & 0 & 0 & 0 & c_0c_5d_0 & 0 & 0 \\ 0 & 0 & -c_0c_5d_5 & 0 & 0 & 0 & 0 & -c_0c_5 \\ 0 & c_0c_5d_5 & 0 & -c_0c_5d_0 & 0 & 0 & 0 & -2c_0^2c_5 \\ 0 & 0 & c_0c_5d_0 & 0 & 0 & 0 & 0 & c_0^2 \\ 0 & 0 & 0 & 0 & c_0c_5 & 2c_0^2c_5 & -c_0^2 & 0 \end{bmatrix}$$

The Pfaffian of this matrix is $\mathrm{Pfaff}(A_{r_o-1,o}) = c_0^3c_5^3(c_0d_5 - c_5d_0)^2$, resp. $\mathrm{Pfaff}(A_{r_\epsilon-2,\epsilon}) = c_0^3c_5^3(c_0d_5 - c_5d_0)^2$. Thus the determinant is $\mathrm{Det}(A_{r_o-1,o}) = c_0^6c_5^6(c_0d_5 - c_5d_0)^4$, resp. the determinant is $\mathrm{Det}(A_{r_\epsilon-2,\epsilon}) = c_0^6c_5^6(c_0d_5 - c_5d_0)^4$. By hypothesis, c_0, c_5 are nonzero and (c_0, c_5) is linearly independent from (d_0, d_5). Thus each determinant is nonzero.

For $i = r_o$, resp. $i = r_\epsilon - 1$, denote by A_i the matrix of $\widetilde{\omega}_e : E_i \to E_i^\vee$ with respect to the ordered basis $\mathbf{v}_{i,1}, \mathbf{v}_{i,2}$ and the dual ordered basis of $E_i^\vee$. The computation gives,

$$A_{r_o,o} = \begin{bmatrix} 0 & c_0c_5 \\ -c_0c_5 & 0 \end{bmatrix},$$

resp.

$$A_{r_\epsilon-1,\epsilon} = \begin{bmatrix} 0 & c_0c_5 \\ -c_0c_5 & 0 \end{bmatrix}.$$

Visibly the Pfaffian of this matrix is c_0c_5 and the determinant is $c_0^2c_5^2$. By hypothesis, c_0, c_5 are nonzero, thus each determinant is nonzero.

In the odd case, since the determinant of each matrix $A_{i,o}$ is nonzero, the kernel of $\widetilde{\omega}_e$ is trivial. This proves Theorem 1.2 in case $e \geq 5$ is an odd integer. From this point on, suppose that $e \geq 6$ is even.

Denote by $A_{r_\epsilon,\epsilon}$ the matrix of $\widetilde{\omega}_e : E_{r_\epsilon,\epsilon} \to E_{r_\epsilon,\epsilon}^\vee$ with respect to the ordered basis $\mathbf{v}_{r_\epsilon,1,\epsilon}, \mathbf{v}_{r_\epsilon,2,\epsilon}, \mathbf{v}_{r_\epsilon,3,\epsilon}$ and the dual ordered basis of $E_{r_\epsilon,\epsilon}^\vee$. The computation gives,

$$A_{r_\epsilon,\epsilon} = \begin{bmatrix} 0 & -c_0c_5d_5 & 0 \\ c_0c_5d_5 & 0 & -c_0c_5d_0 \\ 0 & c_0c_5d_0 & 0 \end{bmatrix}$$

This matrix is singular: the kernel contains the vector $\mathbf{w} = d_{0,\epsilon}\mathbf{v}_{r_\epsilon,1} + d_{5,\epsilon}\mathbf{v}_{r_\epsilon,3}$, i.e. $(d_{0,\epsilon}X_0^{2r_\epsilon} + d_{5,\epsilon}X_1^{2r_\epsilon})X_0X_1\mathbf{e}_2$. So this vector is in the kernel of $\widetilde{\omega}_e$. Consider the quotient vector space $V' = H^0(\mathbb{P}^1, N_{C/X})/\mathbb{C}\{\mathbf{w}\}$. There is an induced alternating bilinear pairing $\widetilde{\omega}'_e$ on V'. Since $w' \in E_{r_\epsilon,\epsilon}$, there is an induced direct sum decomposition $V' = \bigoplus_{i=0}^{r_\epsilon} E'_{i,\epsilon}$ by pairwise orthogonal subspaces where for $i = 0, \dots, r-1$ the quotient map $E_{i,\epsilon} \to E'_{i,\epsilon}$ is an isomorphism. And $E'_{r_\epsilon,\epsilon}$ has as basis the images of the vectors $\mathbf{v}_{r_\epsilon,1,\epsilon}, \mathbf{v}_{r_\epsilon,2,\epsilon}$ provided $d_{5,\epsilon} \neq 0$, and has as basis the images of the vectors $\mathbf{v}_{r_\epsilon,2,\epsilon}\mathbf{v}_{r_\epsilon,3,\epsilon}$ provided $d_{0,\epsilon} \neq 0$.

