A NOTE ON FANO MANIFOLDS WHOSE SECOND CHERN CHARACTER IS POSITIVE

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Abstract. This note outlines some first steps in the classification of Fano manifolds for which $c_1^2 - 2c_2$ is positive or nef.

1. Introduction

This note lists the few known examples of Fano manifolds X for which the second graded piece of the Chern character is positive, $\mathrm{ch}_2(T_X) = (C_1^2 - 2C_2)(T_X)/2$. There are also many non-examples. Presumably there are many more positive examples. They do not seem easy to find.

Theorem 1.1. *In the following cases X is Fano and $ch_2(T_X)$ is ample, positive or nef.*

1. *For every projective and weighted projective space, $ch_2(T_X)$ is ample.*
2. *For a Grassmannian $X = Grass(k, n)$ of k-dimensional subspaces of a fixed n-dimensional space with $2k \leq n$, $ch_2(T_X)$ is ample if $k = 1$, positive if $n = 2k$ or $n = 2k + 1$, and nef if $n = 2k + 2$.*
3. *A complete intersection $Y = D_1 \cap \cdots \cap D_r$ in X is Fano if $(C_1(T_X) - ([D_1] + \cdots + [D_r]))|_Y$ is ample. And $ch_2(T_Y)$ is ample, resp. positive, weakly positive, nef, if $(ch_2(T_X) - 1/2([D_1]^2 + \cdots + [D_r]))|_Y$ is ample, resp. positive, weakly positive, nef.*
4. *In particular, for a complete intersection of type $(d_1, \ldots, d_r)$ in a n-dimensional weighted projective space, $ch_2(T_X)$ is ample, resp. nef, if $d_1^2 + \cdots + d_r^2 < n+1$, resp. $\leq n + 1$.*
5. *A product $X \times Y$ of Fano manifolds X and Y is Fano, and $ch_2(T_{X\times Y})$ is nef if $ch_2(T_X)$ and $ch_2(T_Y)$ are nef.*
6. *A projective bundle $Y = \mathbb{P}(E)$ over a Fano manifold X associated to an extension E of $\mathcal{O}_X$ by an invertible sheaf L, is Fano if $c_1(T_X) - c_1(L)$ is ample. And $ch_2(T_Y)$ is nef if $ch_2(T_X) + C_1(L)^2/2$ is nef.*
7. *In particular, let (n, d, a) be integers such that $d \geq 1$, $n \geq (d^2 + d + 1)/2$, and $n - d \geq a \geq \lceil \sqrt{\max(0, d^2 - n - 1)} \rceil$. Let X be a degree d hypersurface in $\mathbb{P}^n$, and let $E = (\mathcal{O}_{\mathbb{P}^n}(-a) \oplus \mathcal{O}_{\mathbb{P}^n})|_X$. Then $Y = \mathbb{P}(E)$ is Fano and $ch_2(T_Y)$ is nef.*

Theorem 1.2. *In the following cases $ch_2(T_X)$ is not ample.*

1. *For a Grassmannian $Grass(k, n)$ with $2k \leq n$, $ch_2(T_X)$ is not ample if $k > 1$, and it is not nef if $(n - 2)/2 > k > 1$.*
2. *For a product $X \times Y$ of positive-dimensional Fano manifolds, $ch_2(T_{X\times Y})$ is not weakly positive.*

(3) *For a projective bundle $Y = \mathbb{P}(E)$ over a positive-dimensional Fano manifold X, $ch_2(T_Y)$ is not weakly positive. Moreover, if $rk(E) > 2$, then Y is nef only if the restriction to every curve in X is semistable.*
(4) *For a blowing up Y of $\mathbb{P}^n$ in a nonempty, codimension 2 center, $ch_2(T_Y)$ is not nef.*

Following are the definitions of nef, weakly positive, positive and ample for cycles of codimension greater than one.

Notation 1.3. Let X be a projective variety over an algebraically closed field. For every integer $k \geq 0$, denote by $N_k(X)$ the finitely-generated free Abelian group of k-cycles modulo numerical equivalence, and denote by $N^k(X)$ the k^{th} graded piece of the quotient algebra $A^*(X)/\text{Num}^*(X)$, cf. [Ful84, Example 19.3.9]. For every $\mathbb{Z}$-module B, denote $N_k(X)_B := N_k(X) \otimes B$, resp. $N^k(X)_B := N^k(X) \otimes B$. Denote by $NE_k(X) \subset N_k(X)$ the semigroup generated by nonzero, effective k-cycles. For B a subring of $\mathbb{R}$, denote by $NE_k(X)_B$ the $B_{>0}$-semigroup in $N_k(X)_B$ generated by $NE_k(X)$.

Definition 1.4. A class in $N^k(X)_{\mathbb{R}}$ is *nef* if it pairs nonnegatively with every element in $\overline{NE}_k(X)$. The corresponding cone is denote $\text{Nef}^k(X)$. A class is *weakly positive* if it pairs positively with every element in $NE_k(X)$. The corresponding cone is denoted $\text{WPos}^k(X)$. A class is *positive* if it is contained in the interior of $\text{Nef}^k(X)$; the interior of $\text{Nef}^k(X)$ is denoted $\text{Pos}^k(X)$. The *ample cone* is the $\mathbb{R}_{>0}$-semigroup generated by the image of the cup-product map, $(\text{Pos}^1(X))^k \to N^k(X)_{\mathbb{R}}$. It is denoted $\text{Ample}^k(X)$, and its elements are *ample* classes.

Remark 1.5. There are obvious inclusions,

$$\text{Ample}^k(X) \subset \text{Pos}^k(X) \subset \text{WPos}^k(X) \subset \text{Nef}^k(X).$$

For $k = 1$, $\text{Ample}^1(X) = \text{Pos}^1(X)$ by definition. Moreover, by Kleiman's criterion, this is the $\mathbb{R}_{>0}$-semigroup generated by first Chern classes of ample invertible sheaves. For $k > 1$, it can happen that $\text{Ample}^k(X) \neq \text{Pos}^k(X)$; for instance, because $(N^1(X))^{\otimes k} \to N^k(X)$ is not surjective. There are also examples where $\text{Pos}^k(X) \neq \text{WPos}^k(X)$ and $\text{WPos}^k(X) \neq \text{Nef}^k(X)$.

