GLOBAL SECTIONS OF SOME VECTOR BUNDLES ON KONTSEVICH MODULI SPACES

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Abstract. On the Kontsevich moduli space of unpointed stable maps to $\mathbb{P}^1$ of genus 0 and degree e, there is a tautological vector bundle of rank $e-1$. Global sections of tensor powers of this vector bundle arise when considering holomorphic contravariant tensors on Kontsevich spaces of stable maps to more general projective varieties. The computation of the global sections is reduced to an explicit combinatorial problem.

1. Introduction

sec-intro

This note concerns the computation of the global sections of tensor powers of a tautological vector bundle on the Kontsevich moduli space of unpointed stable maps to $\mathbb{P}^1$ of genus 0 and degree e. This computation is a small part of a larger project whose goal is to prove the conjecture (apparently due to Fano) that there exist smooth Fano hypersurfaces that are not unirational.

The strategy of the proof is as follows (cf. [3]). For a Fano hypersurface X and every integer $e > 0$, the idea is to prove that on $\overline{\mathcal{M}}_{0,0}(X,e)$ there are many *holomorphic contravariant tensors*, i.e. sections of tensor powers of the cotangent bundle. Because the restriction of a holomorphic contravariant tensor to a rational curve is 0, these tensors "bound" the rational curves on $\overline{\mathcal{M}}_{0,0}(X,e)$, which in turn "bound" the rational surfaces on X. The goal is to produce enough tensors to prove that through a very general point of X there is no rational surface, thus proving X is not unirational. In her thesis, Beheshti has done this for $e = 1$ and X a smooth hypersurface of degree n or $n-1$ in $\mathbb{P}^n$: she proves that for a point of X in the complement of countably many codimension 2 subvarieties, there is no rational surface ruled by lines contained in X containing the point. Beheshti's result appears to extend to the case $e \leq n$.

Two techniques are known for producing tensors on $\overline{\mathcal{M}}_{0,0}(X,e)$. The first uses the correspondence coming from the universal stable map over $\overline{\mathcal{M}}_{0,0}(X,e)$ to obtain holomorphic $(p,0)$-forms on $\overline{\mathcal{M}}_{0,0}(X,e)$ from holomorphic $(p+1,1)$-forms on X, cf. [1]. The second produces pluricanonical forms on $\overline{\mathcal{M}}_{0,0}(X,e)$ by computing the canonical divisor class and proving it is *big*, cf. [3]. Neither of these techniques produce enough tensors when e is large.

A third technique, to which this note is directly relevant, is to produce tensors *from the inside out*. There is a closed substack Y of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizing degree e covers of lines on X. The technique is to produce tensors on the formal neighborhood of Y in $\overline{\mathcal{M}}_{0,0}(X,e)$. The idea is to prove these tensors algebraize, or

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to at least understand the formal tensors well enough to guess which come from algebraic tensors (hopefully suggesting some fourth technique for proving these tensors algebraize).

Behind this is the fact that Y is *much simpler* than $\overline{\mathcal{M}}_{0,0}(X,e)$: The stack Y is the total space of a fibration whose base is the Fano variety of lines on X, and whose fiber is isomorphic to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,e)$. Everything is known about the Fano variety of lines on X. And quite a bit is known about $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,e)$. The restriction of the cotangent bundle of $\overline{\mathcal{M}}_{0,0}(X,e)$ to Y has a filtration where the subquotients are the pullback of the cotangent bundle on the Fano variety of lines, the relative cotangent bundle of the fibration, and the conormal bundle of Y in $\overline{\mathcal{M}}_{0,0}(X,e)$. The conormal bundle is essentially the tensor product of the pullback from the Fano variety of lines of a well-understood vector bundle with the vector bundle $\mathcal{R}_Y^\vee$ considered in this note.

The vector bundle $\mathcal{R}_Y^\vee$ is a relative version of a tautological vector bundle $\mathcal{R}^\vee$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,e)$. The vector bundle $\mathcal{R}^\vee$ is very natural, but it has not been studied before. The question here is the most basic one: What are the global sections of $\mathcal{R}^\vee$? More generally, what are the global sections of a tensor power of $\mathcal{R}^\vee$, and what are the global sections of the tensor product of a power of $\mathcal{R}^\vee$ with a power of the cotangent bundle? Forming the direct sum of global sections of every power of $\mathcal{R}^\vee$ gives a (non-commutative) ring, and forming the direct sum of global sections of the tensor product of every power of $\mathcal{R}^\vee$ with a fixed power of the cotangent bundle gives a left-module over this ring. The question is to describe this ring and to describe the module over this ring: in what degrees is the ring generated, in what degrees are the relations, what are the dimensions of the graded pieces of this ring, in what degrees is the module generated, etc?

sec-gc

2. The general construction

subsec-unptd

2.1. **Unpointed curves.** Let M be a Deligne-Mumford stack over B. Let $\pi : C \to B$ be a proper, flat, representable 1-morphism of relative dimension 1. Let $g : C \to \mathbb{P}E$ be a 1-morphism. There is a natural map of coherent $\mathcal{O}_M$-modules,

$$\phi^\dagger_{\pi,g} : E^\vee \otimes_{\mathcal{O}_B} \mathcal{O}_M \to \pi_* g^* \mathcal{O}_{\mathbb{P}E}(1),$$

constructed as follows. There is a natural map of coherent sheaves on $M \times_B \mathbb{P}E$,

$$(\pi,g)^\# : \mathcal{O}_{M\times_B \mathbb{P}E} \to (\pi,g)_*\mathcal{O}_C.$$

This induces a map of coherent sheaves,

$$(\pi,g)^\#_1 : \mathrm{pr}^*_{\mathbb{P}E}\mathcal{O}_{\mathbb{P}E}(1) \to (\pi,g)_* g^*\mathcal{O}_{\mathbb{P}E}(1).$$

