RATIONAL CONNECTIVITY AND SECTIONS OF FAMILIES OVER CURVES

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1. Introduction

1.1. **Statement of results.** We will work over the field of complex numbers, although our results hold over any uncountable algebraically closed field of characteristic zero. A *variety* will be a reduced scheme of finite type, but not necessarily irreducible. Recall that a property is said to hold at a general point of a variety V if it holds for all points in a dense open subset of V. A property is said to hold at a very general point of V if it holds at all points in a countable intersection of dense open subsets of V.

A variety V is said to be *rationally connected* if two general points of V can be joined by a rational curve. In [GHS], it is proved that a one-parameter family of rationally connected varieties always has a rational section: explicitly, we have the following theorem.

Theorem 1.1. *Let $\pi : X \to C$ be a proper morphism of complex varieties, with C a smooth connected curve. If the general fiber of π is rationally connected, then π has a section.*

The goal of this paper is to state and prove a converse to this statement (cf. Theorem 1.3 below).

We should first of all discuss what we mean by this, inasmuch as the literal converse of Theorem 1.1 is clearly false. To this end, let's focus on the question: under what circumstances does a family $\pi : X \to B$ of varieties have the property that its restriction to a general curve $C \subset B$ has a section?

This is certainly the case if the family $\pi : X \to B$ has a global rational section. It is also the case by Theorem 1.1 if the general fiber of π is rationally connected, and by extension it is the case if X contains a subvariety $Z \subset X$ dominating B and whose fiber over a general point of B is rationally connected. (We can think of the case where the family $\pi : X \to B$ has a global rational section as a special case of this, a single point being a rationally connected variety!) In this paper, we will prove that in fact these are the only circumstances under which it may occur. To make this claim precise, we start by making the following definition.

Definition 1.2. *Let $\pi : X \to B$ be an arbitrary morphism of complex varieties. By a* ***pseudosection*** *of π we will mean a subvariety $Z \subset X$ such that the restriction $\pi|_Z : Z \to B$ is dominant with rationally connected general fiber.*

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Our main result is the following.

Theorem 1.3. *Let B be any irreducible variety. For every positive integer d there exists a bounded family $\mathcal{H}_d$ of maps $h : C \to B$ from smooth irreducible curves to B such that for any proper morphism $\pi : X \to B$ of relative dimension d or less, if $h : C \to B$ is a map parametrized by a very general point of $\mathcal{H}_d$, the pullback*

$$\pi_C : X_C = X \times_B C \to C$$

has a section if and only if π has a pseudosection.

If B is normal and quasi-projective, we can take $\mathcal{H}_d$ to be the family of smooth one-dimensional linear sections of B under a sufficiently positive projective embedding.

What we are saying here, in other words, is that if we have any family of varieties $\pi : X \to B$ satisfying the condition that every one-parameter subfamily has a section, it does so by virtue of the fact that X contains a family of rationally connected varieties.

As a corollary of this, we will at the end of the paper settle a question left hanging in [GHS]: whether or not the statement of Theorem 1.1 holds for the larger class of $\mathcal{O}$-*acyclic* varieties—that is, varieties X with $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. In fact, it was suggested by Serre in a letter to Grothendieck ([GS], p. 152) that this might be the case (though Serre immediately adds that it is "sans doute trop optimiste"). In Section 5, we show this does not hold: specifically, by applying Theorem 1.3 to the universal family over a parameter space of Enriques surfaces with a particular polarization, we will deduce the following result.

Corollary 1.4. *There exists a one-parameter family $X \to C$ of Enriques surfaces with no rational section.*

While this method produces the *existence* of a family of Enriques surface over a complex curve with no section, it does not provide an answer to the following natural questions.

- What are the possible base curves for such families? (For example is there a pencil of Enriques surfaces with no section?)
- Can such a family be defined over a number field?
- Are there local obstructions to the existence of sections?

A recent result of Guillaume Lafon [L] clarifies the situation greatly. He constructs Enriques surfaces over $\mathbb{Q}(t)$ that do not have rational points over $\mathbb{C}((t))$.

We remark, however, that the Enriques surfaces we produce have points everywhere locally, so the existence of local obstructions is not the only reason that Enriques surfaces over function fields can fail to have rational points.

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2. Stable maps and stable sections

Our proof of Theorem 1.3 involves an induction on the relative dimension of $f : X \to B$ where the base case (fiber dimension zero) is proved by a version of the Lefschetz hyperplane theorem. In the course of the proof we will need to use specializations of irreducible curves in B. There are several possible compactifications of the Chow variety of irreducible curves in B, but the one we will use is the

Kontsevich space of stable maps. The reader who is unfamiliar with stable maps is referred to the article [FP]. A stable map to X is a morphism $h : C \to X$ for which C is a connected, projective curve which has at-worst-nodes as singularities and such that the morphism h has a finite automorphism group.

There is an equivalence relation on stable maps which is the obvious one, and the Kontsevich moduli space of stable maps is the corresponding coarse moduli space of equivalence classes of stable maps.

If X is a quasi-projective variety, we let $\overline{M}_g(X, \beta)$ denote the Kontsevich space of stable maps $h : C \to X$ such that C has arithmetic genus g and such that the push-forward fundamental class $h_*[C]$ equals $\beta \in H_2(X, \mathbb{Z})$. This is a quasi-projective coarse moduli scheme which is projective if X is projective (cf. [FP]). We will often not need the decorations, so let $\overline{M}(X) := \cup_{g,\beta} \overline{M}_g(X, \beta)$ denote the space of all stable maps to X.

Given a morphism of quasi-projective schemes $\pi : X \to B$, there is an induced map on Kontsevich spaces, $\overline{M}(\pi) : \overline{M}(X) \to \overline{M}(B)$. (Technically we must restrict to those stable maps with $g > 1$ or with $\pi_*\beta \neq 0$, but this will always be the case for us when we apply $\overline{M}(\pi)$.) This map takes the moduli point $h : C \to X$ to the point $\pi \circ h : \tilde{C} \to B$, where $\tilde{C}$ is the stabilization of C relative to the morphism $\pi \circ h$. It is obtained by contracting those components of C of genus 0 which are contracted by $\pi \circ h$ and meet the rest of C in fewer than 3 points. If π is projective, then $\overline{M}(\pi)$ is projective on each connected component of $\overline{M}(X)$.

Definition 2.1. *If $h : C \to X$ and $h' : C' \to X$ are stable maps, we say that h' is a **submap** of h if there is an element $i \in \overline{M}(C)$ such that $[h'] = \overline{M}(h)([i])$.*

This terminology is suggested by the special case where h' factors as a composition of h with a closed embedding $i : C' \to C$. We use the more general notion for convenience.

The following definition makes precise what it means for one stable map to be a submap of a specialization of another stable map.

Definition 2.2. *Given a quasi-projective variety X, and two families of stable maps to X, D and E, we say that D **dominates** E if a general map parametrized by E can be realized as a submap of a specialization of maps parametrized by D. Precisely, for a general point of E parametrizing a stable map $h : C \to X$, we can find a family of stable maps over* $\mathrm{Spec}(\mathbb{C}[[t]])$ *such that h is a submap of the map over the special fiber, and the map over the generic fiber is pulled back by a morphism from* $\mathrm{Spec}(\mathbb{C}((t)))$ *to D.*

Although the definition here requires only that a general map parametrized by E be obtainable as a submap of a specialization, it is equivalent to require that *every* map parametrized by E arises in this way. To see this, consider the family of stable maps parametrized by D.

$$\begin{tikzcd} \mathcal{C}_D \arrow[r, "h_D"] \arrow[d, "\pi_D"'] & X \\ D & \end{tikzcd}$$

Now define $\overline{M}^0(\mathcal{C}_D)$ to be the open and closed subset of $\overline{M}(\mathcal{C}_D)$ parametrizing stable maps to fibers of π_D. The condition that E is dominated by D is equivalent to the statment that a general point of E maps to a point in $\overline{M}(X)$ which is in the closure of $\overline{M}(h_D)(V)$ for some irreducible component V of $\overline{M}^0(\mathcal{C}_D)$. However, if a general point of E maps to a point in such a closure, then so does every point of E.

We need a criterion for when a family of stable maps to a projective variety is dominated by a family of embedded complete intersection curves. We will repeatedly make use of the following criterion.

Lemma 2.3. *Suppose E is a family of stable maps to a normal, irreducible, projective variety B such that for a general map $h : C \to B$ parametrized by E, there is an open subset $U \subset B$ contained in the smooth locus of B and such that:*

1. *$h^{-1}(U)$ is dense in C,*
2. *$h^{-1}(U)$ contains all the nodes of C, and*
3. *the restricted map $h^{-1}(U) \to U$ is a closed embedding.*

Then E is dominated by the family of one dimensional linear sections of B under a sufficiently positive projective embedding.