First consider the case, $d_{5,\epsilon} \neq 0$. Denote by $A'_{r_\epsilon,\epsilon}$ the matrix of $\widetilde{\omega}'_e : E'_{r_\epsilon,\epsilon} \to (E'_{r_\epsilon,\epsilon})^\vee$ with respect to the ordered basis $\mathbf{v}'_{r_\epsilon,1,\epsilon}, \mathbf{v}'_{r_\epsilon,2,\epsilon}$ and the dual ordered basis of $(E'_{r_\epsilon,\epsilon})^\vee$. The computation gives,

$$A'_{r_\epsilon,\epsilon} = \begin{bmatrix} 0 & -c_0c_5d_5 \\ c_0c_5d_5 & 0 \end{bmatrix}$$

The Pfaffian of this matrix is $c_0c_5d_5$ and the determinant is $c_0^2c_5^2d_5^2$. By hypothesis c_0, c_5, d_5 are nonzero, thus the determinant is nonzero.

The remaining case is that $d_0 \neq 0$. Again denote by $A'_{r_\epsilon,\epsilon}$ the matrix of $\widetilde{\omega}'_e : E'_{r_\epsilon,\epsilon} \to (E'_{r,\epsilon})^\vee$ with respect to the ordered basis $\mathbf{v}'_{r_\epsilon,2,\epsilon}, \mathbf{v}'_{r_\epsilon,3,\epsilon}$ and the dual ordered basis of $(E'_{r_\epsilon,\epsilon})^\vee$. The computation gives,

$$A'_{r_\epsilon,\epsilon} = \begin{bmatrix} 0 & -c_0c_5d_0 \\ c_0c_5d_0 & 0 \end{bmatrix}$$

The Pfaffian of this matrix is $c_0c_5d_0$ and the determinant is $c_0^2c_5^2d_0^2$. By hypothesis c_0, c_5, d_0 are nonzero, thus the determinant is nonzero.

In both cases, the determinant of the restriction of $\widetilde{\omega}'_e$ to each subspace $E'_{i,\epsilon}$ is nonzero. Thus the kernel of $\widetilde{\omega}_e$ is spanned by $\mathbf{w} = (d_{0,\epsilon}X_0^{2r_\epsilon} + d_{5,\epsilon}X_1^{2r_\epsilon})X_0X_1\mathbf{e}_2$. In particular, the kernel of $\widetilde{\omega}_e$ is 1-dimensional. This proves Theorem 1.2 in case $e \geq 6$ is an even integer.

8. Comments and Questions

There are some generalizations of Theorem 1.2 to stable maps with marked points. The cubic hypersurface $X \subset \mathbb{P}^5$ and the stable map $f : \mathbb{P}^1 \to \mathbb{P}^5$ are the same as in Section 7. The marked points are $[0 : 1], [1 : 0] \in \mathbb{P}^1$. The same method as Section 7 proves the following.

Theorem 8.1. *Let $X \subset \mathbb{P}^5$ a smooth cubic hypersurface, let $\overline{\mathcal{M}}_{0,n}(X, e)$ denote the Kontsevich moduli space of pointed stable maps to X of arithmetic genus 0 and degree e, and let $\overline{M}_{e,n}$ be a nonsingular projective model of the coarse moduli space. Denote by $ev : \overline{M}_{e,n} \to X^n$ evaluation at each marked point. And denote by T_{ev}, the kernel of the derivative, $dev : T_{\overline{M}_{e,n}} \to ev^* T_{X^n}$.*

There is a canonical section $\omega_e \in H^0(\overline{M}_{e,n}, \Omega^2_{\overline{M}_{e,n}})$. Suppose that X is general.