The map $\phi^\dagger_{\pi,g}$ is the map obtained by applying $(\mathrm{pr}_M)_*$ to $(\pi,g)^\#_1$. The subject of this note is the coherent sheaf,

$$\mathcal{R}_{\pi,g} = \mathrm{Coker}(\phi^\dagger_{\pi,g}).$$

Assume that $(\pi,g) : C \to M\times_B \mathbb{P}E$ is surjective and generically finite of degree e.

prop-1

Proposition 2.1. *Suppose the total derived pushforward $R(\pi,g)_*\mathcal{O}_C$ is a perfect, bounded complex. Suppose that for every geometric point p of M, $h^0(C_p,\mathcal{O}_{C_p}) = 1$ and $h^1(C_p,\mathcal{O}_{C_p}) = 0$. Then*

(i) $(\pi,g)^\#$ *is injective and the cokernel is a locally free sheaf of rank $e-1$,*

(ii) $\phi^{\dagger}_{\pi,g}$ *is injective and* $\mathcal{R}_{\pi,g}$ *is locally free of rank* $e-1$*, and*

(iii) *the cokernel of* $(\pi,g)^{\#}_1$ *is isomorphic to* $pr_M^*\mathcal{R}_{\pi,g}$.

Proof. **(i)** The claim is that $R(\pi,g)_*\mathcal{O}_C$ is quasi-isomorphic to a locally free sheaf concentrated in degree 0; in particular $(\pi,g)_*\mathcal{O}_C$ is a locally free sheaf. A perfect, bounded complex concentrated in nonnegative degrees is quasi-isomorphic to a locally free sheaf concentrated in degree 0 iff all the higher cohomology sheaves are zero. By a standard downward induction argument, it suffices to prove that for every geometric point $p \in M$, all the higher cohomology sheaves of $R(\pi,g)_*\mathcal{O}_C \otimes^{\mathbf{L}}_{\mathcal{O}_M} \kappa(p)$ are zero.

Denote by $g_p : C_p \to \mathbb{P}E_p$ the fiber of (π,g) over p. The higher cohomology sheaves are the same as the higher direct image sheaves $R^i(g_p)_*\mathcal{O}_{C_p}$. Every fiber of g_p has dimension ≤ 1. Thus $R^i(g_p)_*\mathcal{O}_{C_p} = (0)$ for $i > 1$. It remains to prove that $R^1(g_p)_*\mathcal{O}_{C_p}$ is zero. Since g_p is finite over a dense open subset of $\mathbb{P}E_p$, $R^1(g_p)_*\mathcal{O}_{C_p}$ is a torsion sheaf. By the Leray spectral sequence, $H^0(\mathbb{P}E_p, R^1(g_p)_*\mathcal{O}_{C_p})$ is a subspace of $H^1(C_p, \mathcal{O}_{C_p})$, which is zero. A torsion sheaf on $\mathbb{P}E_p$ with no global sections is the zero sheaf. So $R^1(g_p)_*\mathcal{O}_{C_p} = (0)$, proving the claim.

Item (i) claims that $(\pi,g)^{\#}$ is injective and the cokernel is locally free. The domain and target of $(\pi,g)^{\#}$ are locally free sheaves. So, by the local flatness criterion, it suffices to prove that for every geometric point p of $M \times_B \mathbb{P}E$, the map $(\pi,g)^{\#} \otimes \kappa(p)$ is injective. This map is the inclusion of scalars $\kappa(p)$ in the ring of global sections of the fiber $(\pi,g)^{-1}(p)$. Because (π,g) is surjective, the fiber is nonempty. Thus inclusion of scalars is injective.

(ii) and (iii) Because $h^0(\mathcal{O}) = 1$ for every geometric fiber of π, by the same type of argument as above, $(\mathrm{pr}_M)_*(\pi,g)^{\#}$ is an isomorphism of $\mathcal{O}_M$-modules. In particular, the cokernel of $(\pi,g)^{\#}$ is a locally free sheaf of rank $e-1$ such that both $(\mathrm{pr}_M)_*$ and $R^(\mathrm{pr}_M)*$ are zero. Combined with Grothendieck's Lemma, this implies that the cokernel is of the form $\mathrm{pr}_M^*\mathcal{R}' \otimes \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(-1)$ for some locally free $\mathcal{O}_M$-module of rank $e-1$, $\mathcal{R}'$. Therefore there is a short exact sequence,

$$0 \to \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(1) \to (\pi,g)_*g^*\mathcal{O}_{\mathbb{P}E}(1) \to \mathrm{pr}_M^*\mathcal{R}' \to 0.$$

Item (ii) follows by applying $(\mathrm{pr}_M)_*$. In particular, $\mathcal{R}' = \mathcal{R}_{\pi,g}$, which is (iii). $\square$

rmk-1

Remark 2.2. If π is a local complete intersection morphism, then also (π,g) is a local complete intersection morphism so that $R(\pi,g)_*\mathcal{O}_C$ is a perfect, bounded complex. In particular, if π is a proper, flat family of connected, at-worst-nodal curves of arithmetic genus 0, the hypotheses of Proposition 2.1 hold.

Let $\pi' : C' \to M$ and $g' : C' \to \mathbb{P}E$ be a second pair satisfying the hypotheses of Proposition 2.1. Let $h : C \to C'$ be a 1-morphism of M-stacks such that g is equivalent to $g' \circ h$. Then there is an induced morphism of $\mathcal{O}_M$-modules, $h^* : \mathcal{R}_{\pi',g'} \to \mathcal{R}_{\pi,g}$.

