ARTIN'S AXIOMS, COMPOSITION AND MODULI SPACES

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Abstract. We prove Artin's axioms for algebraicity of a stack are compatible with composition of 1-morphisms. Consequently, some natural stacks are algebraic. One of these is a common generalization of Vistoli's Hilbert stack and the stack of branchvarieties defined by Alexeev and Knutson.

1. Introduction

Many moduli functors in algebraic geometry, properly interpreted, are *algebraic stacks*, also called *Artin stacks* (please note, following Artin, we do not assume diagonal morphisms are quasi-compact). In [Art74], Artin gave axioms for algebraicity involving deformation-obstruction theory and compatibility with completion.

There exist natural stacks $\mathcal{Y}$ where the completion axiom fails but all other axioms hold. Sometimes there exists a stack $\mathcal{X}$ for which the completion axiom holds and a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ representable by algebraic stacks. Intuitively $\mathcal{X}$ should satisfy all Artin's axioms, and thus be algebraic. In other words, Artin's axioms should be compatible with *composition* of 1-morphisms of stacks. The difficulty is that, given a relative obstruction theory for f and an obstruction theory for $\mathcal{Y}$, there may exist no "extension" obstruction theory for $\mathcal{X}$.

Existence of an extension obstruction theory is circumvented using Propositions 2.9 and 2.10. The main result is the following version of [Art74, Theorem 5.3].

Proposition 1.1. *Let S be an excellent scheme and let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of limit preserving stacks in groupoids over (Aff/S) for the étale topology. Let $O_{\mathcal{Y}}$ be an obstruction theory for $\mathcal{Y}$ and let O_f be a relative obstruction theory for f. The stack $\mathcal{X}$ is algebraic if,*

(1) *Conditions* [Art74, (S1,2)] *hold for deformations and automorphisms of $\mathcal{Y}$ and $\mathcal{X}$ (or equivalently, $\mathcal{Y}$ and f).*
(2) *For any complete local $\mathcal{O}_S$-algebra $\widehat{A}$ with residue field of finite type over S, the canonical map*
$$\mathcal{X}(Spec\ \widehat{A}) \to \varprojlim \mathcal{X}(Spec\ \widehat{A}/\mathfrak{m}^n)$$
is faithful, and has a dense image, i.e., the projection to $\mathcal{X}(Spec\ \widehat{A}/\mathfrak{m}^n)$ is essentially surjective for every n.
(3) *Automorphisms, deformations and obstructions of $\mathcal{Y}$ and f satisfy the conditions in Notations 2.4 and 2.21.*
(4) *If the object a_0 of $\mathcal{X}(Spec\ A_0)$ is algebraic, and if ϕ is an automorphism of a_0 inducing the identity in $\mathcal{X}(Spec\ k(y))$ for a dense set of finite type points y of $Spec\ A_0$, then ϕ equals Id_{a_0} on a non-empty open subset of $Spec\ A_0$.*

A consequence is algebraicity of some natural stacks.

Proposition 1.2. *Let S be an excellent scheme and let $\mathcal{Y}$ be a limit preserving algebraic stack over (Aff/S) with finite diagonal. The stack $\mathcal{H}$ parametrizing triples (X, L, g) of a proper algebraic space X, a 1-morphism $g : X \to \mathcal{H}$, and an invertible, g-ample $\mathcal{O}_X$-module L is a limit preserving algebraic stack over (Aff/S) with quasi-compact, separated diagonal.*

This stack $\mathcal{H}$ is a common generalization of Vistoli's *Hilbert stack*, [Vis91], and the stack of *branchvarieties* defined by Alexeev and Knutson, [AK06]. The proof of Proposition 1.2 gives a new proof of algebraicity in each of these special cases.

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2. Artin's axioms and obstruction theory

All hypotheses regarding obstructions in [Art74] trace back to the proofs of [Art74, Proposition 4.3 and Theorem 4.4]. This section describes lifting properties, why the lifting properties are compatible with composition, how Artin uses these lifting properties to prove openness of versality, and finally, how Artin uses obstruction theory to prove the lifting properties.

2.1. **Relative deformation situations.** Relative deformation situations and obstruction theories are studied in [Ols04, Appendix A]. Following are the basic definitions.

Let S be a locally Noetherian algebraic space (for later applications, it will be necessary that S is excellent). An *infinitesimal extension* is a surjective homomorphism of Noetherian $\mathcal{O}_S$-algebras with nilpotent kernel, $A' \xrightarrow{q} A$. An *extension pair* is a pair of infinitesimal extensions $(A' \xrightarrow{q} A \xrightarrow{q_0} A_0)$ such that A_0 is reduced and the kernel M of q is annihilated by the kernel of $q_0 \circ q$.

A *morphism of infinitesimal extensions* (u', u) is a Cartesian diagram of $\mathcal{O}_S$-algebras,

$$\begin{array}{ccc} A' & \xrightarrow{q_A} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{q_B} & B \end{array}$$

whose rows are infinitesimal extensions. *Morphisms of extension pairs* (u', u, u_0) are defined analogously.

Let $\mathcal{X}$ be a stacks in groupoids over (Aff/S) for the étale topology. An *infinitesimal extension over $\mathcal{X}$* is a datum (q, a) of a reduced infinitesimal extension $A \xrightarrow{q_0} A_0$ and an object a of $\mathcal{X}(\text{Spec } A)$. A *morphism of infinitesimal extensions over $\mathcal{X}$*,

$$(u, u_0, \phi) : (A \xrightarrow{q_{A,0}} A_0, a) \to (B \xrightarrow{q_{B,0}} B_0, b),$$

is a morphism (u, u_0) of infinitesimal extensions together with a morphism $\phi : b \to a$ in $\mathcal{X}$ mapping to $u^* : \text{Spec } B \to \text{Spec } A$.

Definition 2.1. A *deformation situation over $\mathcal{X}$* is a datum $(A' \xrightarrow{q} A \xrightarrow{q_0} A_0, a)$ of an extension pair (q, q_0) and an object a of $\mathcal{X}(\text{Spec } A)$.

A *morphism of deformation situations over $\mathcal{X}$* is a datum (u', u, u_0, ϕ) of a morphism (u', u, u_0) of extension pairs together with a morphism $\phi : b \to a$ in $\mathcal{X}$ mapping to u^*.

By the axioms for a stack, given a deformation situation (q, q_0, a) and a morphism of extension pairs (u', u, u_0), there exists a morphism ϕ so that (u', u, u_0, ϕ) is a morphism of deformation situations over $\mathcal{X}$. A *clivage normalisé* determines a choice of ϕ, cf. [Gro03, Définition VI.7.1]. From this point on, a clivage normalisé is assumed given. The image of (u', u, u_0, ϕ) is called the *base change* of (q, q_0, a) by (u', u, u_0).

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over (Aff/S). An *infinitesimal extension over* f is a datum $(\widetilde{q}, a)$ of a morphism $\widetilde{q}$ of $\mathcal{Y}$ mapping to a reduced infinitesimal extension $A \xrightarrow{q_0} A_0$ and an object a of $\mathcal{X}(\mathrm{Spec}\ A)$ mapping to the target of $\widetilde{q}$. Equivalently, it is a 1-morphism $\mathrm{Spec}\ A \to \mathcal{Y}$ and an infinitesimal extension over the 2-fibered product $\mathrm{Spec}\ A \times_{\mathcal{Y}} \mathcal{X}$. Morphisms of infinitesimal extensions over f, deformation situations over f, morphisms of deformation situations over f, and base change are defined analogously.

If $\mathcal{Y}$ satisfies the Schlessinger-Rim criterion [Art74, (S1)], there are well-defined relative analogues of the Schlessinger-Rim criterion for f.

Lemma 2.2. [Ols04, §A.14] *Let $\mathcal{Y}$ be a stack in groupoids over (Aff/S) satisfying* [Art74, (S1)]. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over (Aff/S).*

(i) *The stack in groupoids $\mathcal{X}$ satisfies* [Art74, (S1)] *if and only if f satisfies the relative analogue of* [Art74, (S1)].

(ii) *Let $g : \mathcal{Z} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids. If $\mathcal{X}$ and $\mathcal{Z}$ satisfy* [Art74, (S1)], *then also the 2-fibered product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ satisfies* [Art74, (S1)].

Proof. This is largely verified in [Ols04, §A.14]. The details are left to the reader. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over (Aff/S), each of which satisfies [Art74, (S1)]. For each reduced $\mathcal{O}_S$-algebra A_0 and object a_0 of $\mathcal{X}(\mathrm{Spec}\ A_0)$, [Ols04, §A.15] gives a natural 7-term exact sequence of automorphism and deformation groups. This implies the following.

Lemma 2.3. [Ols04, §A.15]

(i) *Assuming $\mathcal{Y}$ satisfies* [Art74, (S2)], *$\mathcal{X}$ satisfies* [Art74, (S2)] *if and only if f satisfies the relative analogue of* [Art74, (S2)].

(ii) *If $g : \mathcal{Z} \to \mathcal{Y}$ is a 1-morphism of stacks in groupoids such that $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ each satisfy* [Art74, (S1,2)], *then also $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ satisfies* [Art74, (S1,2)].

Proof. This is largely verified in [Ols04, §A.15]. The details are left to the reader. $\square$

Notation 2.4. [Art74, (4.1)], [Ols04, §A.11] There are relative analogues of the conditions on automorphisms and deformations. (The relative analogues of conditions on obstructions are stated in Notation 2.21.)