2. Projective spaces, Grassmannians, products and complete intersections

2.1. **Projective spaces.** The simplest example is $\mathbb{P}^n$ for $n \geq 2$. Denote by $h \in N^1(\mathbb{P}^n)$ the first Chern class of $\mathcal{O}_{\mathbb{P}^n}(1)$. Using the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0,$$

the Chern character of $T_{\mathbb{P}^n}$ is $(n+1)e^h - 1$. In particular, $\text{ch}_k(T_X) = (n+1)h^k/k!$ for every $k = 1, \dots, n$. So $\text{ch}_k(T_X)$ is ample for $k = 1, \dots, n$.

Weighted projective spaces work the same way provided we consider the space as a smooth Deligne-Mumford stack.

2.2. **Grassmannians.** Let X be the Grassmannian $\text{Grass}(k, n)$ of k-dimensional subspaces of a fixed n-dimensional space. Since $\text{Grass}(k, n) \cong \text{Grass}(n-k, n)$, assume $2k \leq n$ without loss of generality. Denote by $\mathcal{O}_X^{\oplus n} \to S_k^\vee$ the universal rank k quotient. There is an analogue of the Euler sequence,

$$0 \longrightarrow Hom_{\mathcal{O}_X}(S_k^\vee, S_k^\vee) \longrightarrow (S_k^\vee)^{\oplus n} \longrightarrow T_X \longrightarrow 0.$$

The Chern classes of $S_k^\vee$ are the Schubert classes,

$$C_m(S_k^\vee) = \sigma_{1^m} = \sigma_{1,\dots,1}.$$

Therefore, by standard Chern class computations

$$\mathrm{ch}(T_X) = (n-k)k + n\sigma_1 + \left[\frac{n+2-2k}{2}\sigma_2 - \frac{n-2-2k}{2}\sigma_{1,1}\right].$$

In particular, if $n > 2k+2$, then $\mathrm{ch}_2(T_X)$ has negative intersection with the effective Schubert cycle dual to $\sigma_{1,1}$. If $n = 2k+2$, $\mathrm{ch}_2(T_X)$ has intersection number 0. If $n = 2k$ or $n = 2k+1$, $\mathrm{ch}_2(T_X)$ has positive intersection number with every irreducible surface in X. But it is not a multiple of σ_1^2, thus it is not ample.

2.3. **Products.** For a product $X \times Y$, there is an isomorphism

$$T_{X\times Y} \cong \mathrm{pr}_X^* T_X \oplus \mathrm{pr}_Y^* T_Y.$$

Therefore there is an equation

$$\mathrm{ch}(T_{X\times Y}) = \mathrm{pr}_X^* \mathrm{ch}(T_X) + \mathrm{pr}_Y^* \mathrm{ch}(T_Y).$$

In particular $C_1(T_{X\times Y}) = \mathrm{pr}_X^* C_1(T_X) + \mathrm{pr}_Y^* C_1(T_Y)$ is ample if $C_1(T_X)$ and $C_1(T_Y)$ are ample. Similarly, $\mathrm{ch}_2(T_{X\times Y})$ is nef if $\mathrm{ch}_2(T_X)$ and $\mathrm{ch}_2(T_Y)$ are nef. However, for every curve $A \subset X$ and every curve $B \subset Y$, the intersection number of $\mathrm{ch}_2(T_{X\times Y})$ with $A \times B$ is 0. Therefore $\mathrm{ch}_2(T_{X\times Y})$ is not weakly positive.

2.4. **Complete intersections.** Let Y be a smooth complete intersection of divisors $D_1, \dots, D_r$ in X. There is an exact sequence

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow \oplus_{i=1}^r \mathcal{O}_X(D_i)|_Y \longrightarrow 0.$$

Therefore there is an equation

$$\mathrm{ch}(T_Y) = \left[\mathrm{ch}(T_X) - \sum_{i=1}^r e^{[D_i]}\right]|_Y.$$

In other words, for every integer m,

$$\mathrm{ch}_m(T_Y) = \left[\mathrm{ch}_m(T_X) - \frac{1}{m!}\sum_{i=1}^r [D_i]^m\right]|_Y.$$

Therefore Y is Fano if $(C_1(T_X) - ([D_1] + \cdots + [D_r]))|_Y$ is ample. And $\mathrm{ch}_2(T_Y)$ is ample, resp. positive, weakly positive, nef, if $(\mathrm{ch}_2(T_X) - 1/2([D_1]^2 + \cdots + [D_r]))|_Y$ is ample, resp. positive, weakly positive, nef.

In particular, taking X to be an n-dimensional weighted projective spaces, and taking $[D_i] = d_i h$ for each $i = 1, \dots, r$, the Chern character of T_Y is $(n+1)e^h - 1 - \sum_{i=1}^r e^{d_i h}$. Thus $\mathrm{ch}_k(T_Y) = 1/k!(n+1-(d_1^k + \cdots + d_r^k))h^k$ for $k = 1, \dots, n-r$. In particular, if $d_1^2 + \cdots + d_r^k < n+1$, resp. $\leq n+1$, then $\mathrm{ch}_2(T_Y)$ is ample, resp. nef.

3. Projective bundles

One way to produce new examples of Fano manifolds is to form the projective bundle of a vector bundle of "low degree" over a given Fano manifold.