Let $u : N \to M$ be a 1-morphism of Deligne-Mumford stacks. Denote by C_N the 2-fiber product, $N \times_M C$. Denote by $\pi_N : C_N \to N$ and $g_N : C_N \to \mathbb{P}E$ the induced 1-morphisms. There is a canonical isomorphism of $\mathcal{O}_N$-modules, $u^*\mathcal{R}_{\pi,g} \to \mathcal{R}_{\pi_N,g_N}$.

notat-1

Notation 2.3. For $M = \overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, e)$ and for π and g as in Section 1 denote $\mathcal{R}_{\pi,g}$ by $\mathcal{R}_{\mathbb{P}E,e}$, or simply $\mathcal{R}$ when there is no risk of confusion.

subsec-ptd

2.2. **Pointed curves.** Let M, C, π and g be as in Subsection 2.1. Let $\sigma : M \to C$ be a section of π. Denote by $\tau : M \to M \times_B \mathbb{P}E$ the composition $(\pi, g) \circ \sigma$. Denote by $\mathcal{I}$ the ideal sheaf of $\sigma(M)$ in C. Denote by $\mathcal{O}_{M\times_B \mathbb{P}E}(-\tau)$ the invertible ideal sheaf of $\tau(M)$ in $M \times_B \mathbb{P}E$.

There is a natural map of coherent $\mathcal{O}_M$-modules,

$$\phi^\dagger_{\pi,g,\sigma} : (\mathrm{pr}_M)_*(\mathrm{pr}^*_{\mathbb{P}E}\mathcal{O}_{\mathbb{P}E}(1) \otimes (-\tau)) \to \pi_*(g^*\mathcal{O}_{\mathbb{P}E}(1) \otimes \mathcal{I}),$$

constructed as follows. There is a natural map of coherent sheaves on $M \times_B \mathbb{P}E$,

$$(\pi, g)^\#_\sigma : \mathcal{O}_{M\times_B\mathbb{P}E}(-\tau) \to (\pi, g)_*\mathcal{I}.$$

This induces a map of coherent sheaves,

$$(\pi, g)^\#_{\sigma,1} : \mathrm{pr}^*_{\mathbb{P}E}\mathcal{O}_{\mathbb{P}E}(1)(-\tau) \to (\pi, g)_*(g^*\mathcal{O}_{\mathbb{P}E}(1) \otimes \mathcal{I}).$$

The map $\phi^\dagger_{\pi,g,\sigma}$ is the map obtained by applying $(\mathrm{pr}_M)_*$ to $(\pi, g)^\#_{\sigma,1}$. As above, assume that (π, g) is surjective and generically finite of degree e.

prop-2

Proposition 2.4. *Suppose the total derived pushforward $R(\pi, g)_*\mathcal{O}_C$ is a perfect bounded complex. Suppose that for every geometric point p of M, $h^0(C_p, \mathcal{O}_{C_p}) = 1$ and $h^1(C_p, \mathcal{I} \otimes \kappa(p)) = 0$. Then*

(i) *$(\pi, g)^\#_\sigma$ is injective and the cokernel is canonically isomorphic to the cokernel of $(\pi, g)^\#$, and*
(ii) *$\phi^\dagger_{\pi,g,\sigma}$ is injective and the cokernel is canonically isomorphic to $\mathcal{R}_{\pi,g}$.*

Proof. Because $\mathcal{O}_C$ and $\sigma_*\mathcal{O}_M$ are flat over M, also $\mathcal{I}$ is flat over M. Therefore $\mathcal{I} \otimes \kappa(p)$ is the ideal sheaf of $\sigma(p)$ in C_p. Because h^1 of this sheaf is zero, also $h^1(C_p, \mathcal{O}_{C_p}) = 0$. So the hypotheses of Proposition 2.1 hold. Therefore $(\pi, g)_*\mathcal{O}_C$ is locally free.

Associated to the short exact sequence,

$$0 \to \mathcal{I} \to \mathcal{O}_C \to \sigma_*\mathcal{O}_M \to 0,$$

there is a long exact sequence of higher direct images,

$$0 \to (\pi, g)_*\mathcal{I} \to (\pi, g)_*\mathcal{O}_C \to \tau_*\mathcal{O}_M \to R^1(\pi, g)_*\mathcal{I} \to 0.$$

To prove that $R^1(\pi, g)_*\mathcal{I}$ is zero, it suffices to prove that for every geometric point p of M, the fiber $R^1(\pi, g)_*\mathcal{I} \otimes \kappa(p)$ is zero. The fiber is $R^1(g_p)_*(\mathcal{I} \otimes \kappa(p))$. Because g_p is generically finite, this is a torsion sheaf. By the Leray spectral sequence, the space of global sections of this sheaf is a subspace of $H^1(C_p, \mathcal{I} \otimes \kappa(p))$, which is zero by hypothesis. A torsion sheaf on $\mathbb{P}E_p$ with no global sections is the zero sheaf. Therefore $R^1(\pi, g)_*\mathcal{I} = (0)$.

There is a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{M\times_B\mathbb{P}E}(-\tau) \arrow[r] \arrow[d, "(\pi,g)^\#_\sigma"'] & \mathcal{O}_{M\times_B\mathbb{P}E} \arrow[r] \arrow[d, "(\pi,g)^\#"] & \tau_*\mathcal{O}_M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & (\pi, g)_*\mathcal{I} \arrow[r] & (\pi, g)_*\mathcal{O}_C \arrow[r] & \tau_*\mathcal{O}_M \arrow[r] & 0
\end{tikzcd}$$

By Proposition 2.1, $(\pi, g)^\#$ is injective and the cokernel is canonically isomorphic to $\pi^*_M\mathcal{R} \otimes \pi^*_{\mathbb{P}E}\mathcal{O}_{\mathbb{P}E}(-1)$. Therefore, by the Snake Lemma, $(\pi, g)^\#_\sigma$ is injective and the cokernel is canonically isomorphic to $\pi_M^*\mathcal{R}\otimes\pi_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(-1)$. This proves (i), and (ii) follows by applying $(\mathrm{pr}_M)_*$. □

3. The case $e = 2$

sec-e2

In the special case that $e = 2$, the Deligne-Mumford stack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, e)$ has a particularly simple description. This leads to a complete description of the global sections of each $(\mathcal{R}^\vee)^{\otimes a}$. The symmetric square, $\mathrm{Sym}^2(\mathbb{P}E/M)$ is canonically isomorphic to the Hilbert scheme of degree 2 divisors on fibers of $\mathbb{P}E$, i.e. $\mathbb{P}\mathrm{Sym}^2(E^\vee)$. There is a *branch morphism*, i.e. a 1-morphism $\mathrm{br} : \overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, 2) \to \mathbb{P}\mathrm{Sym}^2(E^\vee)$, cf. [2].