Proof. Let $h : C \to B$ be a general element of our family. We embed B in a projective space and find an integer a such that the ideal of the reduced image curve $h(C)$ is generated by polynomials of degree a. The sufficiently positive embedding required is the ath Veronese re-embedding. Now we know that we can realize $h(C)$ as an intersection of hyperplanes. Since $h(C)$ is a local complete intersection in a neighborhood of the image of each of the nodes of C, if we choose $\dim(B) - 1$ generic hyperplane sections of B which contain $h(C)$, their intersection will agree with $h(C)$ in a neighborhood of the image each of these nodes. Moreover, by Bertini's Theorem, this intersection will be smooth away from $h(C)$. We conclude that the intersection of $\dim(B) - 1$ generic hyperplane sections of B containing $h(C)$ will be a generically reduced curve C' which contains $h(C)$ as a subcurve and such that there exists an open set $U' \subset U$ containing the images of all of the nodes of C such that $h^{-1}(U') \to U' \cap C'$ is an isomorphism.

Now we choose any one parameter family of smooth complete intersections in B whose flat limit is C', i.e., a morphism from $\mathrm{Spec}(\mathbb{C}[[t]])$ to the Hilbert scheme of complete intersections in B whose general fiber maps to a smooth complete intersection and whose special fiber maps to C'. We think of the general fiber as a stable map and perform stable reduction to the corresponding map $\mathrm{Spec}(\mathbb{C}[[t]]) \to \overline{M}(B)$. Denote the special fiber of the stable reduction by $\overline{h} : \overline{C} \to B$. Since $U' \cap C'$ is already at-worst-nodal, there is an open subset $V \subset \overline{C}$ such that $\overline{h} : V \to U' \cap C'$ is an isomorphism. In other words, we have a factorization $i : h^{-1}(U') \to \overline{C}$ of $h : h^{-1}(U') \to B$. Since every point of C in $C - h^{-1}(U')$ is smooth, we can apply the valuative criterion of properness to extend this factorization to a morphism $i : C \to \overline{C}$. So $h : C \to B$ is a submap of $\overline{h} : \overline{C} \to B$, which shows that E is dominated by the family of smooth curves in B which are complete intersections of $d - 1$ hyperplanes.

□

We remark that the lemma above is not the most general result, but to prove a stronger version would lead us too far astray. We leave it to the interested reader to prove that in the above lemma it suffices to assume that for the general map $h : C \to B$ parametrized by E, the preimage of the smooth locus, $h^{-1}(B_{\text{smooth}})$, is a dense open set which contains every node of C.

Although our main concern is to understand sections of a map $\pi : X \to B$ over smooth curves in B, the specialization methods we use force us to consider the more general notion of stable maps. Similarly, we need to replace the notion of section over a curve with an object which specializes as we specialize the base curve.

Definition 2.4. *Given a morphism $\pi : X \to B$, and a stable map $h : C \to B$, we define a **stable section** of π over $[h]$ to be a stable map $\tilde{h} : \tilde{C} \to X$ such that $\overline{M}(\pi)([\tilde{h}]) = [h]$.*

Notice that for a stable section $\tilde{h}$, the class $\pi_*(\tilde{h})_*[\tilde{C}]$ is just $h_*[C]$ and $g(C') = g(C)$. Since h is a stable map, either $g(C) > 1$ or $h_*[C]$ is nonzero, i.e., the map $\overline{M}(\pi)$ really is defined in a neighborhood of $[\tilde{h}]$.

If C is a smooth connected projective curve in B, then a stable section of π over C is simply a section of π over C with some trees of π-contracted rational curves in X attached. The notion is more interesting when C has nodes. In this case, the existence of a stable section over C does not guarantee the existence of any sections over C whatsoever. For example, a typical stable section $\tilde{h}$ over a curve $C = C_1 \cup C_2$, where $t_1 \in C_1$ is glued to $t_2 \in C_2$, would consist of sections $\tilde{h}_1$ and $\tilde{h}_2$ of π over C_1 and C_2 separately together with a tree of π-contracted rational curves in X joining $\tilde{h}_1(t_1)$ to $\tilde{h}_2(t_2)$ (and some trees of π-contracted rational curves attached elsewhere). Such rational curves are exactly the sort which are contracted under the stabilization process associated with $\overline{M}(\pi)$.

The point of this definition is that given a family of curves in the base B specializing to some stable map $h : C \to B$ (possibly reducible), and given an honest section over the generic curve in this family, then we cannot conclude the existence of a section of π over h, but we do conclude the existence of a stable section of π over h. In other words, the existence of a stable section is preserved under specialization. This follows immediately from the properness of the irreducible components of $\overline{M}(X)$. Another elementary fact is the following lemma.

Lemma 2.5. *If $h' : C' \to B$ is a submap of $h : C \to B$, then the existence of a stable section of $\pi : X \to B$ over h implies the existence of a stable section of π over h'.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
& \tilde{C} \arrow[d, "q"'] \arrow[dr, "\tilde{h}"] & \\
\hat{C} \arrow[r, "i"] \arrow[dr, "q'"'] & C \arrow[dr, "h"] & X \arrow[d, "\pi"] \\
& C' \arrow[r, "h'"] & B
\end{tikzcd}$$

where the morphisms q and q' are isomorphisms except for possibly contracting some trees of rational curves. To define a stable section of h' one can simply complete the upper left hand corner of this diagram with a curve C'' giving a stable section of q over i. To construct such a C'', note that over each irreducible component of $\hat{C}$ the pullback of $\tilde{C}$ admits a section. To complete this to a stable section, we simply need to find, for each node of $\hat{C}$, a tree of rational curves in $\tilde{C}$ connecting the image point of one branch of the node to the image point of the other. Since every fiber of q is either a point or a tree of rational curves, this is trivial.

□

Taken together, we get the following fundamental lemma.

Lemma 2.6. *If D and E are families of stable maps to B, with D dominating E and if a general map parametrized by D admits a stable section, then so does every map parametrized by E.*

As an application, we can strengthen the easy direction of our main theorem.

Proposition 2.7. *If $\pi : X \to B$ is a morphism of projective varieties with B smooth and if π admits a pseudosection, then for any smooth curve C, and for any morphism $h : C \to B$, the pullback family $X_C \to C$ admits a section.*

Proof. Since any map from a smooth curve factors through the normalization of its image, it suffices to prove this statement for maps birational onto their image (note that since C and B are smooth, such maps satisfy the hypotheses of Lemma 2.3.) Let Z be a pseudosection of π. We already know by Theorem 1.1 that the proposition is true for any smooth curve such that the general fiber of Z over the curve is rationally connected. In particular it holds for a generic complete intersection curve in B under any projective embedding. By Lemma 2.3, our map h can be realized as a submap of a limit of such curves. Then Lemma 2.6 implies that π admits a stable section over $[h]$. Since C is smooth, this implies that X_C admits a section over C.

□

3. Proof of main theorem

In our proof of Theorem 1.3, we begin by assuming that B is normal and that both X and B are projective. After handling this "special" case, we give the (easy) argument which reduces the general case to the special case.

We will prove the theorem by induction on the relative dimension of X over B. We start with the case of relative dimension zero.

Proposition 3.1. *Let $B \subset \mathbb{P}^n$ be an irreducible normal variety and $\pi : X \to B$ a generically finite proper morphism. Then π admits a rational section if and only if π admits a section when restricted to a general one-dimensional linear section of B.*

Proof. It suffices to prove this when X is irreducible and π is dominant. We are also free to assume that X is normal and π is proper, since the statement depends only on the birational geometry of X. Let b be the dimension of B, and let G denote the family of codimension $b-1$ linear subspaces of $\mathbb{P}^n$. A standard application of Bertini's Theorem shows that there is a dense open subset U_B of G, such that for any linear space L parametrized by U_B, the intersection $L \cap B$ is a smooth, one-dimensional, irreducible subvariety. Applying a stronger characteristic zero version of Bertini's Theorem such as [Har, III.10.9, Ex III.11.3] to the morphism from X to $\mathbb{P}^n$ (obtained by composing π with the inclusion), we can find a dense open subset $U_X \subset G$ such that for any linear space L parametrized by a point of U_X, the pullback to X of L is a smooth, irreducible, one-dimensional subvariety. Choosing a subspace L parametrized by a point in $U_X \cap U_B$ we find that the restricted morphism

$$\pi^{-1}(L \cap B) \to (L \cap B)$$

is a finite morphism of smooth proper curves, so it admits a section if and only if this morphism has degree 1. For a general L, the degree of this morphism agrees with the degree of π and the result follows.

□

We remark that it is also possible to prove this lemma by applying a suitable version of the Lefschetz hyperplane theorem for fundamental groups such as [Ham, Thm. 3] or [GM, Thm. 1.2, Part II].

In handling the case of positive relative dimension, one of the main ingredients needed is the following bend-and-break lemma for sections.