Lemma 3.1. *Let E be a vector bundle on X of rank r. Denote by $\pi : \mathbb{P}E \to X$ the associated projective bundle. The graded pieces of the Chern character of $T_{\mathbb{P}E}$ are, $c_1(T_{\mathbb{P}E}) = r\zeta + \pi^*(c_1(T_X) + c_1(E))$ and $ch_2(T_{\mathbb{P}E}) = r\zeta^2/2 + \pi^* c_1(E)\zeta + \pi^*(ch_2(T_X) + ch_2(E))$, where ζ equals $c_1(\mathcal{O}_{\mathbb{P}E}(1))$.*

Proof. There is an Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}E} \longrightarrow \pi^*E \otimes \mathcal{O}_{\mathbb{P}E}(1) \longrightarrow T_{\mathbb{P}E/X} \longrightarrow 0.$$

Therefore $\mathrm{ch}(T_{\mathbb{P}E/X}) = \pi^*\mathrm{ch}(E)e^{\zeta} - 1$, i.e.,

$$(r + \pi^*c_1(E) + \pi^*\mathrm{ch}_2(E) + \dots)(1 + \zeta + \zeta^2/2 + \dots) - 1 =$$

$$[r-1] + [r\zeta + \pi^*c_1(E)] + [r\zeta^2/2 + \pi^*c_1(E)\zeta + \pi^*\mathrm{ch}_2(E)] + \dots$$

Using the exact sequence,

$$0 \longrightarrow T_{\mathbb{P}E/X} \longrightarrow T_{\mathbb{P}E} \longrightarrow \pi^*T_X \longrightarrow 0,$$

$\mathrm{ch}(T_{\mathbb{P}E})$ equals $\mathrm{ch}(T_{\mathbb{P}E/X})+\pi^*\mathrm{ch}(T_X)$. Thus $\mathrm{ch}_1(T_{\mathbb{P}E/X}) = r\zeta+\pi^*(c_1(T_X)+c_1(E))$ and,

$$\mathrm{ch}_2(T_{\mathbb{P}E}) = r\zeta^2/2 + \pi^*c_1(E)\zeta + \pi^*(\mathrm{ch}_2(T_X) + \mathrm{ch}_2(E)).$$

□

Proposition 3.2. *Let X be a smooth Fano manifold and let E be a vector bundle on X of rank r. The projective bundle $\mathbb{P}E$ is Fano if and only if there exists $\epsilon > 0$ such that for every irreducible curve $B \subset X$,*

$$\mu_B^1(E|_B) - \mu_B(E|_B) \leq (1-\epsilon)deg_B(-K_X)/r,$$

where μ_B and μ_B^1 are the slopes from Definition 5.2, resp. Definition 5.3.

Proof. The invertible sheaf $\omega_{\mathbb{P}E}^\vee$ is π-relatively ample. By hypothesis, $\omega_X^\vee$ is ample. By Lemma 5.4, $\omega_{\mathbb{P}E}^\vee$ is ample iff there exists a real number $\epsilon > 0$ such that

$$\deg_B(g^*\omega_{\mathbb{P}E}^\vee) \geq \epsilon \deg_B(g^*\pi^*\omega_X^\vee),$$

for every finite morphism $g : B \to \mathbb{P}E$ of a smooth, connected curve to X for which $\pi \circ g$ is also finite. Using the universal property of $\mathbb{P}E$, this holds iff for every finite morphism $f : B \to X$ and every invertible quotient $f^*E^\vee \to L^\vee$,

$$\deg_B(g^*\omega_{\mathbb{P}E}^\vee) \geq \epsilon \deg_B(g^*\pi^*\omega_X^\vee),$$

where $g : B \to \mathbb{P}E$ is the associated morphism. By Lemma 3.1, $\deg_B(\omega_{\mathbb{P}E}^\vee)$ equals $rc_1(L^\vee) + c_1(f^*E) + c_1(f^*T_X)$, i.e.,

$$r[c_1(f^*T_X)/r - (\mu_B(L) - \mu_B(f^*E))].$$

So, finally, $\omega_{\mathbb{P}E}^\vee$ is ample iff there exists $\epsilon > 0$ such that for every finite morphism $f : B \to X$ and every invertible quotient $f^*E^\vee \twoheadrightarrow L^\vee$,

$$\mu_B(L) - \mu_B(f^*E) \leq (1-\epsilon)\deg_B(f^*c_1(T_X))/r.$$

Taking the supremum over covers of B and invertible quotients of the pullback of E, this is,

$$\mu_B^1(f^*E) - \mu_B(f^*E) \leq (1-\epsilon)\deg_B(-f^*K_X)/r.$$

Since every finite morphism $f : B \to X$ factors through its image, it suffices to consider only irreducible curves B in X. □

For $r = 2$, there is a necessary and sufficient condition for $\mathrm{ch}_2(T_{\mathbb{P}E})$ to be nef.

Proposition 3.3. *Let E be a vector bundle on X of rank 2. Denoting by $\pi : \mathbb{P}E \to X$ the projection, $ch_2(T_{\mathbb{P}E}) = \pi^*(ch_2(T_X) + 1/2(c_1^2 - 4c_2)(E))$. Therefore $ch_2(T_{\mathbb{P}E})$ is nef iff $ch_2(T_X) + 1/2(c_1^2 - 4c_2)(E)$ is nef. If $dim(X) > 0$, $ch_2(T_{\mathbb{P}E})$ is not weakly positive.*

Proof. By Lemma 3.1, $\mathrm{ch}_2(T_{\mathbb{P}E})$ equals $\zeta^2 + \pi^* c_1(E)\zeta + \pi^*(\mathrm{ch}_2(T_X) + \mathrm{ch}_2(E))$. By definition of the Chern classes of E, $\zeta^2 + \pi^* c_1(E)\zeta + \pi^* c_2(E)$ equals 0. So the class above is $-\pi^* c_2(E) + \pi^*(\mathrm{ch}_2(T_X) + \mathrm{ch}_2(E))$. Finally, $\mathrm{ch}_2(E) - c_2(E)$ equals $1/2(c_1^2 - 2c_2)(E) - c_2(E) = 1/2(c_1^2 - 4c_2)(E)$. $\square$

Applying Proposition 3.2 and Proposition 3.3 to the vector bundle $E = L^\vee \oplus \mathcal{O}_X$ gives Theorem 1.1(6).