The morphism (π, g) is finite and flat of degree 2. The trace morphism gives a splitting of the short exact sequence,

$$0 \to \mathcal{O}_{M\times_B\mathbb{P}E} \to (\pi, g)_*\mathcal{O}_C \to \mathrm{pr}_M^*\mathcal{R} \otimes \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(-1) \to 0.$$

Then the multiplication map on $(\pi, g)_*\mathcal{O}_C$ induces a morphism,

$$(\mathrm{pr}_M^*\mathcal{R} \otimes \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(-1))^{\otimes 2} \to \mathcal{O}_{M\times_B\mathbb{P}E}.$$

Computing étale locally, or even formally locally, this morphism is an isomorphism onto the ideal sheaf of the branch divisor. By definition of the branch morphism, this ideal sheaf is,

$$\mathrm{pr}_M^*\mathrm{br}^*\mathcal{O}_{\mathbb{P}\mathrm{Sym}^2(E^\vee)}(-1) \otimes \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(-2).$$

The upshot is that there is a canonical isomorphism of coherent sheaves,

$$\mathrm{pr}_M^*(\mathcal{R})^{\otimes 2} \to \mathrm{pr}_M^*\mathrm{br}^*\mathcal{O}_{\mathbb{P}\mathrm{Sym}^2(E^\vee)}(-1).$$

Applying $(\mathrm{pr}_M)_*$ gives an isomorphism of invertible sheaves on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, 2)$,

$$\mathcal{R}^{\otimes 2} \to \mathrm{br}^*\mathcal{O}_{\mathbb{P}\mathrm{Sym}^2(E^\vee)}(-1).$$

Therefore there is a 1-morphism from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, 2)$ to the Deligne-Mumford stack Y over $\mathbb{P}\mathrm{Sym}^2(E^\vee)$ parametrizing square roots of $\mathcal{O}(-1)$.

Conversely, on $Y \times_B \mathbb{P}E$, one can reverse the logic above to construct an algebra whose relative Spec is a finite flat scheme over $Y \times_B \mathbb{P}E$, say $C_Y \to Y \times_B \mathbb{P}E$. Computing locally, this is a family of stable maps to $\mathbb{P}E$. So there is an induced 1-morphism $Y \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, 2)$. It is straightforward to compute that these 1-morphisms define an equivalence of stacks. In particular, the space of global sections of $(\mathcal{R}^\vee)^{\otimes a}$ is canonically isomorphic to the space of global sections of the same power of the square root of $\mathcal{O}(1)$. Therefore,

$$H^0(\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, 2), (\mathcal{R}^\vee)^{\otimes a}) \cong \begin{cases} \mathrm{Sym}^{\frac{a}{2}}(\mathrm{Sym}^2 E), & a \text{ is even} \\ (0), & a \text{ is odd} \end{cases}$$

4. The global quotient description of an open substack

subsec-gq

Unfortunately, for $e \geq 3$, there is no such simple description of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, e)$. However, there is a dense open substack that is a global quotient by the action of $\mathbf{PGL}_2$ of a dense open subscheme of a projective space. The idea is to first compute the spaces of sections over this open subset by invariant theory, and then to determine which sections extend to regular sections on all of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, e)$.

Let e be an integer ≥ 2. For $i = 1, \dots, \lfloor \frac{e}{2} \rfloor$, denote by $\Delta_i \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, e)$ the irreducible Cartier divisor whose general point parametrizes a stable map whose domain has two irreducible components, one of degree i over $\mathbb{P}E$ and one of degree $e-i$ over $\mathbb{P}E$. Define $U \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B,e)$ to be the complement of $\Delta_2 \cup \cdots \cup \Delta_{\lfloor \frac{e}{2} \rfloor}$. In case $e=2$ or 3, U is all of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B,e)$. In every case, U intersects Δ_1.

Let F be a locally free sheaf of rank 2 on B. This sheaf is auxiliary; the simplest choice is $F = \mathcal{O}_B \oplus \mathcal{O}_B$. Denote by $P = P_{e_1}$ the Hilbert scheme over B of flat families of closed subschemes of $\mathbb{P}E \times \mathbb{P}F$ whose fibers are in the complete linear system of $\mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(1) \otimes \mathrm{pr}_{\mathbb{P}F}^*\mathcal{O}_{\mathbb{P}F}(e)$, i.e.

$$P = \mathbb{P}H^0(\mathbb{P}E \times_B \mathbb{P}F, \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(1) \otimes \mathrm{pr}_{\mathbb{P}F}^*\mathcal{O}_{\mathbb{P}F}(e)) = \mathbb{P}(E^\vee \otimes \mathrm{Sym}^e F^\vee).$$

Denote by $\mathcal{D} = \mathcal{D}_e$ the Cartier divisor in $P \times \mathbb{P}E \times \mathbb{P}F$ that is the universal closed subscheme. There is a resolution of the structure sheaf,

$$0 \to \mathrm{pr}_P^*\mathcal{O}_P(-1) \otimes \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(-1) \otimes \mathrm{pr}_{\mathbb{P}F}^*\mathcal{O}_{\mathbb{P}F}(-e) \to \mathcal{O} \to \mathcal{O}_{\mathcal{D}} \to 0.$$