Lemma 3.2. *Let $\pi : X \to C$ be a proper morphism with C a smooth connected curve. Let $p \in X$ be an arbitrary point. If there is a positive dimensional family of sections of π passing through p, then there is a rational curve in X passing through p which is contracted by π.*

Proof. This lemma is a version of Mori's Bend-and-Break Lemma from [Mo] which in turn relies on the Rigidity Lemma [Mu, p. 43].

Let $q = \pi(p)$. Suppose we have a one-parameter family of sections passing through p. This gives us a rational map $f : D \times C \dashrightarrow X$ over C whose restriction to a general fiber $\{d\} \times C$ is a section passing through p. Suppose, by way of contradiction, that f is a regular morphism in a neighborhood of $D \times \{q\}$. Since f contracts $D \times \{q\}$, by the Rigidity Lemma it also contracts $D \times \{c\}$ for all $c \in C$, i.e., our family is constant which is a contradiction. Hence, f is not regular near $D \times \{q\}$. So there is at least one point of indeterminacy in $D \times \{q\}$. We may form the minimal blow-up of $D \times C$ necessary to resolve the indeterminacy locus of f.

The exceptional divisor of this blow-up is a tree of rational curves which intersects the proper transform of $D \times \{q\}$ and which is mapped to a tree of π-contracted rational curves in X. Therefore some rational curve in the exceptional divisor maps to a π-contracted rational curve which meets p.

□

We will apply this lemma in two ways. The first application is to get a uniform bound on the dimensions of spaces of sections. First we need a definition.

Definition 3.3. *If $\pi : X \to B$ is a proper morphism, we define the **rational curve locus**, $V(\pi)$, to be the union of all π-contracted rational curves in X.*

Let us pause to describe what sort of object $V(\pi)$ is. If π is projective and we fix a relatively ample divisor H on X, then for each integer d, the locus of all π-contracted rational curves of H-degree less than or equal to d is a Zariski closed subset of X by [D, Sec. 5.6]. Hence, we see that $V(\pi)$ is a countable union of closed subvarieties.

The next lemma applies Lemma 3.2 to give a uniform bound on the dimensions of spaces of sections. Let $\pi : X \to C$ be a projective morphism of relative dimension d. Let Σ be an irreducible variety parametrizing a family of sections of π.

Lemma 3.4. *If there exists a section parametrized by Σ whose image is not contained in $V(\pi)$, then $\dim(\Sigma) \leq d$.*

Proof. Choose a very general point c of C and set $X_c = \pi^{-1}(c)$. Let $ev_c : \Sigma \to X_c$ be the map which evaluates a section at c. Our hypotheses ensure that $\dim(X_c) = d$ and that $ev_c(\Sigma) \not\subset V(\pi)$. Lemma 3.2 then implies that ev_c is generically finite onto its image, yielding the desired bound.

□

Let $B \hookrightarrow \mathbb{P}^N$ be a closed immersion. Let $D \subset \mathrm{Grass}(N-b+1, \mathbb{P}^N)$ be the dense open subset parametrizing linear spaces Λ such that $\Lambda \cap B$ is smooth, irreducible, and one-dimensional. Let $\rho_D : C_D \to D$ denote the universal family of intersections $\Lambda \cap B$ and let $\rho_D^{(2)} : C_D^{(2)} \to D$ denote the fiber product of C_D with itself over D. Denote by $h_D : C_D \to B$ the obvious map.

Definition 3.5. *A **triangle** is a stable map $h : C \to X$ such that*

1. *C has three irreducible components C_1, C_2, and C_3,*
2. *C has three nodes $q_{1,2} = C_1 \cap C_2$, $q_{1,3} = C_1 \cap C_3$, and $q_{2,3} = C_2 \cap C_3$, and*
3. *for each $i = 1, 2, 3$ the map $h|_{C_i} : C_i \to X$ is a closed immersion whose image is in D.*

*The **space of triangles**, is the locally closed subvariety $T(B) \subset C_D^{(2)} \times C_D^{(2)} \times C_D^{(2)}$ which is the subset of triples*

$$((C_1, q_{1,3}, q_{1,2}), (C_2, q_{2,1}, q_{2,3}), (C_3, q_{3,2}, q_{3,1})) \in C_D^{(2)} \times C_D^{(2)} \times C_D^{(2)}$$

such that $q_{i,j} = q_{j,i}$ for $1 \leq i < j \leq 3$, and such that $q_{i,j} \neq q_{i,k}$ for each triple of distinct integers $1 \leq i, j, k \leq 3$. The universal family of triangles over $T(B)$ is the family of curves $C = C_1 \cup C_2 \cup C_3$ obtained by identifying the points $q_{i,j}$ as above.

Of course, $T(B)$ depends not only on B, but also on a choice of projective embedding of B. There is a morphism $\mathrm{ev} : T(B) \to B \times B \times B$ which maps a triple

$$((C_1, q_{1,3}, q_{1,2}), (C_2, q_{2,1}, q_{2,3}), (C_3, q_{3,2}, q_{3,1}))$$

to $(q_{1,2}, q_{1,3}, q_{2,3})$.

Let $q \in B$ be a point. Define $D_q \subset D$ to be the closed subset where $q \in h(C)$. Define $T(B)_q \subset T(B)$ to be the closed set where $q \in h(C_1 \cup C_2 \cup C_3)$. For each $i = 1, 2, 3$ define $T(B)_{q,i} \subset T(B)_q$ to be the closed subset where $q \in h(C_i)$.

Lemma 3.6. (1) *The morphism ev is flat with irreducible fibers.*
(2) *The variety $T(B)$ is irreducible.*
(3) *For each $i = 1, 2, 3$, the variety $T(B)_{q,i}$ is irreducible.*
(4) *The triangle associated to a general point of $T(B)$ satisfies the hypotheses of Lemma 2.3.*

Proof. Let $\Delta \subset B \times B \times B$ denote the union of the three big diagonals, i.e., the set of triples $(q_{1,2}, q_{1,3}, q_{2,3})$ such that some $q_{i,j} = q_{i',j'}$. Given $(q_{1,2}, q_{1,3}, q_{2,3}) \in B \times B \times B - \Delta$, the fiber of *ev* over this point is an open subset of the product of Grassmannians which parametrizes triples $(\Lambda_1, \Lambda_2, \Lambda_3) \in D \times D \times D$ such that $\mathrm{span}(q_{1,2}, q_{1,3}) \subset \Lambda_1$, $\mathrm{span}(q_{2,1}, q_{2,3}) \subset \Lambda_2$ and $\mathrm{span}(q_{3,1}, q_{3,2}) \subset \Lambda_3$. So $\mathrm{ev} : T(B) \to B \times B \times B - \Delta$ is an open subset of a fiber product of three $\mathrm{Grass}(N - b - 1, \mathbb{P}^{N-2})$-bundles. Therefore it is flat and has irreducible fibers.

Statement (2) follows from (1) and the fact that $B \times B \times B$ is irreducible. Statement (3) follows by an argument similar to that for (1). For definiteness, suppose $i = 1$. There is a projection $\mathrm{pr}_1 : T(B)_{q,1} \to D$

$$((C_1, q_{1,3}, q_{1,2}), (C_2, q_{2,1}, q_{2,3}), (C_3, q_{3,2}, q_{3,1})) \mapsto C_1.$$

The image is a subset of the subvariety D_q. This subvariety is isomorphic to an open subset of $\mathrm{Grass}(N - b, \mathbb{P}^{N-1})$ and so is irreducible (possibly empty). Define $C^{(2)}_{D,q}$ to be the preimage of D_q in $C^{(2)}_D$. This is also irreducible since $C^{(2)}_D \to D$ is flat with irreducible fibers. There is a projection $T(B)_{q,1} \to C^{(2)}_{D,q}$. For each $(C_1, q_{1,3}, q_{1,2}) \in C^{(2)}_{D,q}$ the fiber in $T(B)_{q,1}$ is an open subset of a fiber product of two $\mathrm{Grass}(N - b - 1, \mathbb{P}^{N-2})$-bundles over B: the map to B corresponds to the choice of $q_{2,3}$ and the Grassmannian bundles correspond to the choice of Λ_2 and Λ_3 containing $\mathrm{span}(q_{2,1}, q_{2,3})$ and $\mathrm{span}(q_{3,1}, q_{3,2})$ respectively. Thus $T(B)_{q,1} \to C^{(2)}_{D,q}$ is flat with irreducible fibers. Therefore $T(B)_{q,1}$ is irreducible.

Finally (4) is trivial to verify.

□

Our second application of Lemma 3.2 is in the proof of the following lemma, which is the main step in the proof of Theorem 1.3.

Lemma 3.7. *Let $\pi : X \to B$ be a morphism of projective varieties with B normal and irreducible, and let $p \in X$ be any point such that $p \notin V(\pi)$. Suppose also that p is not contained in the closure of the image of any rational section of π. Then* *a very general triangle passing through $q = \pi(p)$ admits no stable sections passing through p.*

Proof. It suffices to exhibit a single triangle with this property. Choose an irreducible subfamily $H \subset D$ of curves passing through q, such that for general $b \in B$ a finite (but positive) number of members of H pass through b. Our strategy is to show that if we construct a triangle by choosing two very general members of H (which will necessarily intersect at q) as C_1 and C_2, and a very general member of D which intersects C_1 and C_2 as C_3, then the result will be a triangle satisfying the desired property.