Finally, for $r > 2$, there is a necessary condition for $\mathrm{ch}_2(T_{\mathbb{P}E})$ to be nef.

Proposition 3.4. *Let E be a vector bundle of rank $r > 2$ on X. If $ch_2(T_{\mathbb{P}E})$ is nef, then the pullback of E to every smooth, projective, connected curve is semistable. Also, $ch_2(T_{\mathbb{P}E})$ is not weakly positive if $dim(X) > 0$ and if the pullback of E to some curve is strictly semistable, e.g., if X contains a rational curve.*

Proof. If the pullback of E to some smooth, projective, connected curve is not semistable, then by Corollary 5.11, there exists a smooth, projective, connected curve B, a morphism $f : B \to X$, and a rank 2 locally free subsheaf F of f^*E such that f^*E/F is locally free and $\mu_B(F) > \mu_B(E)$. There is an induced morphism $g : \mathbb{P}F \to \mathbb{P}E$ such that $\pi \circ g = f \circ \pi$. By Lemma 3.1, $g^*\mathrm{ch}_2(T_{\mathbb{P}E})$ equals $r\xi^2/2 + \pi^* f^* c_1(E)\xi + \pi^* f^*(\mathrm{ch}_2(T_X) + \mathrm{ch}_2(E))$, where ξ equals $c_1(\mathcal{O}_{\mathbb{P}F}(1))$. Since B is a curve, $f^*(\mathrm{ch}_2(T_X) + \mathrm{ch}_2(E))$ equals 0. Also, by definition of the Chern classes of F, $\xi^2 + \pi^* c_1(F)\xi = 0$. Substituting in,

$$g^*\mathrm{ch}_2(T_{\mathbb{P}E}) = 1/2\pi^*(2c_1(f^*E) - rc_1(F))\xi.$$

In particular, $\deg_{\mathbb{P}F}(g^*\mathrm{ch}_2(T_{\mathbb{P}E}))$ equals $1/2(2\deg_B(c_1(f^*E)) - r\deg_B(F))$. This equals $r(\mu_B(f^*E) - \mu_B(F))$, which is negative by construction. Therefore $\mathrm{ch}_2(T_{\mathbb{P}E})$ is not nef. $\square$

Remark 3.5. A vector bundle on a product of projective spaces whose restriction to every curve is semistable is of the form $L^{\oplus r}$, where L is an invertible sheaf, [OSS80, Thm. 3.2.1]. In this case, $\mathbb{P}E$ is also a product of projective spaces.

Corollary 3.6. *Let X be a Fano manifold. For every vector bundle E on X of rank $r > 1$, $ch_2(T_{\mathbb{P}E})$ is not weakly positive.*

4. Blowings up

Let X be a smooth, connected, projective variety, let $i : Y \hookrightarrow X$ be the closed immersion of a smooth, connected subvariety of X of codimension c. Denote by $\nu : \widetilde{X} \to X$ the blowing up of X along Y. Denote by $\pi : E \to Y$ the exceptional divisor. Denote by $j : E \to \widetilde{X}$ the obvious inclusion. Then $E = \mathbb{P}N_{Y/X}$ and $i^*\mathcal{O}_{\widetilde{X}}(E)$ is canonically isomorphic to $\mathcal{O}_{\mathbb{P}N}(-1)$.

Lemma 4.1. *The graded pieces of the Chern character of $\widetilde{X}$ are, $c_1(T_{\widetilde{X}}) = \nu^* c_1(T_X) - (c-1)[E]$ and $ch_2(T_{\widetilde{X}}) = \nu^* ch_2(T_X) + (c+1)[E]^2/2 - i_*\pi^* c_1(N_{Y/X})$*

Proof. Using the short exact sequence,

$$0 \longrightarrow \nu^*\Omega_X \longrightarrow \Omega_{\widetilde{X}} \longrightarrow j_*\Omega_\pi \longrightarrow 0,$$

$\mathrm{ch}(\Omega_{\widetilde{X}})$ equals $\nu^*\mathrm{ch}(\Omega_X) + \mathrm{ch}(j_*\Omega_\pi)$. Grothendieck-Riemann-Roch for the morphism j gives,

$$\mathrm{ch}(Rj_*a) = j_*(\mathrm{ch}(a))(1 - e^{-[E]})/[E].$$

Using the Euler sequence for Ω_π,

$$0 \longrightarrow \Omega_\pi \longrightarrow \pi^* N^\vee_{Y/X} \otimes \mathcal{O}_{\mathbb{P}N}(-1) \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

$\mathrm{ch}(\Omega_\pi)$ equals $\pi^*\mathrm{ch}(N^\vee_{Y/X})i^*(1+e^{[E]})-1$. Putting the pieces together gives the lemma. $\square$

When is $\widetilde{X}$ Fano? Denote by $\mathcal{C}_1$ the collection of finite morphisms $g : B \to X$ from a smooth, connected curve to X whose image is not contained in Y. Denote by $\mathcal{C}_2$ the collection of finite morphisms $g : B \to Y$ from a smooth, connected curve to Y. The following result is well-known.

Proposition 4.2. *Let h be the first Chern class of an ample invertible sheaf on X, e.g., $h = c_1(T_X)$ if X is Fano. The blowing up $\widetilde{X}$ is Fano iff there exists $\epsilon > 0$ such that,*

(i) *for every $g : B \to X$ in $\mathcal{C}_1$,*

$$deg_B(g^{-1}Y) \leq \frac{1}{c-1}(deg_B(g^*c_1(T_X)) - \epsilon deg_B(g^*h)),$$

and

(ii) *for every $g : B \to Y$ in $\mathcal{C}_2$,*

$$\mu^1_B(g^*N_{Y/X}) \leq \frac{1}{c-1}(deg_B(g^*c_1(T_X)) - \epsilon deg_B(g^*h)).$$

The proof is similar to the proof of Proposition 3.2. Using an analogue of Proposition 3.3, no blowing-up of $\mathbb{P}^n$ is a Fano manifold with ch_2 nef.