The morphism $(\mathrm{pr}_P, \mathrm{pr}_{\mathbb{P}E}) : \mathcal{D} \to P \times \mathbb{P}E$ is finite and flat of degree e. There is a maximal open subscheme of P over which $\mathcal{D}$ is a family of stable maps to $\mathbb{P}E$. Denote this open subscheme by V. The complement of V in P is a closed set that has codimension ≥ 3. There is an induced 1-morphism $f : V \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B,e)$. By consideration of divisors on $\mathbb{P}E \times \mathbb{P}F$, the image of f is in U. Thus consider f as a 1-morphism, $f : V \to U$.

There is an induction argument that proves that $f : V \to U$ is surjective on geometric points (the induction is on the number of irreducible components of the domain curve). There is an action m of $\mathrm{Aut}(\mathbb{P}F) = \mathbf{PGL}(F)$ on P. The open subscheme V is $\mathbf{PGL}(F)$-invariant. The 1-morphism f is $\mathbf{PGL}(F)$-invariant, in the sense that there is a 2-isomorphism between the following 1-morphisms,

$$\mathbf{PGL}(F) \times_B V \xrightarrow{\mathrm{pr}_V} V \xrightarrow{f} U,$$
$$\mathbf{PGL}(F) \times_B V \xrightarrow{m} V \xrightarrow{f} U.$$

There is an induced 1-morphism $f' : \mathrm{PGL}(F) \times_B V \to V \times_U V$ such that $\mathrm{pr}_1 \circ f' = m$ and $\mathrm{pr}_2 \circ f' = \mathrm{pr}_V$. By a straightforward (but tedious) argument, f' is an isomorphism of schemes. So there is an induced 1-morphism,

$$u : [\mathbf{PGL}(F) \backslash V] \to U.$$

The claim is that u is an equivalence of stacks. This can be checked after base-change by the surjective, smooth morphism $f : V \to U$. But the base change is the identity morphism Id_V by construction. Because u is an equivalence of stacks, there is a canonical isomorphism between the space of sections of $(\mathcal{R}^\vee)^{\otimes a}$ over U and the space of $\mathrm{PGL}(F)$-invariant sections of $f^*(\mathcal{R}^\vee)^{\otimes a}$ over V. This can be computed by invariant theory.

sec-invt

5. Invariant theory description

The morphisms $\mathrm{pr}_P : \mathcal{D} \to P$ and $\mathrm{pr}_{\mathbb{P}E} : \mathcal{D} \to \mathbb{P}E$ satisfy the hypotheses of Proposition 2.1. Denote by $\mathcal{R}_P$ the associated locally free sheaf. Twisting the short exact sequence in the last section, there is a short exact sequence,

$$0 \to \mathrm{pr}_P^*\mathcal{O}_P(-1) \otimes \mathrm{pr}_{\mathbb{P}F}^*\mathcal{O}_{\mathbb{P}F}(-e) \to \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(1) \to \mathrm{pr}_{\mathbb{P}E}^*\mathcal{O}_{\mathbb{P}E}(1) \otimes \mathcal{O}_{\mathcal{D}} \to 0.$$

Applying $(\mathrm{pr}_P)_*$, gives an isomorphism of $\mathcal{R}_P$ with $\mathcal{O}_P(-1)\otimes_{\mathcal{O}_B} H^1(\mathbb{P}F, \mathcal{O}_{\mathbb{P}F}(-e))$. Therefore the dual, $\mathcal{R}_P^\vee$ is isomorphic to,

$$\mathcal{R}_P^\vee \cong \mathcal{O}_P(1)\otimes_{\mathcal{O}_B} H^0(\mathbb{P}F, \mathcal{O}_{\mathbb{P}F}(e)\otimes\omega_{\mathbb{P}F/B}) \cong \mathcal{O}_P(1)\otimes_{\mathcal{O}_B}\left[\mathrm{Sym}^{e-2}(F^\vee)\otimes\wedge^2(F^\vee)\right].$$

This uses the canonical isomorphism, $\omega_{\mathbb{P}F/B}\cong\mathcal{O}_{\mathbb{P}F}(-2)\otimes_{\mathcal{O}_B}\wedge^2(F^\vee)$.
By definition of f, there is an isomorphism of $V\times_P\mathcal{D}$ with the pullback by f of the universal stable map. Therefore there is a canonical isomorphism of $f^*\mathcal{R}^\vee$ with the restriction of $\mathcal{R}_P$ to V. So there is a canonical isomorphism between the space of sections of $(\mathcal{R}^\vee)^{\otimes a}$ on U with the space of $\mathbf{PGL}(F)$-invariant sections of $(\mathcal{R}_P^\vee)^{\otimes a}$ on V. Because the complement of V has codimension ≥ 3, every section on V extends to a global section. So there is a canonical isomorphism,

$$\begin{aligned}&H^0(U,(\mathcal{R}^\vee)^{\otimes a})\cong H^0(P,(\mathcal{R}_P^\vee)^{\otimes a})^{\mathbf{PGL}(F)} = \\ &\left(\mathrm{Sym}^a(E\otimes\mathrm{Sym}^e F)\otimes\left[\mathrm{Sym}^{e-2}(F^\vee)\otimes\wedge^2(F^\vee)\right]^{\otimes a}\right)^{\mathbf{PGL}(F)}.\end{aligned}$$

A few words about this. First, there is an obvious action of $\mathbf{GL}(F)$ on the space of global sections. By inspection the center of $\mathbf{GL}(F)$ acts trivially on this space, so there is an induced action of $\mathbf{PGL}(F)$. But it is more convenient to consider this as a $\mathbf{GL}(F)$-representation and compute the $\mathbf{GL}(F)$-invariants.