(4.1.i) The functors, Aut_f, resp. D_f, are compatible with étale localization: For every morphism $(\widetilde{u}, \widetilde{u}_0, \phi)$ of infinitesimal extensions over f, if A_0 is a finite-type $\mathcal{O}_S$-algebra and u is étale, then the following associated natural transformations of functors are isomorphisms,

$$\mathrm{Aut}_{f,a_0}(A_0 + M) \otimes_{A_0} B_0 \to \mathrm{Aut}_{f,b_0}(B_0 + M \otimes_{A_0} B_0),$$

$$D_{f,a_0}(M) \otimes_{A_0} B_0 \to D_{f,b_0}(M \otimes_{A_0} B_0).$$

(4.1.ii) The functors Aut_f and D_f are compatible with completions: For every finite-type $\mathcal{O}_S$-algebra A_0 and every maximal ideal $\mathfrak{m}$ of A_0, the following natural maps are isomorphisms,

$$\mathrm{Aut}_{f,a_0}(A_0+M)\otimes_{A_0}\widehat{A}_0 \to \varprojlim \mathrm{Aut}_{f,a_0}((A_0+M)/\mathfrak{m}^n),$$
$$D_{f,a_0}(M)\otimes_{A_0}\widehat{A}_0 \to \varprojlim D_{f,a_0}(M/\mathfrak{m}^n M).$$

(4.1.iii) For every infinitesimal extension $(\widetilde{q}_0, a)$ over f with A a finite type $\mathcal{O}_S$-algebra, there is an open dense set of points of finite type $p \in \mathrm{Spec}\, A_0$ so that the following maps are isomorphisms,

$$\mathrm{Aut}_{f,a_0}(A_0+M)\otimes_{A_0} k(p) \to \mathrm{Aut}_{f,a_0}(k(p)+M\otimes_{A_0} k(p)),$$
$$D_{f,a_0}(M)\otimes_{A_0} k(p) \to D_{f,a_0}(M\otimes_{A_0} k(p)).$$

Lemma 2.5. *With notation as in Lemma 2.2 and Lemma 2.3, assume $\mathcal{X}$ and $\mathcal{Y}$ satisfy* [Art74, (S1,2)] *and assume automorphisms, respectively deformations, of $\mathcal{Y}$ satisfy* [Art74, (4.1)]. *Automorphisms, resp. deformations, of $\mathcal{X}$ satisfy* [Art74, (4.1)] *if and only if automorphisms, resp. deformations, of f satisfy 2.21. Also, if $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ are 1-morphisms of stacks in groupoids over (Aff/S) whose automorphisms, resp. deformations, satisfy* [Art74, (S1,2),(4.1)], *then also the automorphisms, resp. deformations, of $\mathcal{X}\times_{\mathcal{Y}}\mathcal{Z}$ satisfy* [Art74, (S1,2),(4.1)].

Proof. This also largely follows from [Ols04, §A.15]. The details are left to the reader. ☐

2.2. **Lifting properties.** Let $(B' \xrightarrow{q} B \xrightarrow{q_0} B_0)$ be an extension pair and let $(\widetilde{q}, \widetilde{q}_0, b_{\mathcal{X}})$ be a deformation situation over f mapping to (q, q_0). The deformation situation is *algebraic*, resp. *reduced*, *integral*, if B_0 is finite type, resp. reduced, integral.

Localization. The lifting properties involve localization. Let $q : B' \to B$ be an infinitesimal extension. Images and inverse images of multiplicative systems under q are again multiplicative. Let S' be a multiplicative subset of B' and let S'_q be the multiplicative subset $q^{-1}(q(S'))$. The localization of B' with respect to S' equals the localization of B' with respect to S'_q. Moreover, $S^{-1}B' \to q(S)^{-1}B$ is an infinitesimal extension, and the associated graded pieces of the kernel are the localizations $q(S)^{-1}(N^i/N^{i+1})$. Therefore, the localizations of B' are in 1-to-1 bijection with the localizations of B. Moreover, a localization of B is finitely generated if and only if the associated localization of B' is finitely generated.

Let $(B' \xrightarrow{\widetilde{q}_B} B \xrightarrow{\widetilde{q}_{B,0}} B_0, b_{\mathcal{X}})$ be an integral deformation situation. Let A_0 be the fraction field of B_0, and let A and A' be the associated localizations of B and B' respectively. Let $A' \xrightarrow{q_A} A \xrightarrow{q_{A,0}} A_0$ denote the associated extension pair. Denote by $(a'_{\mathcal{Y}} \xrightarrow{\widetilde{q}_A} a_{\mathcal{Y}} \xrightarrow{q_{A,0}} a_{\mathcal{Y},0}, a_{\mathcal{X}})$ the base change over A', etc., of the deformation situation $(\widetilde{q}_B, \widetilde{q}_{B,0}, b)$.

Definition 2.6. A *generic lift* of the deformation situation $(\widetilde{q}_B, \widetilde{q}_{B,0}, b_{\mathcal{X}})$ is a morphism $a'_{\mathcal{X}} \xrightarrow{r_A} a_{\mathcal{X}}$ in $\mathcal{X}$ over $\widetilde{q}_A$. An *integral lift* of the generic lift is a morphism $b'_{\mathcal{X},\mathrm{new}} \xrightarrow{r_B} b_{\mathcal{X}}$ in $\mathcal{X}$ satisfying the following conditions.

(i) The image of r_B in (Aff/S) is an infinitesimal extension $B'_{\mathrm{new}} \xrightarrow{q_{B,\mathrm{new}}} B$ sitting between B' and $q_A^{-1}(B)$.
(ii) The image $\widetilde{q}_{B,\mathrm{new}}$ of r_B in $\mathcal{Y}$ is the base-change of $\widetilde{q}_B$.

(iii) And the base-change of r_B by $(q_{B,\text{new}}, q_{B,0}) \to (q_A, q_{A,0})$ is r_A.

Pushouts of M. Given a deformation situation $(\widetilde{q}, \widetilde{q}_0, b_{\mathcal{X}})$ over f mapping to an extension pair $(B' \xrightarrow{q} B \xrightarrow{q_0} B_0)$, for every surjection of B_0-modules, $M \to N$, there is a surjection of $\mathcal{O}_S$-algebras $B' \xrightarrow{u} B'_N$ whose kernel is the kernel of $M \to N$. This gives a morphism of deformation situations,

$$(\widetilde{u}', \text{Id}, \text{Id}, \text{Id}) : (\widetilde{q}, \widetilde{q}_0, b_{\mathcal{X}}) \to (\widetilde{q}_N, \widetilde{q}_0, b_{\mathcal{X}}),$$

such that the image of $\widetilde{q}_N$ is the extension $B'_N \to B$ and the image of $\widetilde{u}'$ is u.

Let $\mathfrak{a}$ be a radical ideal in B_0. Let R denote the semilocalization of B_0 at the generic points of $\mathfrak{a}$. Assume M is a finite type $B_0/\mathfrak{a}$-module. A localization A_0 of B_0 is $\mathfrak{a}$-*generic* if Spec $A_0 \cap$ Spec $(B_0/\mathfrak{a})$ is dense in Spec $(B_0/\mathfrak{a})$, i.e., A_0 is isomorphic to a B_0-subalgebra of R. An $\mathfrak{a}$-*generic quotient* of M is a pair (A_0, N) of an $\mathfrak{a}$-generic localization A_0 and a surjection $M \otimes_{B_0} A_0 \to N$. It is *finite type* if A_0 is a finite type B_0-algebra. It is *projective* if N is a projective $A_0/\mathfrak{a}A_0$-module. It is *extending* if for the associated deformation situation $(\widetilde{q}_{A,N}, \widetilde{q}_{A,0}, a_{\mathcal{X}})$ over (A'_N, A, A_0), there is a lifting of $\widetilde{q}_{A,N}$ to $\mathcal{X}$.

Definition 2.7. Let $(\widetilde{q}, \widetilde{q}_0, b_{\mathcal{X}})$ be a deformation situation over f whose kernel M is a finite type $B_0/\mathfrak{a}$-module for a radical ideal $\mathfrak{a}$. A *generic extender* is a finite type, projective, extending, $\mathfrak{a}$-generic quotient (A_0, N) such that for every finite type, projective, $\mathfrak{a}$-generic quotient (C_0, P), the quotient $P \otimes_{C_0} R$ factors through the quotient $N \otimes_{A_0} R$.

A generic extender is *compatible with étale extension* if for every étale homomorphism $v : B' \to B'_{\text{ét}}$, the pair $(A_0 \otimes_{B_0} B_{\text{ét},0}, N \otimes_{B_0} B_{\text{ét},0})$ is a $v(\mathfrak{a})$-generic extender for the base change of the deformation situation to $B'_{\text{ét}}$.

A generic extender is *compatible with closed points* if for every closed point $A_0 \to k(y)$ of Spec A_0 and every surjection $M \otimes_{B_0} k(y) \to N_y$, there exists a lift $\widetilde{q}_{\mathcal{X},N_y}$ of the base change of $\widetilde{q}$ to A'_{N_y} if and only if N_y is a quotient of $N \otimes_{A_0} k(y)$.

Lemma 2.8. *Assume $\mathcal{X}$ and $\mathcal{Y}$ each satisfy* [Art74, (S1)]. *Then for every deformation situation and radical ideal $\mathfrak{a}$ there exists a generic extender.*

Proof. For every finite type, projective, extending, $\mathfrak{a}$-generic quotient (C_0, P), consider the quotient $M \otimes_{B_0} R \to P \otimes_{C_0} R$. This system of quotients has an inverse limit. Because $M \otimes_{B_0} R$ has finite length, the inverse limit is equal to the inverse limit of a finite subsystem. Thus, it suffices to prove the following. For every pair of finite type, projective, extending, $\mathfrak{a}$-generic quotients $(A_{0,1}, N_1)$ and $(A_{0,2}, N_2)$, there exists a finite type, projective, extending, $\mathfrak{a}$-generic quotient (A_0, N) such that both quotients $N_1 \otimes_{A_{0,1}} R$ and $N_2 \otimes_{A_{0,2}} R$ factor through $N \otimes_{A_0} R$.

Replace B_0 by the finite type, $\mathfrak{a}$-generic localization $A_{0,1} \otimes_{B_0} A_{0,2}$ and replace B and B' by the associated localizations. Consider the induced map $M \to N_1 \oplus N_2$. After a further finite type, $\mathfrak{a}$-generic localization, the cokernel P is a projective $B_0/\mathfrak{a}$-module. Denote by N the image of M in $N_1 \oplus N_2$. Of course B'_N equals the fiber product $B'_{N_1} \times_{B'_P} B'_{N_2}$.

Let D denote the functor $D_{f,\widetilde{q}_0,b_{\mathcal{X},0}}$. For $i = 1, 2$, let $\widetilde{q}_{\mathcal{X},i} : b_{\mathcal{X},i} \to b_{\mathcal{X}}$ be a lift to $\mathcal{X}$ of the base change of $\widetilde{q}$ over B'_{N_i}. The base change of $b_{\mathcal{X},i}$ over B'_P is a lift to $\mathcal{X}$ of the base change of $\widetilde{q}$ over B'_P. Therefore the 2 base changes differ by an element d in $D(P)$.