5. Theorems about vector bundles on curves

There are two theorems in this section. The first theorem goes back to Shou-Wu Zhang, though possibly it is older. A much more sophisticated arithmetic analogue was also proved by Shou-Wu Zhang in [Zha95, Theorem 1.10]. The second theorem in this section is a variation of the first theorem.

Definition 5.1. Let B be a smooth, projective curve. A *cover* of B is a finite, flat morphism $f : C \to B$ of constant, positive degree. A *vector bundle* on B is a locally free $\mathcal{O}_B$-module of constant rank.

Definition 5.2. Let B be a smooth, projective curve. For every non-zero vector bundle E on B, the *slope* is,

$$\mu_B(E) = \deg(E)/\mathrm{rank}(E) = \chi(B,E)/\mathrm{rank}(E) - \chi(B,\mathcal{O}_B).$$

For every cover $f : C \to B$ and every non-zero vector bundle E on C, the *B-slope* is,

$$\mu_B(f,E) := \deg(E)/(\deg(f)\mathrm{rank}(E)) = \mu_B(f_*E) - \mu_B(f_*\mathcal{O}_C).$$

When there is no chance of confusion, this is denoted simply $\mu_B(E)$.

For every cover $g : C' \to C$, $f \circ g : C' \to B$ is a cover and $\mu_B(f \circ g, g^*E)$ equals $\mu_B(f,E)$.

Definition 5.3. Let B be a smooth, projective curve and let E be a vector bundle on B of rank $r > 0$. For every integer $1 \leq k \leq r$, define $\mu_B^k(E)$ to be,

$$\sup\{-\mu_B(f, F^\vee) | f : C \to B \text{ a cover }, f^*E^\vee \to F^\vee \text{ a rank } k \text{ quotient}\}$$

$$= \sup\{\mu_B(f, F) | f : C \to B \text{ a cover }, F \subset f^*E \text{ a rank } k \text{ subbundle whose cokernel is locally free}\}.$$

Let $f : X \to Y$ be a morphism of projective varieties. Denote by $\mathcal{C}_1$ the collection of all irreducible curves in X not contained in a fiber of f. Denote by $\mathcal{C}_2$ the collection of finite morphisms $g : C \to X$ occurring as the normalization of an irreducible curve in X not contained in a fiber of f. Finally, denote by $\mathcal{C}_3$ the collection of all finite morphisms from smooth, connected curves to X whose image is not contained in a fiber of f.

Lemma 5.4. *Let $f : X \to Y$ be a morphism of projective varieties and let L be an ample invertible $\mathcal{O}_Y$-module. An f-ample invertible $\mathcal{O}_X$-module M is ample iff there exists a real number $\epsilon > 0$ such that for every morphism $g : C \to X$ in $\mathcal{C}_1$, resp. $\mathcal{C}_2, \mathcal{C}_3$, $deg_C(g^*M) \geq \epsilon deg_C(g^*f^*L)$.*

Proof. Because M is f-ample and L is ample, there exists an integer $n > 0$ such that $M \otimes f^*L^{\otimes n}$ is ample. By Kleiman's criterion, M is ample iff there exists a real number $0 < \delta < 1$ such that for every irreducible curve C in X,

$$\deg_C(M) \geq \delta \deg_C(M \otimes f^*L^{\otimes n}).$$

Simplifying, this is equivalent to,

$$\deg_C(M) \geq \frac{n\delta}{1-\delta} \deg_C(f^*L).$$

As M is f-ample, this holds if C is contained in a fiber of f. So M is ample iff the inequality holds for every curve in $\mathcal{C}_1$. Setting $\epsilon = n\delta/(1-\delta)$, $\delta = \epsilon/(n+\epsilon)$, gives the lemma.

Since $\mathcal{C}_2 \subset \mathcal{C}_3$, the condition for $\mathcal{C}_3$ implies the condition for $\mathcal{C}_2$. Since degrees on a curve can be computed after pulling back to the normalization, the condition for $\mathcal{C}_2$ implies the condition for $\mathcal{C}_1$. Finally, for every morphism $g : C \to X$ in $\mathcal{C}_3$, $g(C)$ is in $\mathcal{C}_1$. The inequality for $g(C)$ implies the inequality for C. Thus the condition for $\mathcal{C}_1$ implies the condition for $\mathcal{C}_3$. ☐

Lemma 5.5. *Let B be a smooth, connected, projective curve. A nonzero vector bundle E on B is ample iff there exists a positive real number δ such that for every cover $f : C \to B$ and every invertible quotient $f^*E \to L$, $\mu_B(L) \geq \delta$. In other words, E is ample iff $\mu_B^1(L^\vee) < 0$.*

Proof. Denote by $\pi : \mathbb{P}E^\vee \to B$ the projective bundle associated to $E^\vee$, and denote by $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ the tautological invertible quotient. By definition, E is ample iff $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is an ample invertible sheaf. Of course $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is π-relatively ample. Let M be an invertible $\mathcal{O}_B$-module of degree 1. Then M is ample. By Lemma 5.4, $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is ample iff there exists $\epsilon > 0$ such that for every smooth, connected curve C and every finite morphism $g : C \to \mathbb{P}E^\vee$ such that $\pi \circ g$ is finite, $\deg_C(g^*\mathcal{O}_{\mathbb{P}E^\vee}(1)) \geq \epsilon \deg_C(g^*\pi^*M)$. Of course $\deg_C(g^*\pi^*M) = \deg(\pi \circ g)$. Using the universal property of $\mathbb{P}E^\vee$, this holds iff for every cover $f : C \to B$ and every invertible quotient $f^*E \to L$,

$$\deg_C(L) \geq \epsilon\deg(f) \Leftrightarrow \mu_B(L) \geq \epsilon.$$

$\square$

Lemma 5.6. *For every ample vector bundle E on B, there exists a cover $f : C \to B$, invertible $\mathcal{O}_C$-modules $L_1, \dots, L_r$, and a morphism of $\mathcal{O}_C$-modules, $\phi : f^*E \to (L_1 \oplus \cdots \oplus L_r)$ such that,*