Second, there is an obvious action of the symmetric group $\mathfrak{S}_a$. A full description of $H^0(U,(\mathcal{R}^\vee)^{\otimes a})$ should include a description as a $\mathfrak{S}_a$-representation. Likewise, there is an action of $\mathbf{GL}(E)$, and a full description should describe this action. Finally, there is an obvious product map,

$$H^0(U,(\mathcal{R}^\vee)^{\otimes a})\otimes H^0(U,(\mathcal{R}^\vee)^{\otimes b})\to H^0(U,(\mathcal{R}^\vee)^{\otimes(a+b)}).$$

A full description should describe this product map.

In some sense the formula,

$$H^0(U,(\mathcal{R}^\vee)^{\otimes a})\cong\left(\mathrm{Sym}^a(E\otimes\mathrm{Sym}^e F)\otimes\left[\mathrm{Sym}^{e-2}(F^\vee)\otimes\wedge^2(F^\vee)\right]^{\otimes a}\right)^{\mathbf{GL}(F)},$$

satisfies all of these conditions. In another sense, it satisfies none of them. By inspection, the center of $\mathbf{GL}(E)$ acts by the character $t\mapsto t^a$. Thus the irreducible decomposition of this representation is of the form,

$$\bigoplus_{u=0}^{\lfloor\frac{a}{2}\rfloor}\mathrm{Sym}^{a-2u}(E)\otimes(\wedge^2E)^{\otimes u}\otimes_{\mathbb{Q}} W_{a,u},$$

where $W_{a,u}$ is a finite dimensional $\mathfrak{S}_a$-representation over $\mathbb{Q}$. The irreducible $\mathfrak{S}_a$-representations are in 1-to-1 correspondence with partitions $\lambda=(\lambda_1,\dots,\lambda_k)$ of a; denote by V_λ the corresponding representation. There is an irreducible decomposition,

$$W_{a,u}=\bigoplus_\lambda W_{a,u,\lambda}\otimes V_\lambda,$$

where $W_{a,u,\lambda}$ is a trivial $\mathfrak{S}_a$-representation over $\mathbb{Q}$.

The first goal is to compute, for each (a,u,λ), the dimension of the $\mathbb{Q}$-vector space $W_{a,u,\lambda}$. The second goal is to compute the multiplication map on the direct summands. For each pair of integers $a,b\geq 0$, for each triple of integers $0\leq u\leq\lfloor\frac{a}{2}\rfloor$,

$0 \leq v \leq \lfloor \frac{b}{2} \rfloor$, $0 \leq w \leq \lfloor \frac{a+b}{2} \rfloor$, and for each triple of a partition λ of a, μ of b and ν of $a+b$, there is a map $T_{(a,b),(u,v,w),(\lambda,\mu,\nu)}$,

$$\begin{gathered}\mathrm{Sym}^{a-2u}(E) \otimes \mathrm{Sym}^{b-2v}(E) \otimes (\wedge^2 E)^{\otimes(u+v)} \otimes V_\lambda \otimes V_\mu \otimes W_{a,u,\lambda} \otimes W_{b,v,\mu} \\ \to \mathrm{Sym}^{a+b-2w}(E) \otimes (\wedge^2 E)^{\otimes w} \otimes V_\nu \otimes W_{a+b,w,\nu}.\end{gathered}$$

The multiplication map is the sum of all these maps.

This map is equivariant for all the obvious groups, so the map can be decomposed by plethysm into a sum of smaller maps. For the triple of partitions λ, μ, ν, denote by $N_{\lambda,\mu,\nu}$ the corresponding Littlewood-Richardson coefficient, so that,

$$V_\lambda \otimes V_\mu \cong \bigoplus_\nu (V_\nu)^{\oplus N_{\lambda,\mu,\nu}}.$$

Without loss of generality, suppose $a - 2u \geq b - 2v$. There is an irreducible decomposition,

$$\mathrm{Sym}^{a-2u}(E) \otimes \mathrm{Sym}^{b-2v}(E) = \bigoplus_{k=0}^{b-2v} \mathrm{Sym}^{a+b-2u-2v-2k}(E) \otimes (\wedge^2 E)^{\otimes k}.$$

It is clear that $T_{(a,b),(u,v,w),(\lambda,\mu,\nu)}$ is zero unless $u+v \leq w \leq u+b-v$ and $N_{\lambda,\mu,\nu} \neq 0$. And in this case, the map is equivalent to a map of $\mathbb{Q}$-vector spaces,

$$T'_{(a,b),(u,v,w),(\lambda,\mu,\nu)} : (W_{a,u,\lambda} \otimes M_{b,v,\mu})^{\oplus W_{\lambda,\mu,\nu}} \to W_{a+b,w,\nu}.$$

The second goal is to compute each of these maps.

The final goal is to compute $H^0(\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, e), (\mathcal{R}^\vee)^{\otimes a})$, not the space of sections $H^0(U, (\mathcal{R}^\vee)^{\otimes a})$. This requires understanding for each $i = 2, \dots, \lfloor \frac{e}{2} \rfloor$ the map sending a section over U to its *principal part* along Δ_i. Because of this, the description of the space of sections should be as explicit as possible.