Because $N_1 \oplus N_2 \to P$ is a surjective map of $B_0/\mathfrak{a}$-modules whose image is a projective, it is split. Therefore $D(N_1 \oplus N_2) \to D(P)$ is surjective. So there exist elements d_i of $D(N_i)$ for $i = 1, 2$ such that d is the image of (d_1, d_2). After translating $b_{\mathcal{X},1}$, resp. $b_{\mathcal{X},2}$, by d_1, resp. d_2, the base changes to B'_P agree. Therefore by [Art74, (S1)], there exists an element $b_{\mathcal{X}}$ over B'_N whose base change to B'_{N_i} equals $b_{\mathcal{X},i}$ for $i = 1, 2$. □

2.3. **Lifting properties and compositions.** Let $e : \mathcal{W} \to \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of stacks in groupoids over (Aff/S).

Proposition 2.9. *If for both e and f generic lifts of integral, algebraic deformation situations have integral lifts, then the same holds for $f \circ e$.*

Proof. Given an integral, algebraic deformation situation $(\widetilde{q}_{\mathcal{Y},B}, \widetilde{q}_{\mathcal{Y},B,0}, b_{\mathcal{W}})$ over $f \circ e$ and a generic lift $a'_{\mathcal{W}} \to a_{\mathcal{W}}$, then $e(a'_{\mathcal{W}}) \to e(a_{\mathcal{W}})$ is a generic lift of the integral, algebraic deformation situation $(\widetilde{q}_{\mathcal{Y},B}, \widetilde{q}_{\mathcal{Y},B,0}, e(b_{\mathcal{W}}))$ over f. By hypothesis, this has an integral lift r_f. After replacing B', this integral lift gives a deformation situation $(r_f, \widetilde{q}_{\mathcal{X},B,0}, b_{\mathcal{W}})$ over e. And $a'_{\mathcal{W}} \to a_{\mathcal{W}}$ is a generic lift. By hypothesis, there exists an integral lift r_e. This is also an integral lift for the original generic lift over $f \circ e$. □

Proposition 2.10. *Assume each of $\mathcal{W}$, $\mathcal{X}$ and $\mathcal{Y}$ satisfy* [Art74, (S1,2)] *and their deformations satisfy* [Art74, (4.1.i),(4.1.iii)]. *If for both e and f there exist generic extenders of algebraic deformations compatible with étale extensions and with closed points, the same holds for $f \circ e$.*

Proof. Let $(\widetilde{q}_{\mathcal{Y}}, \widetilde{q}_{\mathcal{Y},0}, b_{\mathcal{W}})$ be a deformation situation over $f \circ e$ whose associated extension pair is $B' \xrightarrow{q} B \xrightarrow{q_0} B_0$. Assume the kernel M of q is a module over $B_0/\mathfrak{a}$ for a radical ideal $\mathfrak{a}$. By Lemma 2.8, there exists a generic extender $(A_0, N_{f\circ e})$ for the deformation situation over $f \circ e$. The problem is to prove the generic extender is compatible with étale extensions and closed points. The main issue is to find an appropriate finite type, $\mathfrak{a}$-generic localization of A_0.

First of all, by [Art74, (4.1.iii)], there is a finite type, $\mathfrak{a}$-generic localization of B_0 so that $D_f(M)$ and $D_{e\circ f}(M)$ are compatible with closed points. Replace B_0 by this localization and replace B and B' by the associated localizations. Denote by $\widetilde{q}_{\mathcal{W},N_{f\circ e}}$ a lift to $\mathcal{W}$ of the base change of $\widetilde{q}_{\mathcal{Y}}$ to $B'_{N_{f\circ e}}$. Replace B_0 by A_0 and replace B and B' by the associated localizations. By hypothesis there exists a generic extender (A_0, N_f) for f compatible with étale extensions and with closed points. Replace B_0 by A_0, etc. Let $b'_{\mathcal{X}} \xrightarrow{\widetilde{q}_{\mathcal{X}}} b_{\mathcal{X}}$ denote a lift to $\mathcal{X}$ of the base change of $\widetilde{q}_{\mathcal{Y}}$ to B'_{N_f}. Because the base change of $\widetilde{q}_{\mathcal{Y}}$ to $B'_{N_{f\circ e}}$ lifts to $e(\widetilde{q}_{\mathcal{W},N_{f\circ e}})$, $N_{f\circ e}$ is a quotient of N_f.

"Calibrate" the lifts $\widetilde{q}_{\mathcal{W},N_{f\circ e}}$ and $\widetilde{q}_{\mathcal{X}}$ as follows. Denote by $\widetilde{q}_{\mathcal{X},N_{f\circ e}}$ the base change of $\widetilde{q}_{\mathcal{X}}$ to $B'_{N_{f\circ e}}$. The difference of $e(\widetilde{q}_{\mathcal{W},N_{f\circ e}})$ and $\widetilde{q}_{\mathcal{X},N_{f\circ e}}$ is an element d of $D_{f,\widetilde{q}_{\mathcal{W},0},b_{\mathcal{X},0}}(N_{f\circ e})$. After a further finite type, $\mathfrak{a}$-generic localization, the surjection $N_f \to N_{f\circ e}$ is split. Therefore there exists an element d' of $D_{f,\widetilde{q}_{\mathcal{W},0},b_{\mathcal{X},0}}(N_f)$ mapping to d. Replace $\widetilde{q}_{\mathcal{X}}$ by its translation by d. Then $e(\widetilde{q}_{\mathcal{W},N_{f\circ e}})$ equals $\widetilde{q}_{\mathcal{X},N_{f\circ e}}$.

Claim 2.11. *It suffices to prove the proposition after replacing M by the quotient N_f, replacing B' by B'_{N_f}, etc.*

For every étale extension $B' \to B'_{\text{ét}}$ and every projective, extending quotient $M \otimes_{B_0} B_{\text{ét},0} \to N_{\text{ét}}$ for $f \circ e$ over $B'_{\text{ét}}$, also $N_{\text{ét}}$ is extending for f over $B'_{\text{ét}}$. By hypothesis, N_f is compatible with étale extension. Thus $M_{\text{ét}}$ is a quotient of N_f.

Similarly, for every closed point y of Spec B_0, and every surjection $M \otimes_{B_0} k(y) \to N_y$, if the base change of $\widetilde{q}_{\mathcal{Y}}$ to B'_{N_y} lifts to $\mathcal{W}$, then it also lifts to $\mathcal{X}$. By hypothesis, N_f is compatible with closed points. Thus N_y is a quotient of N_f. This proves Claim 2.11. Therefore, without loss of generality, replace M by the quotient N_f, replace B' by B'_{N_f}, etc.

Here is the problem: the lift $\widetilde{q}_{\mathcal{X}}$ is not unique. For every lift, by hypothesis, there is a finite type, $\mathfrak{a}$-generic localization and a generic extender over this localization which is compatible with etale extensions and finite points. However, the finite type localization would appear to depend on the choice of lift. As there may be infinitely many lifts, it is a priori possible the coproduct of these localizations is not finite type.

Here is the remedy. Let D denote the module $D_{f,\widetilde{q}_{\mathcal{Y},0},b_{\mathcal{X},0}}(B_0/\mathfrak{a})$. Localize so that D is a finite projective $B_0/\mathfrak{a}$-module. Denote the module $\text{Hom}_{B_0}(D, B_0/\mathfrak{a})$ by $D^\vee$. Then $D_{f,\widetilde{q}_{\mathcal{Y},0},b_{\mathcal{X},0}}(D^\vee)$ is canonically isomorphic to $\text{Hom}_{B_0}(D,D)$. Let β denote the object of $\mathcal{X}$ over $B_0 + D^\vee$ corresponding to Id_D. Of course $(B_0 + D^\vee) \times_{B_0} B'$ equals $B' + D^\vee$. By [Art74, (S1)], there exists an extension $b'_{\mathcal{X},D} \xrightarrow{\widetilde{q}_{\mathcal{X},D}} b_{\mathcal{X}}$ of $\widetilde{q}_{\mathcal{X}}$ over $B' + D^\vee$ such that the image over B' is $\widetilde{q}_{\mathcal{X}}$ and the image over $B_0 + D^\vee$ is β.

Here is the point: for every lift $b'_{\mathcal{X},1} \xrightarrow{\widetilde{q}_{\mathcal{X},1}} b_{\mathcal{X}}$ of $\widetilde{q}_{\mathcal{Y}}$, the difference of $b'_{\mathcal{X},1}$ and $b_{\mathcal{X},1}$ is an element of $D_{f,\widetilde{q}_{\mathcal{Y},0},b_{\mathcal{X},0}}(M)$, i.e., an element d of $\text{Hom}_{B_0}(D^\vee, M)$. There is a unique surjection of B'-algebras, $B' + D^\vee \to B'$ whose restriction to $D^\vee$ is $d : D^\vee \to M$. The image of $b'_{\mathcal{X},D}$ equals $b'_{\mathcal{X},1}$.

By hypothesis, for the ideal $\mathfrak{a}$ the deformation situation $(\widetilde{q}_{\mathcal{X},D}, \widetilde{q}_{\mathcal{X}}, b_{\mathcal{W}})$ over e has a generic extender (A_0, N_D) compatible with étale extensions and closed points. Let $D^\vee \to E$ be the fiber coproduct of $M \oplus D^\vee \to N_D$ and the projection $\text{pr}_{D^\vee} : M \oplus D^\vee \to D^\vee$. After a further finite type, $\mathfrak{a}$-generic localization, E is projective.

Claim 2.12. *The quotient $M \oplus D^\vee \to N_D$ is the direct sum of the quotient $M \to N_{f\circ e}$ and the quotient $D^\vee \to E$.*

The claim holds if and only if it holds after a finite type, $\mathfrak{a}$-generic localization.

Let K_D denote the kernel of the surjection $M \oplus D^\vee \to N_D$. Let $K_{f\circ e}$ denote the kernel of the surjection $M \to N_{f\circ e}$. The composition of $\text{pr}_M : M \oplus D^\vee \to M$ and $M \to N_{f\circ e}$ is extending. Therefore, after a further $\mathfrak{a}$-generic localization, it factors through N_D, i.e., the image $\text{pr}_M(K_D)$ is contained in $K_{f\circ e}$. On the other hand, the quotient $M/\text{pr}_M(K_D)$ is extending. After a finite type, $\mathfrak{a}$-generic localization, it is projective.