(i) *the support of coker(ϕ) is a finite set,*
(ii) *for every $i = 1, \dots, r$, the projection $f^*E \to \oplus_{j\neq i} L_j$ is surjective, and*
(iii) *for every $i = 1, \dots, r$, $\mu_B(L_i) = deg_B(E)$.*

Proof. Denote $r = \text{rank}(E)$. The claim is that for every $k = 1, \dots, r$, there exists a cover $f_k : C_k \to B$, invertible $\mathcal{O}_{C_k}$-modules $L_{k,1}, \dots, L_{k,k}$, and a morphism of $\mathcal{O}_{C_k}$-modules, $\phi_k : f^*E \to (L_{k,1} \oplus \cdots \oplus L_{k,k})$ satisfying (ii) and (iii) above and the following variant of (i): for $k < r$, ϕ_k is surjective and for $k = r$, the support of coker(ϕ_k) is a finite set. The lemma is the case $k = r$. The claim is proved by induction on k.

The base case is $k = 1$. Denote by $\pi : \mathbb{P}E^\vee \to B$ the projective bundle associated to $E^\vee$, and denote by $\pi^*E \twoheadrightarrow \mathcal{O}_{\mathbb{P}E^\vee}(1)$ the tautological invertible quotient. By hypothesis, $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is ample. By Bertini's theorem, for $d_1, \dots, d_{r-1} \gg 0$, there exist effective Cartier divisors $D_1, \dots, D_{r-1}$ with $D_i \in |\mathcal{O}_{\mathbb{P}E^\vee}(d_i)|$ such that the intersection $C_1 = D_1 \cap \cdots \cap D_r$ is a smooth, connected curve, cf. [Jou83]. Denote by $f_1 : C_1 \to B$ the restriction of π. Denote by $\phi_1 : f^*E \to L_{1,1}$ the restriction of $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$. This satisfies (i) because $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ is surjective. It satisfies (ii) trivially. Finally, $\deg(f)$ equals $d_1 \times \cdots \times d_{r-1}$, and $\deg_{C_1}(L_{1,1})$ equals $d_1 \times \cdots \times d_{r-1} \times [c_1(\mathcal{O}_{\mathbb{P}E^\vee}(1))]^r$, i.e., $d_1 \times \cdots \times d_{r-1} \times \deg_B(E)$. Therefore $\mu_B(L_{1,1}) = \deg_B(E)$, i.e., this satisfies (iii).

By way of induction, assume the result is known for $k < r$, and consider the case $k + 1$. Since ϕ_k is surjective, there is an induced closed immersion $\mathbb{P}(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee \hookrightarrow \mathbb{P}(f_k^*E)^\vee$. The image is irreducible and has codimension $r - k \geq 1$. For every $i = 1, \dots, k$, the image of $\mathbb{P}(\oplus_{j\neq i} L_{k,j})^\vee$ is irreducible and has codimension $r - k + 1 \geq 2$. Associated to the finite morphism f_k, there is a finite morphism $\mathbb{P}(f_k^*E)^\vee \to \mathbb{P}E^\vee$. The pullback of an ample invertible sheaf by a finite morphism is ample; hence $\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(1)$ is ample. By Bertini's theorem, for $d_1, \dots, d_{r-1} \gg 0$, there exist effective Cartier divisors $D_1, \dots, D_{r-1}$ with $D_i \in |\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(d_i)|$ such that the intersection $C_{k+1} = D_1 \cap \cdots \cap D_{r-1}$ is a smooth, connected curve, disjoint from $\mathbb{P}(\oplus_{j\neq i} L_j)^\vee$ for every $i = 1, \dots, k$, and either disjoint from $\mathbb{P}(\oplus_i L_i)^\vee$ if $k < r - 1$, or else intersecting $\mathbb{P}(\oplus_i L_i)^\vee$ in finitely many points if $k = r - 1$. Define $g_{k+1} : C_{k+1} \to C_k$ to be the restriction of the projection. Define $f_{k+1} = f_k \circ g_{k+1}$, define $L_{k+1,i} = g_{k+1}^* L_{k,i}$ for $i = 1, \dots, k$, and define $L_{k+1,k+1}$ to be the restriction of $\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(1)$. Define ϕ_{k+1} to be the obvious morphism.

The cokernel of ϕ_{k+1} is supported on the intersection of C_{k+1} with $\mathbb{P}(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee$. By construction, this is empty if $k < r - 1$, and is a finite set if $k = r - 1$. Thus ϕ_{k+1} satisfies (i). By the induction hypothesis, $f_{k+1}^*E \to (L_{k+1,1} \oplus \cdots \oplus L_{k+1,k})$, which is the pullback under g_{k+1} of ϕ_k, is surjective. For $i = 1, \dots, k$, the cokernel of $f_{k+1}^*E \to \oplus_{j\neq i} L_{k+1,j}$ is supported on the intersection of C_{k+1} with the image of $\mathbb{P}(\oplus_{j\neq i} L_{k,j})^\vee)$. By construction, this is empty, i.e., $f_{k+1}^* E \to \oplus_{j\neq i} L_{k+1,j}$ is surjective. Thus ϕ_{k+1} satisfies (ii). Finally, ϕ_{k+1} satisfies (iii) by the same argument as in the base case. The claim is proved by induction on k. $\square$

Theorem 5.7. *For every non-zero vector bundle E on B, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a invertible quotient $f^*E \to L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. In other words, $\mu_B^1(E^\vee) \geq \mu_B(E^\vee)$.*

Proof. Denote $r = \operatorname{rank}(E)$. If $r = 1$, set $f = \mathrm{Id}_B$ and $L = E$. Then L is an invertible quotient of f^*E, and $\mu_B(L)$ equals $\mu_B(E)$ which is less than $\mu_B(E) + \epsilon$. Therefore assume $r > 1$.