sec-comps

6. Computations

To compute the $\mathbb{Q}$-vector spaces $W_{a,w,\lambda}$, it suffices to consider the case $B = \mathrm{Spec}\ \mathbb{Q}$. From now on $B = \mathrm{Spec}\ \mathbb{Q}$. Let F be $\mathcal{O}_B \oplus \mathcal{O}_B$. Let α, β be a basis for F. To simplify notation, the invertible sheaves spanned by α and β will also be written as α and β. This basis determines a maximal torus $\mathbb{G}_m(\alpha) \times \mathbb{G}_m(\beta) \subset \mathbf{GL}(F)$. Every finite dimensional representation M of $\mathbf{GL}(F)$ has an eigenspace decomposition, or *weight decomposition*, with respect to this torus,

$$M = \bigoplus_{l,m \in \mathbb{Z}} M_{l,m}$$

where the torus acts on $M_{l,m}$ by the character $(s,t) \mapsto s^l t^m$. Of course $M^{\mathbf{GL}(F)} \subset M_{0,0}$. Moreover, the subspace $\oplus_{l+m=0} M_{l,m}$ is invariant under the action of $\mathbf{SL}(F)$. The intersection of $\mathbb{G}_m(\alpha) \times \mathbb{G}_m(\beta)$ with $\mathbf{SL}(F)$ is a maximal torus. Choose a positive root for this maximal torus. The action of the positive root determines an injective map,

$$\chi : M_{-1,1} \to M_{0,0},$$

and the following composition is an isomorphism,

$$M^{\mathbf{GL}(F)} \hookrightarrow M_{0,0} \to \mathrm{Coker}(\chi).$$

In the case of $M = H^0(P, (\mathcal{R}_P^\vee)^{\otimes a})$, there is an action of $\mathbf{GL}(E) \times \mathfrak{S}_a$ on each of the weight spaces $M_{l,k}$. Because the category of representations of $\mathbf{GL}(E) \times \mathfrak{S}_a$

is semisimple, to determine the representation $M^{\mathbf{GL}(F)}$, it suffices to determine the representations $M_{0,0}$ and $M_{-1,1}$. The goal of this section is to give a simple combinatorial description of the representations $M_{0,0}$ and $M_{-1,1}$.

As a representation of $\mathbb{G}_m(\alpha) \times \mathbb{G}_m(\beta)$, there is a weight decomposition,

$$\mathrm{Sym}^e(F) \cong \alpha^e \oplus \cdots \oplus \alpha^{e-i}\beta^i \oplus \cdots \oplus \beta^e.$$

Therefore there is a decomposition,

$$\mathrm{Sym}^a(E \otimes \mathrm{Sym}^e(F)) \cong \bigoplus_{\underline{i} \in I_{e,a}} \mathrm{Sym}^{\underline{i}}(E) \otimes \alpha^{l(\underline{i})}\beta^{m(\underline{i})} \otimes \mathbf{b}^{\underline{i}},$$

where $\underline{i}$ is a sequence of nonnegative integers $(i_0, \dots, i_e)$, where $I_{e,a}$ is the set of sequences such that $\sum_{k=0}^{e} i_k = a$, where $l(\underline{i})$ is defined to be $\sum_k (e-k)i_k$, where $m(\underline{i})$ is defined to be $\sum_k k i_k$, where $\mathbf{b}^{\underline{i}}$ is just a placeholder $\mathbf{b}_0^{i_0} \cdot \cdots \cdot \mathbf{b}_e^{i_e}$, and where

$$\mathrm{Sym}^{\underline{i}}(E) = \bigotimes_{k=0}^{e} \mathrm{Sym}^{i_k}(E).$$

Of course $I_{e,a}$ is in bijection with the set of Young tableaux with at most a rows and at most e columns, and $m(\underline{i})$ is the number of squares in the corresponding tableaux.

Similarly, there is a weight decomposition,

$$\mathrm{Sym}^{e-2}(F^\vee) \otimes \wedge^2(F^\vee) \cong A^{e-1}B \oplus \cdots \oplus A^{e-j}B^j \oplus \cdots \oplus AB^{e-1},$$

where (A, B) is the dual ordered basis to (α, β). Therefore there is a decomposition,

$$\left[\mathrm{Sym}^{e-2}(F^\vee) \otimes \wedge^2(F^\vee)\right]^{\otimes a} \cong \bigoplus_{\underline{j} \in J_{e,a}} A^{ea-|\underline{j}|}B^{|\underline{j}|} \otimes \mathbf{c}_{\underline{j}},$$

where $\underline{j}$ is a sequence of integers $(j_1, \dots, j_a)$ such that each $1 \leq j_\kappa \leq e-1$, where $|\underline{j}| = j_1 + \cdots + j_a$, and where $\mathbf{c}_{\underline{j}}$ is a placeholder $\mathbf{c}_{j_1} \otimes \cdots \otimes \mathbf{c}_{j_a}$. The action of $\mathfrak{S}_a$ is the obvious one: for $\sigma \in \mathfrak{S}_a$,

$$\sigma \cdot \mathbf{c}_{j_1} \otimes \cdots \otimes \mathbf{c}_{j_\kappa} \otimes \cdots \otimes \mathbf{c}_{j_a} = \mathbf{c}_{j_{\sigma(1)}} \otimes \cdots \otimes \mathbf{c}_{j_{\sigma(\kappa)}} \otimes \cdots \otimes \mathbf{c}_{j_{\sigma(a)}}.$$

Putting this together, the weight-$(0,0)$ subspace of $M = H^0(P, (\mathcal{R}_P^\vee)^{\otimes a})$ is,

$$M_{0,0} = \bigoplus_{(\underline{i},\underline{j}) \in K_{e,a}} \mathrm{Sym}^{\underline{i}}(E) \otimes \alpha^0\beta^0 \otimes \mathbf{b}^{\underline{i}} \otimes \mathbf{c}_{\underline{j}},$$

where, as before, $\mathbf{b}^{\underline{i}}$ and $\mathbf{c}_{\underline{j}}$ are placeholders, and where $K_{e,a}$ is,

$$K_{e,a} = \left\{ (\underline{i}, \underline{j}) \in I_{e,a} \times J_{e,a} | m(\underline{i}) = |\underline{j}| \right\}.$$