Because $M \to M/\text{pr}_M(K_D)$ is extending and locally free, after a further localization it is a quotient of $N_{f\circ e}$, i.e., $K_{f\circ e}$ is contained in $\text{pr}_M(K_D)$. Therefore $\text{pr}_M(K_D)$ equals $K_{f\circ e}$.

Denote by K'_D the kernel of the surjection $K_D \to K_{f\circ e}$. There is a projection $\text{pr}_{K,D^\vee} : K'_D \to D^\vee$. After a finite type, $\mathfrak{a}$-generic localization, the cokernel E is projective. The composition $K_D \xrightarrow{\text{pr}_{D^\vee}} D^\vee \to E$ factors through $K_D \to K_{f\circ e}$. Denote this morphism by $\delta : K_{f\circ e} \to E$. The claim is equivalent to the vanishing of δ.

By way of contradiction, assume δ is not zero. Then there exists a surjection $E \to Q$ such that $K_{f\circ e} \to Q$ is surjective and Q is a projective $B_0/\mathfrak{a}$-module of length 1. Denote by $K_{f\circ e,Q}$ the kernel of $K_{f\circ e} \to Q$. Take the quotient of M by $K_{f\circ e,Q}$. There is an induced isomorphism $K_{f\circ e}/K_{f\circ e,Q} \to Q$. Inverting this isomorphism and composing with the inclusion gives a map of $B_0/\mathfrak{a}$-modules, $i : Q \to M/K_{f\circ e,Q}$. Define $i' : Q \to M$ to be any $B_0/\mathfrak{a}$-morphism lifting i. Define $\phi : D^\vee \to M$ to be the composition of $D^\vee \to Q$ and the *negative* of i'. Define $(\mathrm{Id}_M, \phi) : M \oplus D^\vee \to M$ to be the induced surjection. The induced morphism $(\mathrm{Id}_M, \phi) : K_D \to M$ annihilates K'_D. So there is an induced map from $K_{f\circ e}$ to M. The restriction to $K_{f\circ e,Q}$ is the inclusion. So there is an induced map from $K_{f\circ e}/K_{f\circ e,Q}$ to $M/K_{f\circ e,Q}$. But, by construction, this is the zero map. Therefore $(\mathrm{Id}_M, \phi)(K_D)$ is strictly contained in $K_{f\circ e}$. The quotient $M \to M/(\mathrm{Id}_M, \phi)(K_D)$ is extending, but does not factor through $M \to N_{f\circ e}$. This contradiction proves that δ is zero, i.e., it proves Claim 2.12.

Claim 2.13. *After this sequence of localizations, the generic extender $M \to N_{f\circ e}$ for $f \circ e$ is compatible with étale extensions and with closed points.*

Étale extensions. By hypothesis, the generic extender $M \oplus D^\vee \to N_D$ is compatible with étale extensions. By Claim 2.13, this surjection is split. It follows that $M \to M_{f\circ e}$ is also compatible with étale extensions.

Closed points. Let $B_0 \to k(y)$ be a closed point, let $M \otimes_{B_0} k(y) \to N_y$ be a surjection, and let $\widetilde{q}_{\mathcal{W},y}$ be a lift to $\mathcal{W}$ of the base change of $\widetilde{q}_{\mathcal{Y}}$ to B'_{N_y}. Because $D_f(M)$ is compatible with closed points, there is a morphism $d : D^\vee \to M$ so that $e(\widetilde{q}_{\mathcal{W},y})$ is the base change of $\widetilde{q}_{\mathcal{X},D}$ associated to the composition of $(\mathrm{Id}_M, d) : M \oplus D^\vee \to M$ and $M \to N_y$. Because $M \oplus D^\vee \to N_D$ is compatible with closed points, N_y is a quotient of N_D, i.e., the kernel of $M \to N_y$ contains $(\mathrm{Id}_M, d)(K_D)$. Because K_D is the direct sum $K_{f\circ e} \oplus K'_D$, in particular $(\mathrm{Id}_M, d)(K_D)$ contains $(\mathrm{Id}_M, d)(K_{f\circ e} \oplus 0)$, i.e., $K_{f\circ e}$. Thus N_y factors through $M \to N_{f\circ e}$. Therefore $M \to N_{f\circ e}$ is compatible with closed points. $\square$

2.4. **Openness of versality.** Let $\mathcal{X}$ be a stack in groupoids over (Aff/S) for the étale topology.

Definition 2.14. The stack $\mathcal{X}$ satisfies *openness of versality* if for every finite type $\mathcal{O}_S$-algebra R and every object v of $\mathcal{X}(\mathrm{Spec}\ R)$, if v is formally versal at p, then there is an open neighborhood of p in Spec R on which v is formally smooth. Openness of versality is also *compatible with étale extensions* if for every étale morphism $e : \mathrm{Spec}\ R^* \to \mathrm{Spec}\ R$, for every point p^* of Spec R^*, if v is formally versal at $e(p^*)$, then e^*v is formally smooth on an open neighborhood of p^* in Spec R^*.

Theorem 2.15. [Art74, Proposition 4.3 and Theorem 4.4] *Assume $\mathcal{X}$ is limit preserving, deformations of $\mathcal{X}$ satisfy* [Art74, (S1,2), (4.1)], *every generic lift of an integral, algebraic deformation situation over $\mathcal{X}$ has an integral lift, and every algebraic deformation situation over $\mathcal{X}$ has a generic extender compatible with étale extensions and with closed points, cf. Definitions 2.6 and 2.7. Then $\mathcal{X}$ satisfies openness of versality compatibly with étale extensions.*

Remark 2.16. The hypotheses of Theorem 2.15 do not include existence or properties of an obstruction theory for $\mathcal{X}$. In practice, an obstruction theory is used to verify existence of integral lifts and generic extenders, etc.

Proof. The entire proof will not be repeated. We only indicate the modifications necessary to replace existence and properties of obstruction theory by the hypotheses above.

First of all, the proof of [Art74, Proposition 4.3] uses obstructions only at the end of the proof. In fact, obstructions are used precisely to prove existence of generic extenders compatibly with étale extensions.

Next, in the proof of [Art74, Theorem 4.4], obstructions are used to prove formal versality is stable under generization. The arguments on [Art74, p. 176] use obstructions precisely to deduce existence of an integral lift b' of the generic lift a' of the deformation situation $(B' \xrightarrow{q} B \xrightarrow{q_0} B_0, b)$. By [Art74, Theorem 3.3], formal versality of v at p implies versality of v at p. Thus, there is a morphism from R to the Henselization of the localization of B' at p inducing a lift from v to the "Henselization" of b' at p, compatibly with the map from v to the "Henselization" of b at p. Because $\mathcal{X}$ is limit preserving, this morphism factors through an étale extension $B' \to B'_{\text{new}}$. Replace the original deformation situation by the base change by this étale extension.

Because the base change of v by $R \to B'$ agrees with b' after localizing at p, and by the same sort of arguments as above, after replacing B' by a subring $B'[t^{-1}M]$ of A', the base change of v equals b'. So the composition of $R \to B'$ with the localization $B' \to A'$ gives a morphism $v \to a'$ making the diagram commute. Therefore v is formally versal at the point x.

The final use of obstructions in the proof of [Art74, Theorem 4.4] is at the bottom of p. 177 and the top of p. 178. Here obstructions are used to construct a generic extender (A_0, N) compatibly with closed points. Replace B_0 by A_0, and replace B, B' by the associated localizations. By the hypothesis on $\mathcal{S}$, the quotient N has nonzero fiber $N \otimes_{B_0} k(y)$ for a dense set of points. Therefore the generic fiber of N is nonzero. Let $M \xrightarrow{\phi} B_0$ be any surjection factoring through $M \to N$. Denote by $B' \to B^*$ the associated surjection. This has the desired property. The rest of the proof goes through precisely as in [Art74]; there is no further use of obstructions. $\square$

Let there be given a sequence of 1-morphisms of stacks in groupoids over (Aff/S),

$$\mathcal{X}_n \xrightarrow{f^n_{n-1}} \mathcal{X}_{n-1} \xrightarrow{f^{n-1}_{n-2}} \dots \xrightarrow{f^2_1} \mathcal{X}_1 \xrightarrow{f^1_0} (\text{Aff}/S).$$

Assume each $\mathcal{X}_i$ satisfies [Art74, (S1,2)].

Corollary 2.17. *For every $i = 1, \dots, n$, assume $\mathcal{X}_i$ is limit preserving. Also, for every $i = 1, \dots, n$, assume f^i_{i-1} satisfies the relative version of* [Art74, (S1,2)] *and the conditions of Notation 2.4. Finally, for every $i = 1, \dots, n$, assume every generic lift of an integral, algebraic deformation situation over f^i_{i-1} has an integral lift, and every algebraic deformation situation over f^i_{i-1} has a generic extender compatible with étale extensions and with closed points. Then every $\mathcal{X}_i$ satisfies openness of versality compatibly with étale extensions.*

Proof. This follows immediately from Lemmas 2.2, 2.3, 2.5 and Propositions 2.9, 2.10. $\square$

2.5. **Artin's representability theorems.** Artin's approximation theorem has been generalized by Conrad and de Jong, [CdJ02]. This gives the following version of Artin's representability theorem.

Corollary 2.18. [Art74, Corollary 5.2], [LMB00, Corollaire 10.8], [CdJ02, Theorem 1.5] *Assume S is an excellent scheme. Let $\mathcal{X}$ be a limit preserving stack in groupoids over (Aff/S) with the étale topology. Then $\mathcal{X}$ is an algebraic stack if*

(1) *$\mathcal{X}$ is relatively representable, i.e., the diagonal 1-morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by locally finitely presented algebraic spaces.*
(2) *The Schlessinger-Rim criteria* [Art74, (S1,2)] *hold.*
(3) *If $\widehat{A}$ is a complete local $\mathcal{O}_S$-algebra with residue field of finite type over S, then $\mathcal{X}(Spec\ \widehat{A}) \to \varprojlim \mathcal{X}(Spec\ \widehat{A}/\mathfrak{m}^n)$ has a dense image.*
(4) *The stack $\mathcal{X}$ satisfies openness of versality compatibly with étale extensions.*

Applying Theorem 2.15, this gives the following version.