We construct a subset $\Omega \subset X$ which is a countable union of subvarieties of X in the following way. For every finite type family of sections of π over curves in H which take the value p at q, we have a map from the base of this family to $\overline{M}(X)$. Form the closure of the image of this map, and define $\tilde{\Omega}$ to be the countable union of all such closed subvarieties of $\overline{M}(X)$ arising from the countably many Chow varieties of sections as above. Notice that $\tilde{\Omega}$ is not necessarily quasi-compact, but it is a closed subset of $\overline{M}(X)$ (which is also not quasi-compact).

We can restrict the universal curve of $\overline{M}(X)$ over $\tilde{\Omega}$, and there is a map from the total space of this universal curve to X. We define Ω to be the image of this map, so Ω is a countable union of closed subvarieties of X. Let $\tilde{\Omega}_0$ be any irreducible component of $\tilde{\Omega}$ and let $\Omega_0 \subset X$ be the (closed) image of the universal curve over $\tilde{\Omega}_0$. Notice that $p \in \Omega_0$, since it is in the image of each section parametrized by a general point of $\tilde{\Omega}_0$.

Consider the restricted morphism $\overline{M}(\pi) : \tilde{\Omega}_0 \to \overline{M}(B)$. The general point of $\tilde{\Omega}_0$ parametrizes a section over a member of H, so the image of $\tilde{\Omega}_0$ under $\overline{M}(\pi)$ is contained in the closure $\overline{H}$ of H in $\overline{M}(B)$. By Lemma 3.2, the morphism $\overline{M}(\pi) : \tilde{\Omega}_0 \to \overline{H}$ is generically finite, since $p \notin V(\pi)$. Therefore the map from the universal curve over $\tilde{\Omega}_0$ to the universal curve over $\overline{H}$ is generically finite. By construction, the evaluation morphism from the universal curve over $\overline{H}$ to B is generically finite. So finally we conclude the restricted morphism $\pi : \Omega_0 \to B$ is generically finite, i.e., for a general point $b \in B$ there are only finitely many preimages of b in Ω_0.

Claim 3.8. *There are no rational sections of π whose image is contained in Ω.*

We will prove this by showing that the closure of any rational section ρ of Ω must contain p, in violation of the hypotheses on π. First observe that ρ factors through one of the subsets $\Omega_0 \subset \Omega$. Now for a general point b in B, $\rho : B \dashrightarrow \Omega_0$ is actually regular in a neighborhood of b and the image $\rho(b)$ lies on some honest section $\tilde{h} : C \to X$ over a curve C in H which contains q and b and such that $\tilde{h}(q) = p$. Since $\pi : \Omega_0 \to B$ is unramified at $\rho(b) = \tilde{h}(b)$ (by genericity of the choice of b), we have that $\tilde{h} : C \to X$ and $\rho|_C : C \dashrightarrow X$ are equal as rational maps. We conclude that $\tilde{h} : C \to X$ factors through the closure of the image of ρ, in particular $p = \tilde{h}(q)$ lies on the closure of the image of ρ.

Applying Lemma 3.1 to each of the countably many components of Ω, we now conclude that for a very general curve C_3 in D, there is no section of

$$\pi : \pi^{-1}(C_3) \to C_3$$

whose image lies in Ω.

We know that for each irreducible component Ω_0 of Ω, p is in $\Omega_0 - V(\pi)$. Thus $\Omega_0 \cap V(\pi)$ is a proper closed subset of Ω_0, and by the finiteness of $\pi : \Omega_0 \to B$, we conclude that for a very general b in B, $\Omega \cap V(\pi) \cap \pi^{-1}(b) = \emptyset$. Choose a very general curve C_3 in D as above, and choose a very general point r on C_3. Then $\Omega \cap \pi^{-1}(r)$ is a countable set disjoint from $V(\pi)$. Given any point in this set, there are at most countably many sections of π over C_3 which take this value at r by Lemma 3.1. Hence there are at most countably many sections of π over C_3 whose value at r is contained in Ω. The image of any such section cannot be contained in Ω and thus meets Ω in at most countably many points with countably many images in C_3. Choosing another point $s \in C_3$ not to lie in any of these countably many countable sets, we conclude that for any section $\tilde{h}_3$ of π over C_3 such that $\tilde{h}_3(r)$ is in Ω, we have that $\tilde{h}_3(s)$ is *not* in Ω.

Now we take our triangle to be $C = C_1 \cup C_2 \cup C_3$ where C_1 and C_2 are members of H which join q to r and q to s respectively. By way of contradiction, suppose there is a stable section $\tilde{h}$ of π over C whose image contains p. As we have discussed, such a stable section consists of honest sections $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$ over C_1, C_2, and C_3 respectively, perhaps with some trees of π-contracted rational curves attached which connect $\tilde{h}_1(r)$ to $\tilde{h}_3(r)$, which connect $\tilde{h}_2(s)$ to $\tilde{h}_3(s)$ and which connect $\tilde{h}_1(q)$ and $\tilde{h}_2(q)$ to p. Since p is not contained in $V(\pi)$, there is no tree of π-contracted rational curves which meets p. Therefore $\tilde{h}_1(q) = \tilde{h}_2(q) = p$. By the definition of Ω, the images $\tilde{h}_1(C_1)$ and $\tilde{h}_2(C_2)$ are therefore necessarily contained in Ω. Since r and s are very general on B, both $\Omega \cap V(\pi) \cap \pi^{-1}(r)$ and $\Omega \cap V(\pi)\pi^{-1}(s)$ are empty, hence no π-contracted rational curves over r or s which meet Ω. In particular, there is no tree of π-contracted rational curves which meets either $\tilde{h}_1(r)$ or $\tilde{h}_2(s)$. So we must have $\tilde{h}_1(r) = \tilde{h}_3(r)$ and $\tilde{h}_2(s) = \tilde{h}_3(s)$. The upshot is that, after pruning any extraneous trees of π-contracted rational curves, we have that $\tilde{h}$ is an honest section of π over the reducible curve C.

But now we have our contradiction: we have seen that for any section $\tilde{h}_3$ of π over C_3 such that $\tilde{h}_3(r)$ is contained in Ω, then $\tilde{h}_3(s)$ is not contained in Ω. On the other hand we have by the last paragraph that $\tilde{h}_3(r) = \tilde{h}_1(r)$ is contained in Ω and also $\tilde{h}_3(s) = \tilde{h}_2(s)$ is contained in Ω. Therefore we conclude there is no stable section $\tilde{h}$ of π over C.

$\square$

Of course, Lemma 3.7 tells us nothing in case the fibers of π are uniruled. Thanks to a construction of Campana and Kollár-Miyaoka-Mori and using Theorem 1.1, we can always reduce to the case that the fibers of π are nonuniruled.

Definition 3.9. *Given a morphism $\pi : X \to B$, the* ***relative MRC fibration*** *is a dominant rational map $\phi : X \dashrightarrow W$ that fits into a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & W \arrow[dl, "\pi'"] \\ B & \end{tikzcd}$$

such that a general fiber of ϕ is rationally connected and a general fiber of π' is not uniruled.

Note that although the general fiber of a rational map is not well-defined, it is well-defined up to birational equivalence. Hence the concept of the rational connectivity of the general fiber of a rational map makes sense. Since the existing references only establish the existence of the absolute MRC fibration (the case where $B = \mathrm{Spec}(K)$ for some field K), we point out that the simplest way to construct this relative version is to simply use the absolute MRC fibration for the fiber over the generic point of B, and choose for W any B-model for the resulting K-scheme. A very readable account of the construction in the case of $B = \mathrm{Spec}(K)$ can be found in [D]. While K is assumed to be algebraically closed there, this hypothesis is never actually used in the construction.

Finally, we warn the reader that the definition of the MRC fibration is usually more complicated. The equivalence of our definition with the usual one is an easy consequence of Theorem 1.1.

Before applying Lemma 3.7 to the proof of our main theorem, we note a corollary which is interesting in its own right.

Corollary 3.10. *If B is a normal, connected, quasi-projective variety, and $\mathcal{H}$ is any family of smooth curves in B which dominates the family of triangles $T(B)$, then for any projective morphism $\pi : X \to B$, the following two conditions are equivalent.*

(1) *A general point of X lies in a pseudosection.*
(2) *For a general curve C parametrized by $\mathcal{H}$, a general point of $X_C = \pi^{-1}(C)$ lies on a section of $\pi : X_C \to C$.*

Note that the existence of such a family $\mathcal{H}$ is ensured by Lemma 2.3.