Certainly an effective version of the following argument can be given, but a simpler argument is by contradiction.

Hypothesis 5.8. For every cover $f : C \to B$ and every invertible quotient $f^*E \to L$, $\mu_B(L)$ is $\geq \mu_B(E) + \epsilon$, i.e., $\mu_B^1(E^\vee) < \mu_B(E^\vee) - \epsilon$.

By way of contradiction, assume Hypothesis 5.8. Let $f : C \to B$ be a connected, smooth cover of degree d. For every $a/d \in \frac{1}{d}\mathbb{Z}$, there exists an invertible sheaf M on C of degree a, and thus $\mu_B(M) = a/d$. In particular, for d sufficiently large, there exists an invertible quotient M such that $0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1)$. Denote $\delta = \mu_B(E) - \mu_B(M)$. Denote $F = f^*E \otimes M^\vee$. Then $\mu_B(F)$ equals δ, and $0 < \delta < \epsilon/(r-1)$.

Let $g : C' \to C$ be any cover and let $g^*F \to N$ be any invertible quotient. Then $f \circ g : C' \to B$ is a cover and $(f\circ g)^*E = g^*F \otimes g^*M \to N \otimes g^*M$ is an invertible quotient. By Hypothesis 5.8,

$$\mu_C(N) = \deg(f)\mu_B(N) = \deg(f)(\mu_B(N \otimes g^*M) - \mu_B(M))$$

$$\geq \deg(f)((\mu_B(E) + \epsilon) - \mu_B(M)) > \deg(f)\epsilon.$$

By Lemma 5.5, F is an ample vector bundle on C. By Lemma 5.6, there exists a cover $g : C' \to C$ and an invertible quotient $g^*F \to P$ such that $\mu_B(P) = r\mu_B(F) = r\delta$. Therefore $L := g^*M \otimes P$ is an invertible quotient of g^*f^*E and,

$$\mu_B(L) = \mu_B(g^*M \otimes P) = \mu_B(M) + r\delta = \mu_B(E) + (r-1)\delta.$$

By hypothesis, $(r-1)\delta < \epsilon$. So $\mu_B(L) < \mu_B(E) + \epsilon$, contradicting Hypothesis 5.8. The proposition is proved by contradiction. $\square$

Corollary 5.9. *For every non-zero vector bundle E on B, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a sequence of vector bundle quotients,*

$$f^*E = E^r \twoheadrightarrow E^{r-1} \twoheadrightarrow \cdots \twoheadrightarrow E^1,$$

such that each E^k is a vector bundle of rank k and $\mu_B(E^k) < \mu_B(E) + \epsilon$.

Proof. The proof is by induction on the rank r of E. If $\operatorname{rank}(E) = 1$, defining $f = \mathrm{Id}_B$ and $E^1 = E$, the result follows. Thus, assume $r > 1$ and the result is known for smaller values of r. By Theorem 5.7, there exists a cover $g : B' \to B$ and a rank 1 quotient $g^*E \to L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. Denote by K the kernel of $g^*E \to L$. Then $\operatorname{rank}(K) = r - 1$ and $\mu_B(K) = (r\mu_B(E) - \mu_B(L))/(r-1)$. By the induction hypothesis, there exists a cover $h : C \to B'$ and a sequence of vector bundle quotients,

$$h^*K = K^{r-1} \twoheadrightarrow \cdots \twoheadrightarrow K^1,$$

such that each K^k is a vector bundle of rank k, and $\mu_{B'}(K^k) \leq \mu_{B'}(K) + \deg(g)\epsilon$. Of course $\mu_B(F) = \mu_{B'}(F)/\deg(g)$ for every F. Thus $\mu_B(K^k) \leq \mu_B(K) + \epsilon$.

Define $f = h \circ g$, define $E^1 = h^*L$, and for every $k = 2, \ldots, r$, define $f^*E \twoheadrightarrow E^k$ to be the unique quotient whose kernel is contained in h^*K and such that $h^*K \to E^k$ has image K^{k-1}. Then $\mu_B(E^1) = \mu_B(L) \leq \mu_B(E) + \epsilon$, and for $k = 2, \ldots, r$,

$$\mu_B(E^k) = 1/k(\mu_B(L) + (k-1)\mu_B(K^{k-1})) < 1/k(\mu_B(L) + (k-1)\mu_B(K) + (k-1)\epsilon) =$$

$$\tfrac{r(k-1)}{(r-1)k}\mu_B(E) + \tfrac{r-k}{(r-1)k}\mu_B(L) + \tfrac{(r-1)(k-1)}{(r-1)k}\epsilon < \mu_B(E) + \tfrac{r-k}{(r-1)k}\epsilon + \tfrac{(r-1)(k-1)}{(r-1)k}\epsilon < \mu_B(E) + \epsilon.$$

□

For semistable bundles in characteristic zero, there is a more precise result. An arithmetic analogue is also proved by Zhang in [Zha95, Theorem 1.10].