Theorem 2.19. [Art74, Theorem 5.3] *Assume S is an excellent scheme. Let $\mathcal{X}$ be a limit preserving stack in groupoids over (Aff/S) with the étale topology. Then $\mathcal{X}$ is an algebraic stack if*

(1) *The Schlessinger-Rim criteria* [Art74, (S1,2)] *hold. Also, for every algebraic element a_0 of $\mathcal{X}(Spec\ A_0)$ and every finite A_0-module M, $Aut_{a_0}(A_0 + M)$ is a finite A_0-module.*
(2) *For any complete local $\mathcal{O}_S$-algebra $\widehat{A}$ with residue field of finite type over S, the canonical map*

$$\mathcal{X}(Spec\ \widehat{A}) \to \varprojlim \mathcal{S}(Spec\ \widehat{A}/\mathfrak{m}^n)$$

is faithful, and has a dense image, i.e., the projection to $\mathcal{X}(Spec\ \widehat{A}/\mathfrak{m}^n)$ is essentially surjective for every n.
(3) *D and $Aut_{a_0}(A_0 + M)$ satisfy* [Art74, (4.1)], *every generic lift of an integral, algebraic deformation situation over $\mathcal{X}$ has an integral lift, and every algebraic deformation situation over $\mathcal{X}$ has a generic extender compatible with étale extensions and with closed points.*
(4) *If the object a_0 of $\mathcal{X}(Spec\ A_0)$ is algebraic, and if ϕ is an automorphism of a_0 inducing the identity in $\mathcal{X}(Spec\ k(y))$ for a dense set of finite type points y of $Spec\ A_0$, then ϕ equals Id_{a_0} on a non-empty open subset of $Spec\ A_0$.*

2.6. **Relative obstruction theory.** Relative obstruction theories are studied in [Ols04, Appendix A]. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism between stacks in groupoids over (Aff/S) both satisfying Schlessinger's conditions [Art74, (S1,2)].

Definition 2.20. [Ols04, Def. A.10] A *relative obstruction theory for f* consists of the following:

(i) an assignment to each reduced infinitesimal extension $(\widetilde{q}_0, a)$ over f of an A_0-linear functor,

$$O_{\widetilde{q}_0,a} : (\text{finite } A_0 - \text{modules}) \to (\text{finite } A_0 - \text{modules}),$$

(ii) an assignment to each morphism $(\widetilde{u}, \widetilde{u}_0, \phi)$ of infinitesimal extensions over f of a natural transformation of A_0-linear functors

$$\theta_{\widetilde{u},\widetilde{u}_0,\phi} : O_{\widetilde{q}_{A,0},a}(M) \otimes_{A_0} B_0 \Rightarrow O_{\widetilde{q}_{B,0},b}(M \otimes_{A_0} B_0),$$

(iii) and an assignment to each deformation situation $(\widetilde{q}, \widetilde{q}_0, a)$ over f of an element $o_{\widetilde{q},\widetilde{q}_0,a}$ of $O_{\widetilde{q},a}(M)$

satisfying the following axioms.

(i) For every deformation situation $(\widetilde{q}, \widetilde{q}_0, a)$ over f, $o_{\widetilde{q},\widetilde{q}_0,a}$ equals 0 if and only if there exists a morphism $a' \to a$ in $\mathcal{X}$ mapping to $\widetilde{q}$.

(ii) For every morphism $(\widetilde{u}', \widetilde{u}, \widetilde{u}_0, \phi)$ of extension pairs over f, $\theta_{\widetilde{u},\widetilde{u}_0,\phi}(o_{\widetilde{q}_A,\widetilde{q}_{A,0},a})$ equals $o_{\widetilde{q}_B,\widetilde{q}_{B,0},b}$.

Notation 2.21. [Art74, (4.1)], [Ols04, §A.11] As in Notation 2.4, there are relative analogues of the conditions on obstruction theory.

(4.1.i) The functor O_f is compatible with étale localization: For every morphism $(\widetilde{u}, \widetilde{u}_0, \phi)$ of infinitesimal extensions over f, if A_0 is a finite-type $\mathcal{O}_S$-algebra and u is étale, then the following associated natural transformation of functors is an isomorphism,

$$O_{f,\widetilde{q}_{A,0},a_0}(M) \otimes_{A_0} B_0 \to O_{f,\widetilde{q}_{B,0},b_0}(M \otimes_{A_0} B_0).$$

(4.1.iii) For every infinitesimal extension $(\widetilde{q}_0, a)$ over f with A a finite type $\mathcal{O}_S$-algebra, there is an open dense set of points of finite type $p \in \text{Spec } A_0$ so that the following map is injective

$$O_{\widetilde{q}_0,a}(M) \otimes_{A_0} k(p) \to O_{\widetilde{q}_0,a}(M \otimes_{A_0} k(p)).$$

2.7. **Obstructions and lifting properties.** Properties of obstructions imply the hypotheses of Theorem 2.15.

Lemma 2.22. [Art74, Lemma 4.6] *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over (Aff/S). Assume both satisfy* [Art74, (S1,2)]. *Let O_f be a relative obstruction theory for f. If $\mathcal{X}$ is limit preserving, if D_f satisfies (4.1.i) of Notation 2.4 and if O_f satisfies (4.1.i) of Notation 2.21, then every generic lift of an integral, algebraic deformation situation over f has an integral lift.*

Proof. Denote by M the kernel of $\widetilde{q}_B$. The obstruction to finding a morphism r_B over $\widetilde{q}_B$ is an element o of the obstruction group $O = O_{f,\widetilde{q}_{B,0},b_{\mathcal{X}}}(M)$. By hypothesis, there is a lift after localizing to the total rings of fractions. These localizations are limits of finite type localizations. Because $\mathcal{X}$ is limit preserving, there exists a lift after a finite type localization. Thus, the image of o in the obstruction group of this finite type localization is zero. Because a finite type localization is an étale morphism, (4.1.i) of Notation 2.21 implies the obstruction group of the localization is the localization of the obstruction group. Therefore, the image of o in $O \otimes_{B_0} A_0$ is zero. So there exists a nonzero element t of B_0 such that $t \cdot o$ is zero in O.

Let B'_{new} be the subring of A' generated by B' and by $t^{-1}M$ inside $M \otimes_{B_0} A_0$, i.e., the kernel of q_A. Denote by $q_{B,\text{new}}$ the restriction of q_A to B'_{new} considered as a morphism with target B. The infinitesimal extension $B'_{\text{new}} \xrightarrow{q_{B,\text{new}}} B$ gives an deformation situation $(q_{B,\text{new}}, q_{B,0})$ which is the image of an obvious morphism from $(q_B, q_{B,0})$. The base change of $(\widetilde{q}_B, \widetilde{q}_{B,0}, b_{\mathcal{X}})$ by this morphism is an deformation situation $(\widetilde{q}_{B,\text{new}}, \widetilde{q}_{B,0}, b_{\mathcal{X}})$ over f. By Axiom (ii) of Definition 2.20, the obstruction o_{new} of this deformation situation is the image of o in $O_{f,\widetilde{q}_{B,0},b_{\mathcal{X}}}(t^{-1}M)$. There is an isomorphism $t^{-1}M \to M$ sending $t^{-1}m$ to m. This defines an isomorphism $O_{f,\widetilde{q}_{B,0},b_{\mathcal{X}}}(t^{-1}M) \to O_{f,\widetilde{q}_{B,0},b_{mcX}}(M)$, and the image of o_{new} is exactly $t \cdot 0$, which is zero. Therefore o_{new} is zero. By Axiom (i) of Definition 2.20, there is a morphism r^*_B of $\mathcal{X}$ over $\widetilde{q}_{B,\text{new}}$.

Replace B' by B'_{new}, etc. By construction, r^*_B satisfies Axioms (i) and (ii) of an integral lift. The base change of r^*_B over $A' \to A$ is a morphism r^*_A over $\widetilde{q}_A$. By [Art74, (S1,2)], there exists an element δ of $D_{f,a_{\mathcal{X}},0}(M \otimes_{B_0} A_0)$ such that $\delta \cdot r^*_A$ equals r_A. By the same type of argument as above, using (4.1.i) of Notation 2.4 for D_f and that $\mathcal{X}$ is limit preserving, there exists a nonzero element t of A_0 and an element d of $D_{f,b_{\mathcal{X}},0}(M)$ such that $t\delta = d$. Thus, after replacing B' by an extension

$B' \to B'_{\text{new}}$ as above and replacing r_B^* by its base change over B'_{new}, there exists an element d of $D_{f,b_{\mathcal{X}},0}(M)$ such that r_A equals $d \cdot r_A^*$. Define r_B to be $d \cdot r_B^*$. This is an integral lift of r_A. $\square$

Lemma 2.23. [Art74, Proof of Theorem 4.4] *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over (Aff/S). Assume both satisfy* [Art74, (S1,2)]. *Let O_f be a relative obstruction theory for f. If O_f satisfies (4.1.i) and (4.1.iii) of Notation 2.21, then every algebraic deformation situation over f has a generic extender compatible with étale extension and with closed points.*

Proof. Because of (4.1.i), we may replace B_0 by any finite type localization (and replace B and B' by the associated localizations). In particular, assume M is a finite projective B_0-module. Similarly, assume $P = O_{f,\widetilde{q}_0,b_{\mathcal{X}}}(B_0)$ is a finite projective B_0-module.

By B_0-linearity of O_f, $O_{f,\widetilde{q}_0,b_{\mathcal{X}}}(M)$ is canonically isomorphic to $M \otimes_{B_0} P$. Moreover, for every surjection $M \to M'$, the map $M \otimes_{B_0} P \to O_{b,\widetilde{q}_0,b_{\mathcal{X}}}(M')$ factors through the surjection $M \otimes_{B_0} P \to M' \otimes_{B_0} P$. By (4.1.iii), after replacing B_0 by a further finite type localization, the induced map $M' \otimes_{B_0} P \to O_{f,\widetilde{q}_0,b_{\mathcal{X}}}(M')$ is injective whenever M' is a direct sum of copies of $k(y)$ for y a closed point.