Proof. Direction (1) $\Rightarrow$ (2) follows from a stronger version of Theorem 1.1: if $\pi : Z \to C$ is a proper morphism from an irreducible variety to a smooth curve, and if a general fiber of π is rationally connected, then a general point of Z is contained in a section of π, cf. [KMM, 2.13] and also [K, Thm. IV.6.10].

Direction (2) $\Rightarrow$ (1) is more interesting. Suppose first that a general fiber of π is not uniruled and that (2) holds. By the properness of the spaces of stable sections of bounded degree, for every curve C parametrized by $\mathcal{H}$, every point of X_C lies on a stable section. Therefore for every triangle $h : C \to B$ in $T(B)$, every point of X_C lies on a stable section. Also a very general point of X is not contained in $V(\pi)$. So, by Lemma 3.7, a very general point of X is contained in a rational section. It follows from a straightforward uncountability argument that a general point of X is contained in a rational section, so (1) holds.

Next suppose that the general fiber of π is uniruled and that (2) holds. Let

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & W \arrow[dl, "\pi'"] \\ B & \end{tikzcd}$$

be a relative MRC fibration. Let $f : X' \to X$ be a proper birational morphism such that $f \circ \phi$ extends to a regular morphism $\phi' : X' \to W$. Let $p \in X$ be a point over which f is an isomorphism. Let C be a curve in $\mathcal{H}$ which contains $\pi(p)$. Let $s : C \to X_C$ be a section which contains p. Then the rational map $f^{-1} \circ s : C \dashrightarrow X'_C$ is defined. Since π is proper and C is smooth, this rational map extends to a section of X'_C. So (2) holds for $X' \to B$. It suffices to prove that (1) holds for X', since the image of a pseudosection in X' is a pseudosection in X. Thus, we will assume from now on that ϕ is a regular morphism.

For any section $s : C \to X_C$, the composition $\phi \circ s : C \to W_C = C \times_B W$ is a section of W_C. Since a general point of X_C is contained in a section s, a general point of W_C is contained in a section $\phi \circ s$, i.e., (2) holds for W. Since a general fiber of π' is not uniruled, a general point of W is contained in a rational section $Z \subset W$. The preimage $\phi^{-1}(W)$ of a rational section through a general point of W is a pseudosection of π passing through a general point of X. So (1) holds for X.

$\square$

Our proof of Theorem 1.3 proceeds similarly. First we will prove the result for maps whose general fiber is not uniruled, and then we will handle the general case by appealing to the relative MRC fibration. In addition we will use an induction on the relative dimension of X over B. We have already considered the case of fiber dimension zero in Proposition 3.1, thus suppose that $d > 0$. By way of induction, assume that we have already constructed a family $\mathcal{H}_{d-1}$ of smooth curves in B which cover B and such that for any morphism $\pi : X \to B$ of relative dimension less than d, π admits a section when restricted over a very general curve in $\mathcal{H}_{d-1}$ if and only if π admits a pseudosection. We construct $\mathcal{H}_d$ as follows. First we construct a family of reducible nodal curves by letting $\tilde{\mathcal{H}}_d$ be the family of maps $f : C \to B$ of the form $C = C_0 \cup C_1 \cup \cdots \cup C_{d+1}$, where $[f_{C_0}]$ is a member of $\mathcal{H}_{d-1}$ and the other C_i, $i = 1, \ldots, d+1$ are triangles which each meet C_0 in a single node which is embedded in the smooth locus of B. Now take $\mathcal{H}_d$ to be any family of smooth curves that dominates $\tilde{\mathcal{H}}_d$. By Lemma 2.3, we can take $\mathcal{H}_d$ to be the family of linear sections of B under a sufficiently positive projective embedding.

We need to check that $\mathcal{H}_d$ satisfies the desired property. Namely, suppose $\pi : X \to B$ is a projective morphism of relative dimension less than or equal to d which does not admit a pseudosection. Then we need to show that over a very general member of $\mathcal{H}_d$, π does not admit a section. By Lemma 2.6, it suffices to check that over a very general member of $\tilde{\mathcal{H}}_d$, π does not admit a stable section.

First we will consider the case where the general fiber of π is not uniruled. In order to later handle the uniruled case, it will be useful for us to prove a statement that seems stronger than necessary. In particular, we will prove a statement that applies to morphisms π which do admit pseudosections. As above, we let $V(\pi)$ be the rational curve locus which is the union of all π-contracted rational curves in X. This is a subset of X which is a countable union of subvarieties. We let $Y(\pi)$ be the union of $V(\pi)$ and the closures of the images of all rational sections of π. This is also a countable union of subvarieties of X. Note that Lemma 3.7 says exactly that for any point p in $X - Y(\pi)$, a very general triangle through $\pi(p)$ admits no stable sections containing p.

Lemma 3.11. *Any stable section of π over a very general member of $\tilde{\mathcal{H}}_d$ has values over C_0 contained in $Y(\pi)$.*

Before proving the lemma, we remark that (given the inductive hypothesis) it immediately implies our theorem in the case where the fibers are not uniruled. If $\pi : X \to B$ is a morphism whose general fiber is not uniruled and which does not admit a pseudosection then $Y(\pi) = V(\pi)$ is a countable union of *proper* subvarieties of X, i.e., it is a countable union of subvarieties $Y(\pi)_0$ of X such that the fiber dimension of $Y(\pi)_0 \to B$ is strictly less than d. By the induction assumption, for a very general C_0 in $\mathcal{H}_{d-1}$, there can be no honest section of π over C_0 contained in any of the subvarieties $Y(\pi)_0$. Thus, by the lemma, there can be no stable section of π over $C = C_0 \cup C_1 \cup \cdots \cup C_{d+1}$.

Proof. We now prove the lemma. We imagine assembling our very general member of $\tilde{\mathcal{H}}_d$ one component at a time. Pick a very general $C_0 \in \mathcal{H}_{d-1}$ and let Σ denote the parameter space of all sections of π over C_0 which are not contained in $Y(\pi)$. This is the complement of a countable union of subvarieties in a countable union of subvarieties of $\overline{M}(X)$. Denote the irreducible components of Σ by Σ^α, and by Lemma 3.4 we conclude that each Σ^α has dimension less than or equal to d.

Our strategy now is simple. The condition that a section over C_0 extends to a stable section over $C_0 \cup C_i$ should impose a condition by Lemma 3.7, and so after imposing $d+1$ conditions there should be no sections left. To prove this, we consider the chain

$$\Sigma_{d+1} \subset \Sigma_d \subset \cdots \subset \Sigma$$

where Σ_i is defined to be the subset of Σ parametrizing sections of π over C_0 which are not contained in $Y(\pi)$ and which can extended to stable sections over $C_0 \cup C_1 \cup \cdots \cup C_i$. That is, if we let $q_1, \ldots, q_{d+1}$ be the very general points at which we attach the triangles, Σ_i parametrizes those sections of π over C_0 whose value at q_j agrees with the value of some stable section of π over C_j for all $j \leq i$. This is a countable union of closed subsets of Σ. We will prove by induction on i that $\dim(\Sigma_i) \leq d - i$ for each i, in particular Σ_{d+1} is empty.

We have already seen that every component of $\Sigma = \Sigma_0$ has dimension at most d, so this establishes the base case $i = 0$. By way of induction, assume that every component of Σ_k has dimension at most $d - k$. Now we want to show the result for $k+1$. For any one of the countably many irreducible components Σ_k^α of Σ_k, for a general point q_{k+1} of C_0, a very general section of π over C_0 parametrized by

Σ_k^α maps q_{k+1} to a point not in $Y(\pi)$. Choosing a very general point q_{k+1}, we can arrange that for every irreducible component Σ_k^α of Σ_k, there is a section $\tilde{h}_0^\alpha$ in Σ_k^α has the property that $p^\alpha = \tilde{h}_0^\alpha(q_{k+1})$ is not in $Y(\pi)$.

Now for each α, for a very general triangle C_{k+1} through q_{k+1}, we conclude by Lemma 3.7 that there is no stable section over C_{k+1} which passes through p^α. So if we choose a very general triangle C_{k+1}, we can arrange that for every α, there is no stable section over C_{k+1} which passes through any of the points p^α. So none of the sections $\tilde{h}_0^\alpha$ extend to a stable section over $C_0 \cup C_{k+1}$. Thus, for each α, $\Sigma_{k+1} \cap \Sigma_k^\alpha$ has dimension strictly less than $\dim(\Sigma_k^\alpha) \le d - k$. Since we have

$$\Sigma_{k+1} = \bigcup_\alpha (\Sigma_{k+1} \cap \Sigma_k^\alpha)$$

we conclude that every irreducible component of Σ_{k+1} has dimension at most $d - k - 1$, as desired. In particular, we conclude that $\Sigma_{d+1} = \emptyset$, i.e., over C_0 every section of π which can be extended to stable sections over C is contained in $Y(\pi)$.

$\square$

As discussed above, Lemma 3.11 proves the induction step in case the fibers of π are not uniruled. So to finish the inductive proof of Theorem 1.3, we are left to consider the case where the fibers of π are uniruled. We argue by contradiction.