1. *(i) If $n = 1$, if $e \geq 5$ is an odd integer, and if $\zeta = (C, p, f : C \to X)$ is a general point of $\overline{M}_{e,1}$, the restriction of ω_e to $T_{ev}|_\zeta$ has a 1-dimensional kernel.*
2. *(ii) If $n = 1$, if $e \geq 6$ is even, $e \geq 6$, and if $\zeta = (C, p, f : C \to X)$ is a general point of $\overline{M}_{e,1}$, the restriction of ω_e to $T_{ev}|_\zeta$ is nondegenerate. Therefore a general fiber of ev has Kodaira dimension ≥ 0 and, in particular, it is not uniruled.*
3. *(iii) If $n = 2$, if $e \geq 5$ is odd, and if $\zeta = (C, p_1, p_2, f : C \to X)$ is a general point of $\overline{M}_{e,2}$, the restriction of ω_e to $T_{ev}|_\zeta$ is nondegenerate. Therefore a general fiber of (ev_1, ev_2) has Kodaira dimension ≥ 0 and, in particular, it is not uniruled.*
4. *(iv) If $n = 2$, if $e \geq 6$ is even, and if $\zeta = (C, p_1, p_2, f : C \to X)$ is a general point of $\overline{M}_{e,2}$, the restriction of ω_e to $T_{ev}|_\zeta$ has a 1-dimensional kernel.*

Proof. Most of the details are left to the reader. The technique is almost identical to the proof of Theorem 1.2 and is roughly as follows: For (i) and (ii), consider the special pairs $([X], [C])$ used in Section 7. In addition, assume that $d_{0,o}, d_{5,o}$ are both nonzero. For the marked point on C, use either $f([0 : 1])$ or $f([1 : 0])$. The tangent space to $T_{\text{ev}}|_\zeta$ is the subspace of sections of $H^0(\mathbb{P}^1, N_{C/X})$ vanishing at $[0 : 1]$, resp. $[1 : 0]$. The form ω_e on this subspace is the form computed in Section 7. In particular, since the space of sections vanishing at $[0 : 1]$ is generated by standard monomial basis vectors of $H^0(\mathbb{P}^1, N_{C/X})$, the direct sum decomposition into pairwise orthogonal subspaces yields a direct sum decomposition of the space of sections vanishing at $[0 : 1]$.

In the odd case, the kernel is generated by $c_{5,o}\mathbf{v}_{0,2,o} + \mathbf{v}_{0,3,o} + d_{5,o}\mathbf{v}_{0,6,o}$. And the induced pairing on the quotient space is nondegenerate. In the even case, the kernel is nontrivial: it is generated by $d_{0,\epsilon}\mathbf{v}_{r_\epsilon,1,\epsilon} + d_{5,\epsilon}\mathbf{v}_{r_\epsilon,3,\epsilon}$ and $\mathbf{v}_{r_\epsilon-2,1} + 2c_{5,\epsilon}\mathbf{v}_{r_\epsilon-2,3} - d_{5,\epsilon}\mathbf{v}_{r_\epsilon-2,6,\epsilon}$. However, under a nontrivial first-order deformation of the pointed curve not changing the map $f : \mathbb{P}^1 \to X$, only moving the point $[0 : 1]$ on $\mathbb{P}^1$, the kernel becomes trivial (this is a simple deformation theory exercise).

Parts (iii) and (iv) are the same. In the odd case, the kernel is trivial. In the even case, the kernel is generated by $d_{0,\epsilon}\mathbf{v}_{r_\epsilon,1,\epsilon} + d_{5,\epsilon}\mathbf{v}_{r_\epsilon,3,\epsilon}$ (no deformation theory is needed). □

Question 8.2. What is the Kodaira dimension of $\overline{M}_e$, resp. what is the dimension of a fiber of the MRC quotient of $\overline{M}_e$, when the form ω_e does have a kernel? If $e \geq 6$ is even, is $\overline{M}_e$ uniruled?