The obstruction class o associated to $(\widetilde{q}, \widetilde{q}_0, b_{\mathcal{X}})$ gives a B_0-linear map, $B_0 \to M \otimes_{B_0} P$. By adjunction, this is equivalent to a B_0-linear map, $\mathrm{Hom}_{B_0}(P, B_0) \to M$. Define $M \to N$ to be the cokernel of this map. For every surjection $M \to N'$, the image of o in $N' \otimes_{B_0} P$ is zero if and only if N' is a quotient of N. Because the image of o in $N \otimes_{B_0} P$ is zero, the image is also zero in $O_{f,\widetilde{q}_0,b_{\mathcal{X}}}(N)$. Therefore there exists a morphism $b_{\mathcal{X},N} \to b_{\mathcal{X}}$ in $\mathcal{X}$ mapping to $\widetilde{q}_N$. So N is a generic extender.

For a quotient M' of $M \otimes_{B_0} k(y)$. the induced map $M' \otimes_{B_0} P \to O_{f,\widetilde{q}_0,b_{\mathcal{X}}}(M')$ is injective. Therefore the image of o in $O_{f,\widetilde{q}_0,b_{\mathcal{X}}}(M')$ is zero if and only if M' is a quotient of N. Thus N is compatible with closed points.

Finally, compatibility with étale extensions follows from (4.1.i). $\square$

Corollary 2.24. *Assume $\mathcal{Y}$ is limit preserving and satisfies* [Art74, (S1,2)] *and f is representable by limit preserving algebraic stacks. Then $\mathcal{X}$ is limit preserving and satisfies* [Art74, (S1,2)]. *Also every generic lift of an integral, algebraic deformation situation over f has an integral lift, and every algebraic deformation situation over f has a generic extender compatible with étale extension and with closed points. Finally, if $\mathcal{Y}$ is relatively representable, then $\mathcal{X}$ is relatively representable.*

Proof. Let A equal $\lim A_i$ and let $a_{\mathcal{X}}$ be an object of $\mathcal{X}(\text{Spec } A)$. Denote by $a_{\mathcal{Y}}$ the image $f(a_{\mathcal{X}})$. Because $\mathcal{Y}$ is limit preserving, there exists an i and an object $a_{\mathcal{Y},i}$ of $\mathcal{Y}(\text{Spec } A_i)$ whose base change is isomorphic to $a_{\mathcal{Y}}$. Form the 2-fibered product,

$$\mathcal{X}_{a_{\mathcal{Y}},i} = \mathcal{X} \times_{f,\mathcal{Y},a_{\mathcal{Y},i}} \text{Spec } A_i.$$

By hypothesis, this is a limit preserving. And $a_{\mathcal{X}}$ gives an object of $\mathcal{X}_{a_{\mathcal{Y}},i}(\text{Spec } A)$. Thus there exists a j and an object $a_{\mathcal{X},j}$ of $\mathcal{X}_{a_{\mathcal{Y}},i}(\text{Spec } A_j)$ whose base change is isomorphic to $a_{\mathcal{X}}$. Therefore $\mathcal{X}$ is limit preserving. By a similar argument, $\mathcal{X}$ satisfies [Art74, (S1,2)].

Given an algebraic deformation situation over f, $(\widetilde{q}, \widetilde{q}_0, b_{\mathcal{X}})$, the 2-fibered product

$$\mathcal{X}_{b'_{\mathcal{Y}}} = \mathcal{X} \times_{f,\mathcal{Y},b'_{\mathcal{Y}}} \text{Spec } B'$$

is a limit preserving algebraic stack. Every base change of this deformation situation over f is the projection of a deformation situation over $\mathcal{X}_{b'_{\mathcal{Y}}}$. Thus it suffices to prove algebraic deformation situations over $\mathcal{X}_{b'_{\mathcal{Y}}}$ have integral lifts and generic extenders, etc.

By [Art74, p. 182], [Ols05a, Remark 1.7], $\mathcal{X}_{b'_{\mathcal{Y}}}$ has an obstruction theory such that [Art74, (S1,2), (4.1)] are satisfied for automorphisms, deformations and obstructions. By Lemmas 2.22 and 2.23, algebraic deformation situations over $\mathcal{X}_{b'_{\mathcal{Y}}}$ have integral lifts, generic extenders, etc.

Finally, if $\mathcal{Y}$ is relatively representable, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable. Since f is relatively representable, $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable. Therefore the composition $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, i.e., $\mathcal{X}$ is relatively representable. $\square$

3. The stack of algebraic spaces

Denote by $\mathcal{X}$ the category whose objects are all pairs (U, X) of an affine scheme U and a U-algebraic space X, and whose morphisms are all pairs,

$$(f_U, f_X) : (U', X') \to (U'', X''),$$

of a morphism $f_U : U' \to U''$ of affine schemes and an isomorphism of U'-algebraic spaces, $f_X : X' \to U' \times_{U''} X''$. The identity morphisms and the composition law are the obvious ones. There is a functor $\mathcal{X} \to (\text{Aff}/\text{Spec } \mathbb{Z})$. This functor is a stack for the étale topology, and even the fpqc topology on (Aff), cf. [LMB00, (3.4.6), (9.4)].

Claim 3.1. *The diagonal 1-morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is not representable.*

Proof. Let k be any field. Consider the 1-morphism $\zeta : \text{Spec }(k) \to \mathcal{X}$ associated to the object $(\text{Spec }(k), \mathbb{A}^1_k)$. There is an associated 1-morphism $(\zeta, \zeta) : \text{Spec }(k) \to \mathcal{X} \times \mathcal{X}$. The 2-fibered product of (ζ, ζ) and Δ is the stack associated to the contravariant functor I on $(\text{Aff}/\text{Spec }(k))$ associating to every k-algebra A the set

$$I(A) := \text{Isom}_A(A[t], A[t]).$$

The claim is that I is not an algebraic space.

Consider the subset,

$$T = \{\phi \in I(k[\epsilon]/\epsilon^2) | \phi \equiv \text{Id}(\text{mod}\epsilon)\}.$$

By direct computation, there is an isomorphism from the k-vector space $k[t]$ to T via $f \mapsto (t \mapsto t + \epsilon f)$. The dimension of $k[t]$ as a k-vector space is countably infinite. Therefore the dimension of T as a k-vector space is countably infinite.

On the other hand, for every algebraic space J over k, and every k-point p of J, the set,

$$T_{J,p} = \{\phi \in J(k[\epsilon]/\epsilon^2) | \phi \equiv p(\text{mod}\epsilon)\},$$

is isomorphic to the k-vector space,

$$\text{Hom}_k(\Omega, k),$$

where Ω is the k-vector space $\Omega_{J/k}/\mathfrak{m}_p\Omega_{J/k}$. If T' is infinite-dimensional, then Ω is also infinite-dimensional. For an infinite-dimensional k-vector space Ω, the dimension of $T' = \text{Hom}_k(\Omega, k)$ is uncountable. Since the dimension of T is countably infinite, I is not an algebraic space over k. $\square$

Denote by $\mathcal{X}_{\text{pr,f,lpf}}$ the category whose objects are all pairs (U, X) of an affine scheme U and a proper, flat, locally finitely presented U-algebraic space. This is a full subcategory of $\mathcal{X}$. The restriction of F is a functor $F : \mathcal{X}_{\text{pr,f,lpf}} \to (\text{Aff})$. As with $\mathcal{X}$, $\mathcal{X}_{\text{pr,f,lpf}}$ is a stack for the étale and fpqc topology.

Proposition 3.2. *The stack $\mathcal{X}_{pr,f,lpf}$ is limit preserving.*

Proof. This follows from results in [Gro67, §8]. □

Proposition 3.3. *The diagonal morphism $\Delta : \mathcal{X}_{pr,f,lpf} \to \mathcal{X}_{pr,f,lpf} \times \mathcal{X}_{pr,f,lpf}$ is representable, separated and locally finitely presented.*

Proof. This follows from [Art69, Theorem 6.1]. □

Denote by $\pi : \mathcal{V}_{\text{pr,f,lpf}} \to \mathcal{X}_{\text{pr,f,lpf}}$ the universal 1-morphism representable by proper, flat, locally finitely presented algebraic spaces.

Proposition 3.4. *Deformations and automorphisms of $\mathcal{X}_{pr,f,lpf}$ satisfy* [Art74, (S1,2), (4.1)]. *There is an obstruction theory $\mathcal{O}$ for $\mathcal{X}_{pr,f,lpf}$ satisfying* [Art74, (4.1)].

Proof. The existence of an obstruction theory satisfying [Art74, (S1)] follows from [Ill71, Proposition III.2.1.2.3] using the relative cotangent complex of π. The condition [Art74, (S2)] for deformations and the analogous condition for automorphisms follow from coherence of the cohomology sheaves of the cotangent complex [Ill71, Corollaire II.2.3.7] together with the finiteness theorem, [BGI71, Théorème III.2.2].

Condition [Art74, (4.1.i)] follows from [Ill71, II.1.2.3.5] and standard results about cohomology and flat base change. Condition [Art74, (4.1.ii)] follows from the theorem on formal functions, [Knu71, Theorem V.3.1]. Condition [Art74, (4.1.iii)] follows from generic flatness, [Gro67, Théorème 6.9.1], and the semicontinuity theorem [Gro63, Théorème 7.7.5]. □

However $\mathcal{X}_{\text{pr,f,lpf}}$ does not satisfy Axiom 3 of [Art74, Corollary 5.2].

Claim 3.5. *There does not exist a pair (Q, f) of an algebraic space Q and a representable, faithful, smooth 1-morphism $f : Q \to \mathcal{X}_{pr,f,lpf}$. In fact, $\mathcal{X}_{pr,f,lpf}$ does not satisfy Axiom 3 of* [Art74, Corollary 5.2].

Proof. If $\mathcal{X}_{\text{pr,f,lpf}}$ did satisfy Axiom 3, then every proper algebraic space over a field would have an effective, formal, versal deformation. By [Art69, Theorem 1.6], every proper algebraic space over a field does have a formal, versal deformation. However, it is not always effective. The following example of a projective, smooth variety in characteristic 0 with no effective, formal, versal deformation is well-known.

Let k be an uncountable, characteristic 0, algebraically closed field. Let X be a smooth anticanonical divisor in $\mathbb{P}^1 \times \mathbb{P}^2$. This is a K3 surface together with an elliptic fibration $\text{pr}_{\mathbb{P}^1} : X \to \mathbb{P}^1$.