By way of contradiction, assume that we have a morphism $\pi : X \to B$ with no pseudosection, but which admits a section when restricted to a very general element of $\mathcal{H}_d$. Let $\phi : X \to W$ be the relative MRC fibration. We may resolve the indeterminacy locus of ϕ by blowing up:

$$\begin{tikzcd}
X' \arrow[d, "f"'] \arrow[dr, "\phi'"] & \\
X \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & W \arrow[dl, "\pi'"] \\
B &
\end{tikzcd}$$

Let $Z \subset X$ denote the fundamental locus of the morphism f (i.e., the image under f of the exceptional divisor of f).

Notice that the relative dimension of $\pi|_Z : Z \to B$ is strictly less than d. Suppose that $\pi|_Z : Z \to B$ admits a section when restricted over a very general curve C in $\mathcal{H}_d$. By Lemma 2.6, we conclude that $\pi|_Z : Z \to B$ admits a stable section when restricted over a stable map in $\widetilde{\mathcal{H}}_d$. In particular, since every curve in $\mathcal{H}_{d-1}$ occurs as the C_0-submap of a stable map in $\widetilde{\mathcal{H}}_d$, we conclude that $Z \to B$ admits a section when restricted over a very general curve C_0 in $\mathcal{H}_{d-1}$. By the induction hypothesis, this implies that there is a pseudosection of $\pi|_Z : Z \to B$. But, in particular, this implies there is a pseudosection of $\pi : X \to B$ which contradicts our assumption. So we conclude that for a very general curve C in $\mathcal{H}_d$, $\pi|_Z : Z \to B$ admits no section when restricted over C. On the other hand, our assumption is that $\pi : X \to B$ does admit a section over C, so there exists a section over C which is not contained in Z. This is the same as a rational section of $\pi \circ f : X' \to B$ over C. Since C is smooth, this rational section of $\pi \circ f$ extends to a regular section of $\pi \circ f$ over C.

Thus we find that $\pi \circ f$ admits a section over a very general curve C in $\mathcal{H}_d$. Now if $\pi \circ f$ admits a pseudosection, so does π by simply taking the image of the pseudosection under f. Therefore we conclude that $\pi \circ f : X' \to B$ admits no pseudosection, but it does admit a section when restricted over a very general curve in $\mathcal{H}_d$. Therefore, as far as deriving a contradiction is concerned, we can replace X by X'. So from now on we assume that $\phi : X \to W$ is a regular morphism.

Let $W' \subset W$ denote the closure of the locus of points over which the fiber of ϕ is not rationally connected. Any rational section of π' not contained in W' gives rise to a pseudosection of π, which does not exist by hypothesis. Therefore all rational sections of π' are contained in W'. Applying Lemma 3.11, we find that over a general member C of $\tilde{\mathcal{H}}_d$, any stable section of π' maps C_0 into the subset $W' \cup V(\pi')$. Thus, any stable section of π over C maps C_0 into the subset $\phi^{-1}(W' \cup V(\pi'))$.

On the other hand, $\phi^{-1}(W' \cup V(\pi'))$ is a countable union of proper subvarieties of X, each of which has relative dimension at most $d-1$ over B. So by the induction hypothesis, every section of $\pi : X \to B$ over the very general curve C_0 in $\mathcal{H}_{d-1}$ has image which is not contained in $\phi^{-1}(W' \cup V(\pi'))$. We conclude that over a very general member C of $\tilde{\mathcal{H}}_d$, π admits no stable section. By Lemma 2.6, π admits no section over a very general member of $\mathcal{H}_d$, and this is a contradiction of our assumptions.

This establishes the inductive step needed and completes the proof of Theorem 1.3.

$\square$

4. The general case

In the last section we proved the main theorem in case B is normal and quasi-projective and $\pi : X \to B$ is projective. In this section we will show how to reduce the general case to this case. We proceed by induction on the dimension.

Suppose that B is an algebraic variety of finite type. By Chow's lemma we can find a projective, birational morphism $B_1 \to B$ such that B_1 is quasi-projective. By Noether normalization, the normalization $B_2 \to B_1$ of B_1 is a finite morphism. Thus $f : B_2 \to B$ is a projective, birational morphism such that B_2 is quasi-projective and normal. Let $\mathcal{H}_d$ be the family of curves C in B_2 constructed in the last section. The restriction of f to a general curve in this family is a nonconstant morphism, and hence a stable map. Therefore, replacing $\mathcal{H}_d$ by a Zariski dense open subset, we may consider $\mathcal{H}_d$ to be a family of stable maps $h : C \to B$ with smooth domain C. The claim is that Theorem 1.3 holds for B and $\mathcal{H}_d$. We will prove this by induction, but before proceeding to the induction argument we introduce a little more notation.

Suppose that $\pi : X \to B$ is a proper morphism of relative dimension at most d which admits no pseudosection. We need to prove that for a very general map $h : C \to B$ in $\mathcal{H}_d$, π admits no section over h. The base change $\pi_2 : X \times_B B_2 \to B_2$ is a proper morphism of relative dimension at most d which admits no pseudosection, since the image under $\pi_1 : X \times_B B_2 \to X$ of a pseudosection of π_2 is a pseudosection of π. By again applying Chow's lemma, we can find a projective, birational morphism $\phi : X_2 \to X \times_B B_2$ such that $\pi_2 \circ \phi : X_2 \to B_2$ is projective. Any pseudosection of $\pi_2 \circ \phi$ maps under ϕ to a pseudosection of π_2. Therefore $\pi_2 \circ \phi$ admits no pseudosection.

Now $\pi_2 \circ \phi : X_2 \to B_2$ satisfies the hypotheses of the last section. By the proof of the main theorem in that section, for a very general curve C in $\mathcal{H}_d$, $\pi_2 \circ \phi$ admits no section over C. Let $Z \subset X$ denote the fundamental locus of the birational, projective morphism $X_2 \to X$, i.e., the locus over which this morphism is not an isomorphism.

If $d = 0$, we are essentially done. The locus $Z \subset X$ is a proper subvariety, and since π is generically finite, also $\pi(Z) \subset B$ is a proper subvariety. If we choose a very general map $h : C \to B$ in $\mathcal{H}_0$, then the image $h(C)$ does not lie in $\pi(Z)$. But then any section of $\pi : X \to B$ over h determines a rational section of $\pi_2 \circ \phi : X_2 \to B_2$ over C. Since C is smooth this rational section extends to a regular section. This contradicts the result of the last section. So we conclude that for a very general map $h : C \to B$ in $\mathcal{H}_0$, there is no section of $\pi : X \to B$ over this map.

Now we proceed by induction via an argument very similar to that in the end of the last section. We have established the base case $d = 0$, so we suppose that $d > 1$. By way of induction, we suppose the theorem has been proved for $d - 1$. Consider $\pi|_Z : Z \to B$. This morphism has fiber dimension at most $d - 1$. By our induction assumption, we conclude that $\pi|_Z : Z \to B$ has no section when restricted over a very general map $h_0 : C_0 \to B$ in $\mathcal{H}_{d-1}$. By Lemma 2.6, we conclude that $\pi|_Z : Z \to B$ has no section when restricted over a very general map $h : C \to B$ in $\mathcal{H}_d$ (since $\mathcal{H}_d$ dominates $\mathcal{H}_{d-1}$). So if we choose a very general map $h : C \to B$ in $\mathcal{H}_d$, then for any section $\tilde{h} : C \to X$ of π over h, we have that $\tilde{h}(C)$ is not contained in Z. So the regular section $\tilde{h}$ determines a rational section of $\pi_2 : X_2 \to B_2$ over C. Since C is smooth, this rational section extends to a regular section. This contradicts the result of the last section. So we conclude that for a very general map $h : C \to B$ in $\mathcal{H}_d$, there is no section of $\pi : X \to B$ over h.

5. Application: families of Enriques surfaces

In this section we will show how to apply Theorem 1.3 to a family of Enriques surfaces to deduce Corollary 1.4, that is, to find a one-parameter family of Enriques surfaces without a section.

5.1. **A family of quartic Enriques surfaces.** The family we will be starting with is the universal family over a parameter space for quartic Enriques surfaces: that is, a family of polarized Enriques surfaces S with a polarization $M \in \mathrm{Pic}(S)$ of self-intersection 4 that includes a general such surface. Now, for the purposes of applying Theorem 1.3 and deducing Corollary 1.4, we can just write down the family as in Definition 5.2 below – we do not need to know that it is actually the generic quartic Enriques surface, and the reader who does not particularly care can jump directly to Definition 5.2 – but since we are going to be working closely with the family it seems worthwhile to take a few paragraphs and establish its origin.