Theorem 5.10 (Zhang). *Let B be a smooth, projective curve over an algebraically closed field of characteristic 0. Let E be a semistable vector bundle on B. Let ϵ be a positive real number. There exists a cover $f : C \to B$, invertible sheaves $L_1, \ldots, L_r$ on C, and a morphism of $\mathcal{O}_C$-modules, $\phi : f^*E \to (L_1 \oplus \cdots \oplus L_r)$ such that,*

(i) *the support of coker(ϕ) is a finite set,*
(ii) *for every $i = 1, \ldots, r$, the projection $f^*E \to \oplus_{j \neq i} L_j$ is surjective,*
(iii) *for every $i = 1, \ldots, r$, $\mu_B(L_i) \leq \mu_B(E) + \epsilon$.*

Proof. Denote $r = \text{rank}(E)$. If r equals 1, the theorem is trivial. Thus assume $r > 1$. As in the proof of Theorem 5.7, there exists a cover $g : C' \to B$ and an invertible sheaf M on C' such that $0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1)$. Denote $\delta = \mu_B(E) - \mu_B(M)$ and denote $F = g^*E \otimes M^\vee$. Then $\mu_B(F)$ equals δ, and $0 < \delta < \epsilon/(r-1)$.

Let $h : C \to C'$ be any cover and let $h^*F \to N$ be an invertible quotient. The composition $g \circ h : C \to B$ is a cover. By Kempf's theorem, [Kem92], which ultimately relies on the theorem that every stable vector bundle admits a Hermite-Einstein metric, $(g \circ h)^*E$ is semistable. (Note, there are counterexamples in positive characteristic.) Therefore h^*F is semistable. So $\mu_C(L) \geq \mu_C(h^*F)$, i.e., $\mu_{C'}(L) \geq \mu_{C'}(F) = \delta$. Thus by Lemma 5.5, F is an ample vector bundle on C'. Thus by Lemma 5.6, there exists a cover $h : C \to C'$, invertible $\mathcal{O}_C$-modules $N_1, \ldots, N_r$, and a morphism of $\mathcal{O}_C$-modules $\psi : h^*F \to (N_1 \oplus \cdots \oplus N_r)$ satisfying (i), (ii) and (iii) of Lemma 5.6. Define $f = g \circ h$, $L_i = N_i \otimes h^*M$ and ϕ is the twist of ψ by Id_{h^*M}. Then ϕ satisfies (i) and (ii). And for every $i = 1, \ldots, r$,

$$\mu_B(L_i) = \mu_B(N_i) + \mu_B(M) = \mu_{C'}(N_i)/\deg(g) + \mu_B(E) - \delta =$$

$$\mu_B(E) + r\delta/\deg(g) - \delta \leq \mu_B(E) + (r-1)\delta/\deg(g) < \mu_B(E) + \epsilon.$$

□

Of course, $\mu_B^r(E)$ equals $\mu_B(E)$. The other values are more interesting.

Corollary 5.11. *The slopes $\mu_B^k(E)$ satisfy $\mu_B^1(E) \geq \mu_B^2(E) \geq \cdots \geq \mu_B^r(E) = \mu_B(E)$. For each $1 \leq k < r$, $\mu_B^k(E) = \mu_B(E)$ iff f^*E is semistable for every cover $f : C \to B$.*

Proof. By Corollary 5.9, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a rank k quotient $f^*E \to E^k$ such that $\mu_B(E^k) < \mu_B(E) + \epsilon$. Thus $\mu_B^k(E) \geq \mu_B(E)$. Applying the same reasoning to rank $k-1$ quotients of rank k quotients of f^*E, $\mu_B^{k-1}(E) \geq \mu_B^k(E)$.

If f^*E is semistable for every cover $f : C \to B$, then every vector bundle quotient of f^*E has slope $\geq \mu_C(f^*E)$, and thus has B-slope $\geq \mu_B(f^*E)$. Therefore $\mu_B^k(E) \leq \mu_B(E)$, i.e., $\mu_B^k(E) = \mu_B(E)$.

Conversely, suppose there is a cover $f : C \to B$ such that f^*E is not semistable. Then there exists a vector bundle quotient $f^*E \twoheadrightarrow F$ such that $\mu_B(F) < \mu_B(E)$. Denote the rank by l. Suppose first that $l \geq k$, and define $\epsilon = \deg(f)(\mu_B(E) - \mu_B(F))$. Then by Corollary 5.9, there exists a cover $g : C' \to C$ and a rank k quotient $g^*F \twoheadrightarrow G$ such that $\mu_C(G) < \mu_C(F) + \epsilon$. Therefore $g^*f^*E \twoheadrightarrow g^*F \twoheadrightarrow G$ is a rank k quotient of g^*f^*E and $\mu_B(G) < \mu_C(F) + (\mu_B(E) - \mu_B(F)) = \mu_B(E)$. Therefore $\mu_B^k(E) > \mu_B(E)$.

Next suppose that $l < k$. Denote by K the kernel of $f^*E \to F$. Then $r\mu_B(E) = l\mu_B(F) + (r-l)\mu_B(K)$. Define,

$$\epsilon = \frac{(r-k)l\deg(f)(\mu_B(E) - \mu_B(F))}{(r-l)(k-l)}.$$

By Corollary 5.9, there exists a cover $g : C' \to C$ and a rank $k-l$ quotient $g^*K \twoheadrightarrow G'$ such that $\mu_C(G') < \mu_C(K) + \epsilon$. Therefore $\mu_B(G') < \mu_B(K) + \epsilon/\deg(f)$. Define $g^*f^*E \to G$ to be the unique vector bundle whose kernel is contained in g^*K and such that the image of $g^*K \to G$ equals G'. Then,

$$k\mu_B(G) = l\mu_B(F) + (k-l)\mu_B(G') < l\mu_B(F) + (k-l)\mu_B(K) + (k-l)\epsilon/\deg(f) =$$

$$l\mu_B(F) + \tfrac{k-l}{r-l}(r\mu_B(E) - l\mu_B(F)) + \tfrac{k-l}{\deg(f)}\epsilon =$$

$$k\mu_B(E) - \tfrac{(r-k)l}{r-l}(\mu_B(E) - \mu_B(F)) + \tfrac{(r-k)l}{r-l}(\mu_B(E) - \mu_B(F)) = k\mu_B(E).$$

Thus $\mu_B(G) < \mu_B(E)$, and therefore $\mu_B^k(E) > \mu_B(E)$. □

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