Because the Schlessinger-Rim criteria [Art74, (S1,2)] hold for $\mathcal{X}$, there is a complete, local k-algebra R and a formal, versal deformation $(X_n)_{n\geq 0}$ of X over R. Because K3 surfaces are unobstructed, R is formally smooth, i.e., R is a power series ring. Also there is a canonical isomorphism,

$$\text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \cong H^1(X, T_X).$$

For every invertible sheaf $\mathcal{L}$ on X, there is an associated first Chern class $C_1(\mathcal{L})$ in $H^1(X, \Omega_X)$. This defines an injective group homomorphism,

$$C_1 : N^1(X) \to H^1(X, \Omega_X),$$

where $N^1(X)$ is the group of numerical equivalence classes of invertible sheaves.

There is a cup-product pairing,

$$H^1(X, \Omega_X) \times H^1(X, T_X) \to H^2(X, \mathcal{O}_X).$$

Because X is a K3 surface, or equivalently by adjunction for the inclusion of X in $\mathbb{P}^1 \times \mathbb{P}^2$, there exists an isomorphism of the dualizing sheaf ω_X with $\mathcal{O}_X$. Using this isomorphism, the cup-product pairing above is equivalent to the pairing for Serre duality, which is a perfect pairing.

Let $\mathcal{L}$ be an invertible sheaf on X and let $\theta : \mathfrak{m}/\mathfrak{m}^2 \to k$ be an element in $H^1(X, T_X)$, considered as a k-algebra homomorphism $R/\mathfrak{m}^2 \to k[\epsilon]/\epsilon^2$. Denote by X_θ the Abelian scheme over $k[\epsilon]/\epsilon^2$ given by $\theta(X_2)$. The invertible sheaf $\mathcal{L}$ is the restriction of an invertible sheaf on X_θ only if $C_1(\mathcal{L}) \cup \theta$ is 0 in $H^2(X, T_X)$. If $\mathcal{L}$ is not numerically trivial, then $C_1(\mathcal{L})$ is nonzero. Then, because the cup product pairing is nondegenerate, set of θ for which $\mathcal{L}$ extends over X_θ is a proper k-subspace $V_{\mathcal{L}}$ of $H^1(X, T_X)$.

By the theorem of the base, the group of $N^1(X)$ of numerical equivalence classes of invertible sheaves is a finitely generated Abelian group. Therefore, the set of subspaces $V_{\mathcal{L}}$ arising from invertible sheaves as above is countable. Because k is uncountable, there exists an element θ of $H^1(X, T_X)$ contained in none of the countably many proper subspaces $V_{\mathcal{L}}$. Therefore every invertible sheaf on X_θ is numerically trivial.

Because R is a power series ring, there is a local k-algebra homomorphism $t : R \to k[\![\epsilon]\!]$ whose associated map $\mathfrak{m}/\mathfrak{m}^2 \to k$ is θ. If $(R/\mathfrak{m}^n, X_n)$ comes from a proper algebraic space X_R over Spec (R), then $t(X_R)$ is a proper, flat algebraic space over Spec $k[\![\epsilon]\!]$. By [Knu71, Corollary 5.20], there is a dense open subspace U of $t(X_R)$ which is an affine scheme. Denote by D' the complement of U in $t(X_R)$. Denote by D the closure in $t(X_R)$ of the generic fiber of $D' \to \text{Spec } k[\![\epsilon]\!]$. Since the closed fiber of $t(X_R)$ is smooth, $t(X_R)$ is smooth over Spec $k[\![\epsilon]\!]$. Also, $t(X_R)$ is separated. Therefore, essentially by [Gro63, Théorème 5.10.5], D is a Cartier divisor in X_R. Moreover, it is flat over Spec $k[\![\epsilon]\!]$. Denote by $\mathcal{L}$ the associated invertible sheaf. By the paragraph above, the restriction of $\mathcal{L}$ to X is numerically trivial.

Because the generic fiber of $t(X_R)$ is a proper, positive-dimensional algebraic space, U is not all of the generic fiber, i.e., the generic fiber of D' is not empty. Because D is proper over Spec $k[\![\epsilon]\!]$ and its image contains the generic point, the closed fiber of D is not empty. Because D is flat over Spec $k[\![\epsilon]\!]$, the closed fiber of D is not all of X. Thus it is a nonempty curve in X. But no nonempty curve in X is numerically trivial: it has positive intersection number with either the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ or the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. This contradicts that the restriction of $\mathcal{L}$ to X is numerically trivial. The contradiction proves there exists no proper algebraic space X_R giving the formal, versal deformation of X. □

Remark 3.6. The counterexample also proves there is no effective, formal, versal deformation of the pair $(X, \text{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1))$. If there were, then the argument above would prove there is a proper, flat algebraic space $t(X_R)$ over Spec $k[\![\epsilon]\!]$ and an invertible sheaf $\mathcal{M}$ on $t(X_R)$ restricting to $\text{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1)$ such that the restriction to X of every invertible sheaf on $t(X_R)$ is numerically equivalent to $\text{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(d)$ for some d. Because $h^1(X, \text{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1))$ is zero, every global section of $\text{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1)$ is the restriction of a global section of $\mathcal{M}$. Because $\text{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1)$ is generated by global sections, there exists a global section of $\mathcal{M}$ that does not vanish identically on D. The generic fiber of the zero locus of this section is a nonempty curve not contained in D. Because it is proper, it is also not contained in U. Thus it has positive intersection number with D. On the other hand, $C_1(\text{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1))^2$ is zero.

This proves the restriction of $\mathcal{L}$ is not numerically equivalent to $\mathrm{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1)$. The contradiction proves there is no effective, versal, deformation of $(X, \mathrm{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(1))$.

The following is also noteworthy.

Claim 3.7. *The diagonal morphism is not quasi-compact.*

Proof. Let k be an algebraically closed field and let $(E, 0)$ be an elliptic curve over Spec (k) whose automorphism group is $\mathbb{Z}/2\mathbb{Z}$. Let $X = E \times E$. For the object (Spec $(k), X)$, the fiber in the diagonal is the scheme $I = \mathrm{Isom}_k(X, X)$. The scheme I surjects smoothly to X by associating to each map f the image $f(0)$. The kernel is isomorphic to the discrete, non-quasi-compact k-group scheme $\mathrm{GL}_2(\mathbb{Z})$ by associating to f the unique matrix

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right),$$

in $\mathrm{GL}_2(\mathbb{Z})$ such that $f(x, y) = (ax + by, cx + dy)$. □

4. Variants of the stack of algebraic spaces

As above, denote by $\pi : \mathcal{V}_{\mathrm{pr,f,lpf}} \to \mathcal{X}_{\mathrm{pr,f,lpf}}$ the universal 1-morphism representable by proper, flat, locally finitely presented algebraic spaces. Denote by Coh_π the category whose objects consist of triples (U, X, E) of an object (U, X) of $\mathcal{X}_{\mathrm{pr,f,lpf}}$ and a locally finitely presented $\mathcal{O}_X$-module E which is $\mathcal{O}_U$-flat, cf. [LMB00, 2.4.4]. Morphisms in Coh_π are data (f_U, f_X, f_E) of a morphism (f_U, f_X) in $\mathcal{X}_{\mathrm{pr,f,lpf}}$ and an isomorphism f_E from E' to the pullback of E''. There is a forgetful functor $G : \mathrm{Coh}_\pi \to \mathcal{X}_{\mathrm{pr,f,lpf}}$.

Proposition 4.1. [LMB00, Théorème 4.6.2.1] *The category Coh_π is a stack for the étale and fpqc topologies on (Aff). Moreover, the functor G is representable by limit preserving algebraic stacks with quasi-compact, separated diagonal.*

Proof. By the proof of the first half of [LMB00, Théorème 4.6.2.1], the diagonal of $\mathrm{Coh}_{X/U}$ is representable by separated, finitely presented algebraic spaces. Note the first half of the proof uses only properness of X/U and does not use cohomological flatness in dimension 0. To be absolutely precise, one has to generalize [Gro63, Corollaire 7.7.8] to the case when $f : X \to Y$ is a proper morphism of algebraic spaces. Using [Art74], it is clear how to do this (perhaps [Gro63, Remarques 7.7.9.iii] anticipates this generalization).

The proof that $\mathrm{Coh}_{X/U}$ is a locally finitely presented Artin stack uses [Art74, Theorem 5.3]. Existence of an obstruction theory for $\mathrm{Coh}_{X/U}$ such that automorphisms, deformations and obstructions satisfy [Art74, (S1,2), (4.1)] is just as in Proposition 3.4, replacing [Ill71, Proposition III.2.1.2.3] by [Ill71, Proposition IV.3.1.5]. Finally, compatibility with completions follows from the Grothendieck existence theorem, [Knu71, Theorem V.6.3]. □

Denote by $\mathcal{X}_{\mathrm{pol}}$ the full subcategory of Coh_π of triples (U, X, L) where L is an invertible sheaf on X ample relative to U.

Proposition 4.2. *The category $\mathcal{X}_{pol}$ is a limit preserving algebraic stack with quasi-compact, separated diagonal.*

Proof. By [Gro63, Théorème 4.7.1], the inclusion functor $\mathcal{X}_{\mathrm{pol}} \to \mathrm{Coh}_\pi$ is representable by open immersions. Therefore $\mathcal{X}_{\mathrm{pol}} \to \mathcal{X}_{\mathrm{pr,f,lpf}}$ is representable by limit preserving algebraic stacks. By Proposition 2.9, Proposition 2.10, Corollary 2.24, Proposition 3.2 and Proposition 3.4, $\mathcal{X}_{\mathrm{pol}}$ satisfies Axioms 1,2 and 4 of Corollary 2.18.