To begin with, since Enriques surfaces have fundamental group $\mathbb{Z}/2\mathbb{Z}$ and have as universal covering space a K3 surface, a quartic Enriques surface S is the quotient of an octic K3 surface (T, L) – that is, a K3 surface T with a polarization $L \in \mathrm{Pic}(T)$ of self-intersection $c_1(L)^2 = 8$ – by an involution τ of T preserving L. For a general octic K3 (T, L), the linear system of sections of L is base-point-free and defines an embedding of T into $\mathbb{P}^5$, and the image surface is the intersection of three quadric hypersurfaces in $\mathbb{P}^5$ with defining equations Q_1, Q_2, and Q_3.

Next, since $\tau^* L \cong L$, the action of τ can be lifted to an action on $H^0(T, L)$, and hence to an involution of $\mathbb{P}^5$ carrying T to itself. Moreover, if we let M be the line bundle on the quotient Enriques surface S obtained by descent, then by Riemann-Roch

$$h^0(S, M) \;=\; \frac{c_1(M)^2}{2} + \chi(\mathcal{O}_S) \;=\; \frac{4}{2} + 1 \;=\; 3$$

the action of τ on $H^0(T, L)$ must have eigenvalues 1 and -1, each with multiplicity 3. We thus have a canonical direct-sum decomposition

$$H^0(T, L) \;=\; \Gamma \oplus \Psi$$

with $\dim \Gamma = \dim \Psi = 3$.

Applying the same principle, we see that the action of τ on $H^0(T, L^2)$ has eigenvalue 1 with multiplicity

$$h^0(S, M^2) \;=\; \frac{c_1(M^2)^2}{2} + \chi(\mathcal{O}_S) \;=\; \frac{16}{2} + 1 \;=\; 9$$

and correspondingly eigenvalue -1 with multiplicity $h^0(T, L^2) - 9 = 18 - 9 = 9$. On the other hand, given that $H^0(T, L) = \Gamma \oplus \Psi$ as above, we can write

$$\mathrm{Sym}^2 H^0(T, L) \;=\; \mathrm{Sym}^2\Gamma \oplus (\Gamma \otimes \Psi) \oplus \mathrm{Sym}^2\Psi$$

with the action of τ on $\mathrm{Sym}^2 H^0(T, L)$ having $(+1)$-eigenspace $\mathrm{Sym}^2\Gamma \oplus \mathrm{Sym}^2\Psi$ of dimension 12 and (-1)-eigenspace $\Gamma \otimes \Psi$ of dimension 9. It follows that the kernel of the restriction map

$$\mathrm{Sym}^2 H^0(T, L) \;\longrightarrow\; H^0(T, L^2)$$

– that is, the vector space of quadrics in $\mathbb{P}^5$ vanishing on T – must be contained in the direct sum $\mathrm{Sym}^2\Gamma \oplus \mathrm{Sym}^2\Psi$. In other words, we can choose homogeneous coordinates

$$[Z, W] \;=\; [Z_0, Z_1, Z_2, W_0, W_1, W_2]$$

on $\mathbb{P}^5$ so that the action of τ is given by

$$\tau \;:\; [Z_0, Z_1, Z_2, W_0, W_1, W_2] \;\mapsto\; [Z_0, Z_1, Z_2, -W_0, -W_1, -W_2]$$

and the defining equations of the double cover T of a general quartic Enriques surface S may be written in the form

$$Q_\alpha(Z,W) \;=\; Q'_\alpha(Z) + Q''_\alpha(W).$$

We are now prepared to write down the families of K3 and Enriques surfaces we will be studying in the sequel. To start with, let Γ and Ψ be 3-dimensional vector spaces and denote by $\mathbb{P}^5$ the projective space of 1-dimensional subspaces of $\Gamma \oplus \Psi$ And let

$$\mathbb{P}^{11} \;=\; \mathbb{P}(\mathrm{Sym}^2\Gamma^\vee \oplus \mathrm{Sym}^2\Psi^\vee)$$

be the projective space of 1-dimensional subspaces of the (12-dimensional) vector space of quadrics on $\mathbb{P}^5$ of the form above. Finally, we let $[Z,W] = [Z_0, Z_1, Z_2, W_0, W_1, W_2]$ be homogeneous coordinates on $\mathbb{P}^5$ with Γ the zero locus of W_0, W_1 and W_2, and Ψ likewise the zero locus of Z_0, Z_1 and Z_2; and we let τ be the involution $[Z_0, Z_1, Z_2, W_0, W_1, W_2] \mapsto [Z_0, Z_1, Z_2, -W_0, -W_1, -W_2]$ of $\mathbb{P}^5$.

Definition 5.1. *By the **principal family of K3 surfaces** we will mean the family $\pi : Y \to B$ with $B = \mathbb{P}^{11} \times \mathbb{P}^{11} \times \mathbb{P}^{11}$ and Y the subvariety of $B \times \mathbb{P}^5$ defined by*

$$Y \;=\; \left\{\; (Q_1, Q_2, Q_3, p) \in B \times \mathbb{P}^5 \mid Q_1(p) = Q_2(p) = Q_3(p) = 0 \;\right\},$$

with $\pi : Y \to B$ the projection on the first factor.

Note that the action of τ on the second factor of $B \times \mathbb{P}^5$ carries Y into itself, so that we can make the following definition.

Definition 5.2. *By the principal family of Enriques surfaces we will mean the family $\pi : X \to B$ with B again as above and X the quotient of the variety Y above by the involution τ of $\mathbb{P}^5$.*

It may be a misnomer to call these families of K3 and Enriques surfaces, since they are only generically that: there are degenerate fibers, and even fibers of dimension greater than 2. But it is convenient to use the term, and we hope the reader will forgive this.

5.2. **Proof of Corollary 1.4.** In order to apply Theorem 1.3 to the principal family of Enriques surfaces and deduce Corollary 1.4, we simply have to show that $X \to B$ admits no pseudosections. We will do this by analyzing the corresponding family $Y \to B$ of K3 surfaces, since their equations are in simpler form. We start with a straightforward result.

Lemma 5.3. *Let $Y \to B$ be the principal family of K3 surfaces of Definition 5.1. The total space Y is smooth, and its Chow ring, $A^*(Y)$, is generated by restrictions of pullbacks of hyperplane classes under the inclusion*

$$Y \;\hookrightarrow\; \mathbb{P}^{11} \times \mathbb{P}^{11} \times \mathbb{P}^{11} \times \mathbb{P}^5.$$

Proof. To start, introduce the variety

$$W \;=\; \{(Q,p) : p \in Q\} \;\subset\; \mathbb{P}^{11} \times \mathbb{P}^5.$$

Via the projection $\eta : W \to \mathbb{P}^5$ on the second factor, W is a $\mathbb{P}^{10}$-bundle over $\mathbb{P}^5$; it is therefore smooth, and its Chow ring is generated over the Chow ring of $\mathbb{P}^5$ by any class whose restriction to the fibers of η is the hyperplane class on $\mathbb{P}^{10}$—for example, the restriction of the pullback of the hyperplane class from $\mathbb{P}^{11}$, via the inclusion $W \hookrightarrow \mathbb{P}^{11} \times \mathbb{P}^5$. Since the total space Y of our principal family of K3 surfaces is (via projection to $\mathbb{P}^5$) simply the triple fiber product

$$Y = W \times_{\mathbb{P}^5} W \times_{\mathbb{P}^5} W$$

the lemma follows.

□

As an immediate corollary of this lemma, we have the following description of cycles $Z \subset X$ of relative dimension 0 over B.

Proposition 5.4. *Let $X \to B$ be the principal family of Enriques surfaces as in Definition 5.2. If $Z \subset X$ is any cycle of codimension 2, the degree of the projection $\pi|_Z : Z \to B$ is divisible by 4.*

Proof. Let $\eta : Y \to X$ be the quotient map. Let T be the class of a general fiber of Y over B. By the preceding Lemma, the class of any cycle in Y is a polynomial (with integer coefficients) in the restrictions to Y of the pullbacks of the hyperplane classes to $\mathbb{P}^{11} \times \mathbb{P}^{11} \times \mathbb{P}^{11} \times \mathbb{P}^5$. But the first three of these classes restrict to 0 on a general fiber, so the class of $\eta^{-1}Z \cdot T$ must be a multiple of the restriction to T of the hyperplane class on $\mathbb{P}^5$ This has degree divisible by 8. As η has degree 2, the proposition follows.

□

As an immediate consequence of Proposition 5.4, we see that *the principal family $X \to B$ of Enriques surfaces has no rational sections*: the image of such a section would give a codimension 2 cycle of X with degree one over B.

In order to show that $X \to B$ admits no pseudosections, it remains to prove that X cannot contain a subvariety $Z \subset X$ whose general fiber over B is an irreducible rational curve. To do this, suppose that Z is such a subvariety. Let $\tilde{Z}$ be a resolution of singularities of Z. We then have a commutative diagram

$$\begin{tikzcd} \tilde{Z} \arrow[r, "f"] \arrow[dr, "\mu"'] & X \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Consider the class $f_*(c_1(\omega_{\tilde{Z}/B}))$ in $A^2(X)$. Since the general fiber of $\tilde{Z}$ over B is a smooth rational curve, this class has degree -2 when restricted to a general fiber of π. This contradicts the fact that all elements of $A^2(X)$ have degree over B divisible by 4.