We are convinced that $\overline{M}_e$ is not uniruled, but we do not have a proof for $e \geq 8$. In case e is 6, we can prove that $\overline{M}_6$ is not uniruled by an *ad hoc* argument. It is possible this could be used as the base case of an induction by considering how the kernel of ω_{e+2} specializes on the boundary divisor $\Delta_{e,2} \subset \overline{\mathcal{M}}_e$.

Proposition 8.3. *If the cubic hypersurface $X \subset \mathbb{P}^5$ is general, then M_6 is not uniruled. More precisely, there exists a rational transformation $f : M_6 \dashrightarrow Hilb_X^{6t}$ whose general fiber is a genus 1 curve which is a leaf of the distribution $Ker(\omega_e)$.*

Here is a rough sketch of the proof. The method of proof is similar to that in [13], but instead of using residual curves in an intersection of X with a cubic scroll, we use residual curves in an intersection of X with a quartic scroll. For a general nondegenerate, rational, degree 6 curve $C \subset \mathbb{P}^5$, there is a unique quartic scroll $\Sigma \subset \mathbb{P}^5$ containing C. If X is general, then X contains no quartic scrolls (although special smooth cubic fourfolds can contain a quartic scroll, [14, Section 4.1.3]). The intersection $\Sigma \cap X$ is a degree 12 curve in Σ that is a local complete intersection (in particular it is Gorenstein) and contains C as a subcurve of degree 6. By Gorenstein liaison, the residual curve C' to C in Σ is a degree 6 curve of arithmetic genus 1, and is a smooth, connected curve if C general. This gives a rational transformation from M_6 to the open subset U of the Chow variety/Hilbert scheme parametrizing degree 6 curves in X of arithmetic genus 1; $[C] \mapsto [C']$. The fiber of this rational transformation containing $[C]$ is isomorphic to $\mathrm{Pic}^2(C')$, i.e., it is a connected, smooth curve of genus 1 (actually it will only be a dense open subset since we are working on the non-complete variety M_6).

On M_6 there is the 2-form ω_6 constructed in Section 4. On U there is a 2-form by the same process as in Section 4 corresponding to the family of degree 6 curves of arithmetic genus 1. On the domain of definition of the rational transformation $M_6 \to U$, form the pullback of the 2-form on U; denote this pullback 2-form by ω'. Over a dense open set of M_6, the curve $\Sigma \cap X$ is a connected, reduced at-worst-nodal curve and the process from Section 4 produces a 2-form ω'' corresponding to this family of curves. The relation between these forms is

$$\omega_6 + \omega' = \omega''$$

on the open, dense locus where all three are defined.

On the other hand, there is a *unirational* space $W \subset \mathrm{Hilb}^{(2t+1)(t+1)}(\mathbb{P}^5)$ parametrizing all smooth, nondegenerate quartic scrolls in $\Sigma \subset \mathbb{P}^5$ (in fact this is a homogeneous space for PGL_6 since any two such scrolls are projectively equivalent). Over a dense open subset of W the process from Section 4 produces a 2-form corresponding to the family of curves whose fiber over $[\Sigma]$ is $\Sigma \cap X$. And ω'' is the pullback of this 2-form by the obvious rational map $M_6 \dashrightarrow W$. Since W is unirational, it does not support any nonzero 2-form, i.e., $\omega'' = 0$. So $\omega_6 = -\omega'$. In particular, the kernel of ω_6 coincides with the kernel of ω'. Since ω' is a pullback by the rational transformation $M_6 \to U$, in particular the tangent space of the fiber of this rational transformation is contained in ω_6. We know the fiber is one-dimensional. By Theorem 1.2, also the kernel of ω_6 is one-dimensional. Thus the kernel of ω_6 at a general point of M_6 is precisely the tangent space to the fiber of $M_6 \to U$. In other words, the foliation determined by the kernel of ω_6 is algebraically integrable on a dense (Zariski) open subset of M_6, the leaf space is (birationally) an open subset U of the Hilbert scheme of smooth, degree 6 curves in X of genus 1, and the projection to the leaf space is (birationally) the rational transformation $M_6 \dashrightarrow U$.

From this it follows that U has Kodaira dimension ≥ 0, in particular it is not uniruled. By the special case of the Iitaka conjecture proved in [17], the Kodaira dimension of M_6 is ≥ 0. In particular, M_6 is not uniruled.

There are lots of missing details in this argument. They each follow by straightforward arguments of projective geometry, and are left to the reader.

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