The proof of Axiom (3) uses the Grothendieck existence theorem. Let $\widehat{A}$ be a complete local algebra and let $(X_n, \mathcal{L}_n)$ be a compatible collection of objects of $\mathcal{X}_{\mathrm{pol}}$ over Spec $\widehat{A}/\mathfrak{m}^n$. For d sufficiently large, $\mathcal{L}_0^{\otimes d}$ is very ample and $h^1(X_0, \mathcal{L}_0^{\otimes d})$ is zero. The compatible system of $\widehat{A}/\mathfrak{m}^n$-modules $H^0(X_n, \mathcal{L}_n^{\otimes d})$ defines a finite free $\widehat{A}$-module. Choosing a basis for this module, there are induced closed immersions $\mathcal{X}_n \to \mathbb{P}^N_{\widehat{A}/\mathfrak{m}^n}$ such that the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ is $\mathcal{L}_n^{\otimes d}$. By the Grothendieck existence theorem, [Gro63, Corollaire 5.1.8], there exists a closed subscheme X of $\mathbb{P}^N_{\widehat{A}}$ whose reductions give the compatible family $\mathcal{X}_n$. By the theorem on formal functions, [Gro63, Théorème 4.1.5], $H^0(X, \mathcal{L}^{\otimes e})$ equals $\varprojlim H^0(X_n, \mathcal{L}_n^{\otimes e})$ for every e. Because the higher cohomologies vanish, and because the X_n are flat over Spec $\widehat{A}/\mathfrak{m}^n$, also $H^0(X, \mathcal{L}^{\otimes e})$ is flat over $\widehat{A}$. Therefore X is flat over Spec $\widehat{A}$. Finally, by [Gro63, Corollaire 5.1.6], there exists an invertible sheaf $\mathcal{L}$ on X whose reductions give $\mathcal{L}_n$. By [Gro63, Théorème 4.7.1], this is ample. $\square$

Remark 4.3. There are, of course, other ways to verify the proposition. One can use the existence of Hilbert schemes to give a smooth morphism from a scheme to $\mathcal{X}_{\mathrm{pol}}$.

Let S be an excellent scheme. Let $\mathcal{Y}$ be an limit preserving, separated algebraic stack over (Aff/S) whose diagonal is representable by finite morphisms. Define $\mathcal{H}$ to be the category whose objects are 4-tuples (U, X, L, g) where (U, X) is an object of $\mathcal{X}_{\mathrm{pr,f,lpf}}$, L is an invertible sheaf on X, and $g : X \to \mathcal{Y}$ is a 1-morphism. A datum is required to satisfy the condition that L is g-relatively ample, i.e., for every affine scheme Spec A and object a of $\mathcal{Y}(\mathrm{SP}A)$, the pullback of L to the 2-fibered product Spec $A \times_{a,\mathcal{Y},g} X$ is ample.

Morphisms in $\mathcal{H}$ are data (f_U, f_X, f_L, f_g) of a morphism (f_U, f_X, f_L) in Coh_π together with a 2-isomorphism from the 1-morphism $g' : X' \to \mathcal{Y}$ to the composite 1-morphism

$$X' \xrightarrow{f_X} X'' \times_{U''} U' \xrightarrow{\mathrm{pr}_{X''}} X'' \xrightarrow{g''} \mathcal{Y}.$$

Proposition 4.4. *The category $\mathcal{H}$ is a limit preserving algebraic stack over (Aff/S) with quasi-compact, separated diagonal.*

Proof. There is a 1-morphism $\mathrm{T} : \mathcal{H} \to \mathcal{C}\wr\langle_\pi$. In fact this factors through the open substack of data (U, X, L) such that L is an invertible sheaf. By [Ols05b, Theorem 1.1], and using [Gro63, Théorème 4.7.1] (locally over the Hom stack, for a quasi-compact smooth cover of $\mathcal{Y}$ whose image contains the local image of g), the 1-morphism T is representable by limit preserving algebraic stacks. Moreover, the diagonal morphism associated to T is quasi-compact and separated. By Proposition 4.1, the composite $G \circ F : \mathcal{H} \to \mathcal{X}_{\mathrm{pr,f,lpf}}$ is representable by limit preserving algebraic stacks with quasi-compact, separated diagonal.

By Proposition 2.9, Proposition 2.10, Corollary 2.24, Proposition 3.2 and Proposition 3.4, $\mathcal{H}$ satisfies Axioms 1,2 and 4 of Corollary 2.18. It only remains to verify Axiom (3).

To prove Axiom (3), apply the straightforward analogue of the argument from Proposition 4.2. Namely, given a compatible family (X_n, L_n, g_n) over Spec $\widehat{A}/\mathfrak{m}^n$, first form the sequence of sheaves $E_{d,n} = (g_n)_* L_n^{\otimes d}$ on Spec $\widehat{A}/\mathfrak{m}^n \times_S \mathcal{Y}$. For d sufficiently large, the natural map $g_n^* E_{n,d} \to L_n^{\otimes d}$ is surjective for every d. For d perhaps larger, the induced morphism $X_n \to \mathrm{ProjSym}(E_{n,d})$ is a closed immersion. These statements are local on $\mathcal{Y}$, and thus can be checked after base change by a smooth morphism Spec $B \to \mathcal{Y}$ whose image contains $g_n(X_n)$. Then the statements follow from the usual versions for schemes.

The sheaves $E_{n,d}$ are coherent with proper support. Therefore, by the analogue of the Grothendieck existence theorem for stacks, [OS03, Proposition 2.1], there exists a coherent sheaf with proper support E_d on Spec $\widehat{A} \times_S \mathcal{Y}$ whose reductions are the sheaves $E_{n,d}$. Denote by P the stack $\mathrm{ProjSym}(E_d)$ (which can be constructed by flat descent, for example). The projection,

$$\mathrm{pr} : P \to \mathrm{Spec}\ \widehat{A} \times_S \mathcal{Y},$$

is representable by proper, finitely presented algebraic spaces. Therefore P is a limit preserving algebraic stack with quasi-compact, separated diagonals. There are natural closed immersions $X_n \to \mathrm{Spec}\ \widehat{A}/\mathfrak{m}^n \times_{\mathrm{Spec}\ \widehat{A}} P$. Again by the Grothendieck existence theorem [OS03, Proposition 2.1], there exists a closed substack X of P whose reductions are X_n. Since the reductions are proper, flat, finitely presented algebraic spaces over Spec $\widehat{A}/\mathfrak{m}^n$, by the same argument as in the proof of Proposition 4.2, X is a proper, flat, finitely presented algebraic space over Spec $\widehat{A}$. Define g to be the restriction of pr to X. By the Grothendieck existence theorem for algebraic spaces, [Knu71, Theorem V.6.3], there exists an invertible sheaf L on X whose reductions are the sheaves L_n. By [Gro63, Théorème 4.7.1] (applied after base change to a quasi-compact smooth cover of $\mathcal{Y}$), L is g-ample. Thus (X, L, g) is an object of $\mathcal{H}(\mathrm{Spec}\ \widehat{A})$ whose reductions are the objects (X_n, L_n, g_n). □

Remark 4.5. There is a stack $\mathcal{H}_{\mathrm{finite}}$ closely related to $\mathcal{H}$ whose objects are data (U, X, g) as above such that $\mathcal{O}_X$ is g-ample. There is a 1-morphism $\mathcal{H}_{\mathrm{finite}}$ sending (U, X, g) to $(U, X, \mathcal{O}_X, g)$. This morphism is representable by affine morphisms by the same proof as for the relative representability of Coh_π in [LMB00, Théorème 4.6.2.1]. Obviously, if $\mathcal{O}_X$ is g-ample if and only if g is representable by finite morphisms.

Denote by $g : X \to \mathcal{H}_{\mathrm{finite}} \times_S \mathcal{Y}$ the universal morphism. There is an open substack of X where g is unramified: namely the complement of the support of the sheaf of relative differentials of g. The complement of this open stack is proper over $\mathcal{H}_{\mathrm{finite}}$. Its image in $\mathcal{H}_{\mathrm{finite}}$ is proper. The complement of the image equals Vistoli's Hilbert stack, cf. [Vis91].

Similarly, using [Gro67, Théorème 12.1.1], there is an open substack of $\mathcal{H}_{\mathrm{finite}}$ over which the fibers of X are geometrically reduced and equidimensional. In the special case that $\mathcal{Y}$ is a projective scheme over S, this locus is the stack of *branchvarieties* of Alexeev and Knutson, cf. [AK06].

References

[AK06] Valery Alexeev and Allen Knutson. Complete moduli spaces of branchvarieties. preprint available at `http://www.math.uga.edu/~valery/`, 2006.

[Art69] M. Artin. Algebraization of formal moduli. I. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 21–71. Univ. Tokyo Press, Tokyo, 1969.

[Art74] M. Artin. Versal deformations and algebraic stacks. *Invent. Math.*, 27:165–189, 1974.

[BGI71] P. Berthelot, A. Grothendieck, and L. Illusie, editors. *Théorie des intersections et théorème de Riemann-Roch*. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre, Lecture Notes in Mathematics, Vol. 225.

[CdJ02] Brian Conrad and A. J. de Jong. Approximation of versal deformations. *J. Algebra*, 255(2):489–515, 2002.

[Gro63] A. Grothendieck. Éléments de géométrie algébrique. III. Étude locale des schémas et des morphismes de schémas. *Inst. Hautes Études Sci. Publ. Math. 11 (1961), 349-511; ibid.*, (17):137–223, 1963. `http://www.numdam.org/item?id=PMIHES_1961__11__5_0`.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. *Inst. Hautes Études Sci. Publ. Math. 20 (1964), 101-355; ibid. 24 (1965), 5-231; ibid. 28 (1966), 5-255; ibid.*, (32):5–361, 1967. `http://www.numdam.org/item?id=PMIHES_1965__24__5_0`.

[Gro03] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Geometric Algebra Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 239.

[Knu71] Donald Knutson. *Algebraic spaces*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 203.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[Ols04] Martin C. Olsson. Semistable degenerations and period spaces for polarized $K3$ surfaces. *Duke Math. J.*, 125(1):121–203, 2004.

[Ols05a] Martin Olsson. Deformation theory of representable morphisms of algebraic stacks. preprint, 2005.

[Ols05b] Martin Olsson. Hom stacks and restriction of scalars. preprint, 2005.

[OS03] Martin Olsson and Jason Starr. Quot functors for Deligne-Mumford stacks. *Comm. Algebra*, 31(8):4069–4096, 2003. Special issue in honor of Steven L. Kleiman.

[Vis91] Angelo Vistoli. The Hilbert stack and the theory of moduli of families. In *Geometry Seminars, 1988–1991 (Italian) (Bologna, 1988–1991)*, pages 175–181. Univ. Stud. Bologna, Bologna, 1991.