We have thus established the following result.

Lemma 5.5. *The principal family $X \to B$ of Enriques surfaces admits no pseudosections.*

Applying Theorem 1.3 we may deduce Corollary 1.4.

6. Application: torsors for Abelian varieties

It follows from Theorem 1.3 that any family $\pi : X \to B$ of smooth, connected, projective curves of positive genus over some smooth variety B has a section over B if and only if the restriction of this family over every curve $C \subset B$ has a section: since the fibers contain no rational curves, every pseudosection is a rational section, and every rational section is everywhere defined. Similarly, we have the following corollary.

Corollary 6.1. *Let B be a smooth variety, let $A \to B$ be an Abelian scheme over B (i.e., a family of Abelian varieties over B), and let $\pi : T \to B$ be a torsor for $A \to B$. Then π is a trivial torsor if and only if for every curve $C \subset B$, the restriction $T_C \to C$ is a trivial torsor for $A_C \to C$.*

Since torsors for an Abelian scheme are classified by étale cohomology with coefficients in the group scheme, we can rephrase Corollary 6.1 by saying that the map

$$H^1_{\text{ét}}(B, A) \to \prod_{C \subset B} H^1_{\text{ét}}(C, A_C)$$

is injective. Note that the Lefschetz hyperplane theorem for Picard groups tells us that the same is true if we replace the Abelian scheme $A \to B$ by the commutative group scheme $\mathbb{G}_m \times B \to B$. It is an interesting question, for which other (possibly noncommutative) group schemes over B does this hold.

7. Further questions and conjectures

In this section we will consider some questions and conjectures raised by Theorem 1.3.

7.1. **Arithmetic question.** For arithmetic questions related to rationally connected varieties, we refer the reader to [GHMS1] and [GHMS2]. Let us just mention the following question which is an analogue of our main theorem in the case of fiber dimension 1. Let K be a number field, let B be a smooth scheme defined over K, and let $\pi : X \to B$ be a proper, smooth morphism of schemes whose geometric fibers are connected curves of positive genus. Suppose that for every number field extension L/K, the induced mapping on rational points $\pi : X(L) \to B(L)$ is surjective – we refer to this property by saying π is *arithmetically surjective*. Does it then follow that $\pi : X \to B$ has a section? We may also ask the same question when the geometric fibers of π are Abelian varieties.

7.2. **Possible extensions.** We would like to take a moment here to discuss possible extensions of Theorem 1.3. To begin with, we interpreted the theorem as stating that a family of varieties $\pi : X \to B$ such that every one-parameter subfamily has a section has this property "by virtue of" the fact that X contains a family of rationally connected varieties. But the statement of the theorem asserts only the existence of a pseudosection in $\pi : X \to B$; it does not assert any direct connection between the sections of $X_C \to C$ over very general curves C and the pseudosection. Accordingly, we could ask the following.

Question 7.1. *Does there exist a family $\mathcal{H}'_d$ of curves on B, whose general member is smooth and irreducible, with the property that for any proper morphism $\pi : X \to$*

B of relative dimension d, for very general $[C] \in \mathcal{H}'_d$ every section of the pullback

$$\pi_C : X_C = X \times_B C \to C$$

lies in a pseudosection of $\pi : X \to B$?

One special case of this question is when $\pi : A \to B$ is an Abelian scheme over a smooth variety B. In this case we are asking whether we can find a family of curves $C \subset B$ such that for a very general member of this family, the map

$$H^0_{\text{ét}}(B, A) \to H^0_{\text{ét}}(C, A_C)$$

is surjective. So, in this case, the question above is an H^0-analogue of the H^1-interpretation of Theorem 1.3.

7.3. **Dependence on d.** A simpler question is whether we can eliminate the dependence of the family $\mathcal{H}_d$ of curves on d. The answer to this is "no." Proving this is not so easy, and for full details the reader is referred to [S]. Here we will just sketch an argument, since it may shed some light on how fast the size of the curves in $\mathcal{H}_d$ have to grow with d.

Briefly, for any e we will write down families of hypersurfaces in $\mathbb{P}^n$ parametrized by $B = \mathbb{P}^2$ with the property that their restriction to any curve $C \subset \mathbb{P}^2$ of degree e or less has a section, but which admits no pseudosections at all. To do this, let m and n be any integers; let $\mathbb{P}^n$ be projective n-space with homogeneous coordinates $[X_0, \ldots, X_n]$ and let $\mathbb{P}^N$ be the projective space parametrizing hypersurfaces of degree m in $\mathbb{P}^n$. Let $\mathcal{X} \subset \mathbb{P}^N \times \mathbb{P}^n \to \mathbb{P}^N$ be the universal hypersurface of degree m in $\mathbb{P}^n$; that is, the zero locus of the polynomial

$$F(a, X) \;=\; \sum a_I X^I$$

which is linear in the coordinates a_I on $\mathbb{P}^N$ and of degree m in the X_i. Finally, let $\mathbb{P}^2 \hookrightarrow \mathbb{P}^N$ be a general map of the form

$$\mathbb{P}^2 \;\xrightarrow{\nu_e}\; \mathbb{P}^{\binom{e+2}{2}-1} \;\to\; \mathbb{P}^N$$

where ν_e is a Veronese map of degree e and the second map is a general linear inclusion; and let

$$\pi \;:\; X = \mathbb{P}^2 \times_{\mathbb{P}^N} \mathcal{X} \;\longrightarrow\; \mathbb{P}^2$$

be the pullback of the universal hypersurface to $\mathbb{P}^2$ via this inclusion.

Now assume that

$$\binom{e+2}{2} \;=\; n+1$$

and that m is large. Consider the following two assertions:

- the restriction of the family $\pi : X \to \mathbb{P}^2$ to any curve $C \subset \mathbb{P}^2$ of degree e or less has a section; but

- the family $\pi : X \to \mathbb{P}^2$ itself has no pseudosection.

The first of these assertions is straightforward to prove: under the inclusion $\mathbb{P}^2 \hookrightarrow \mathbb{P}^N$, the span of a curve $C \subset \mathbb{P}^2$ of degree e or less has dimension $\binom{e+2}{2} - 2 = n-1$ or less. Thus the hypersurfaces appearing as fibers of the restriction $X_C \to C$ of the family $\pi : X \to \mathbb{P}^2$ to C are all linear combinations of n hypersurfaces

$G_1, \ldots, G_n \subset \mathbb{P}^n$, and any point of intersection of these hypersurfaces gives a section of $X_C \to C$.

As for the second assertion, we will not prove it here, but we give a "plausibility argument" which suggests it is true. To begin with, a general fiber of $\pi : X \to \mathbb{P}^2$ is a general hypersurface of degree m in $\mathbb{P}^n$; by a result of Clemens [Cl], for m large this will contain no rational curves. Thus to prove the second assertion we need only show that $\pi : X \to \mathbb{P}^2$ has no rational sections.

Since rational sections over $\mathbb{P}^2$ are tricky to parametrize we will restrict to a general curve $C \subset \mathbb{P}^2$ of degree $e+1$, and present evidence that the restriction $X_C \to C$ has no section. To do this, we start by counting the dimension of the family of sections of the product $C \times \mathbb{P}^n$ there are of a given degree k—that is, graphs of maps $C \to \mathbb{P}^n$ of degree k—and then estimating the number of conditions it imposes on such a section to require it lies on the hypersurface $X_C \subset C \times \mathbb{P}^n$. For the first, a map $C \to \mathbb{P}^n$ of degree k is given by a line bundle L of degree k on C, together with $n+1$ sections of L up to scalars. The line bundles of degree k on C are parametrized by the Jacobian of C, which has dimension

$$g \;=\; \binom{e}{2}.$$

If k is large, moreover, each such line bundle will have $k-g+1$ global sections, so the dimension of the family of maps $C \to \mathbb{P}^n$ of degree k is

$$g + (n+1)(k-g+1) - 1 \;=\; (n+1)(k+1) - ng - 1.$$

Now we count how many conditions it is for the graph of such a map to lie in X_C. This is straightforward: when we pull the polynomial $F(a, X)$ defining the universal hypersurface back to C, the coefficients pull back to sections of $\mathcal{O}_C(e)$ and the coordinates X_i to sections of L, so that the pullback of F is a section of the bundle

$$M \;=\; L^{\otimes m} \otimes \mathcal{O}_C(e).$$

The number of conditions for this section to vanish identically should thus be

$$\begin{aligned} h^0(M) \;&=\; \deg(M) - g + 1 \\ &=\; km + e(e+1) - g + 1 \end{aligned}$$

and the expected dimension of the family of sections of $X_C \to C$ of degree k is accordingly

$$(n+1-m)k - (n-1)(g-1) - e(e+1).$$

In particular, for m large this is negative, suggesting that there should be no sections.

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