And the weight-$(-1,1)$ subspace is,

$$M_{-1,1} = \bigoplus_{(\underline{i},\underline{j}) \in K'_{e,a}} \mathrm{Sym}^{\underline{i}}(E) \otimes \alpha^{-1}\beta^1 \otimes \mathbf{b}^{\underline{i}} \otimes \mathbf{c}_{\underline{j}},$$

where,

$$K'_{e,a} = \left\{ (\underline{i}, \underline{j}) \in I_{e,a} \times J_{e,a} | m(\underline{i}) = 1 + |\underline{j}| \right\}.$$

To relate this to the spaces $W_{a,u}$, choose an ordered basis (ϵ, η) of E. This determines a maximal torus $\mathbb{G}_m(\epsilon) \times \mathbb{G}_m(\eta)$ in $\mathrm{GL}(E)$. The weights of $M_{0,0}$ and $M_{-1,1}$

with respect to this torus are $(a-w,w)$, $w=0,\dots,a$. The weight decompositions are,

$$M_{0,0} = \bigoplus_{u=0}^{a} \bigoplus_{(\underline{h},\underline{i},\underline{j})\in K_{e,a,u}} \epsilon^{a-u}\eta^{u}\otimes\alpha^{0}\beta^{0}\otimes\mathbf{a}^{\underline{h}}\otimes\mathbf{b}^{\underline{i}}\otimes\mathbf{c}_{\underline{j}}$$

$$M_{-1,1} = \bigoplus_{u=0}^{a} \bigoplus_{(\underline{h},\underline{i},\underline{j})\in K'_{e,a,u}} \epsilon^{a-u}\eta^{u}\otimes\alpha^{-1}\beta^{1}\otimes\mathbf{a}^{\underline{h}}\otimes\mathbf{b}^{\underline{i}}\otimes\mathbf{c}_{\underline{j}},$$

where $\underline{h}$ is a sequence of nonnegative integers $(h_0,\dots,h_e)$, where $\mathbf{a}^{\underline{h}}$ is a placeholder $\mathbf{a}_0^{h_0}\cdot\dots\cdot\mathbf{a}_e^{h_e}$, and where,

$$K_{e,a,u} = \{(\underline{h},\underline{i},\underline{j})|(\underline{i},\underline{j})\in K_{e,a},\ \text{each } 0\le h_k\le i_k,\ \text{and } \sum_k h_k = u\},$$

$$K'_{e,a,u} = \{(\underline{h},\underline{i},\underline{j})|(\underline{i},\underline{j})\in K'_{e,a},\ \text{each } 0\le h_k\le i_k,\ \text{and } \sum_k h_k = u\}.$$

For each e,a,u, there is a $\mathfrak{S}_a$-representation $W'_{e,a,u}$, unique up to isomorphism, such that,

$$\bigoplus_{(\underline{h},\underline{i},\underline{j})\in K_{e,a,u}} \mathbf{a}^{\underline{h}}\otimes\mathbf{b}^{\underline{i}}\otimes\mathbf{c}_{\underline{j}} \cong W'_{e,a,u}\oplus \bigoplus_{(\underline{h},\underline{i},\underline{j})\in K'_{e,a,u}} \mathbf{a}^{\underline{h}}\otimes\mathbf{b}^{\underline{i}}\otimes\mathbf{c}_{\underline{j}}.$$

Finally, the relation to $W_{a,u}$ is,

$$W'_{e,a,u}\cong\bigoplus_{0\le u'\le u} W_{a,u}.$$

Because the category of $\mathfrak{S}_a$-representations is semisimple, determining all the representations $W'_{e,a,u}$ determines $W_{a,u}$. Therefore, determining all the $\mathfrak{S}_a$-sets $K_{e,a,u}$ and $K'_{e,a,u}$ determines all the representations $W_{a,u}$. In particular,

$$\dim_{\mathbb{Q}} W_{a,u} = (\#K_{e,a,u}-\#K'_{e,a,u}) - (\#K_{e,a,u-1}-\#K'_{e,a,u-1}),$$

where $K_{e,a,-1}$ and $K'_{e,a,-1}$ are defined to be the empty set.

sec-ce2

7. The case $e=2$ and the case $a=1$

subsec-ce2

7.1. **The case** $e=2$**.** In the case $e=2$, the sets $K_{2,a,u}$ and $K'_{2,a,u}$ can be computed explicitly. For each a, $J_{2,a}=\{(1,\dots,1)\}$. In particular, the action of $\mathfrak{S}_a$ is trivial. Also,

$$K_{2,a} = \left\{(i_0, a-2i_0, i_0)|0\le i_o\le\lfloor\frac{a}{2}\rfloor\right\},$$

$$K'_{2,a} = \left\{(i_0, a-1-2i_0, i_0+1)|0\le i_o\le\lfloor\frac{a-1}{2}\rfloor\right\}.$$

For each $\underline{i}=(i_0,i_1,i_2)$, the number of $\underline{h}$'s is $(i_0+1)(i_1+1)(i_2+1)$. Therefore,

$$\dim_{\mathbb{Q}} M_{0,0} = \sum_{\imath=0}^{\lfloor\frac{a}{2}\rfloor}(i_0+1)^2(a+1-2i_0),$$

$$\dim_{\mathbb{Q}} M_{-1,1} = \sum_{\imath=0}^{\lfloor\frac{a-1}{2}\rfloor}(i_0+1)(i_0+2)(a-2i_0).$$

If a is odd, say $a = 2b+1$, then $\lfloor \frac{a}{2} \rfloor = \lfloor \frac{a-1}{2} \rfloor = b$, and the difference of dimensions is,

$$\dim_{\mathbb{Q}} M_{0,0} - \dim_{\mathbb{Q}} M_{-1,1} = \sum_{i=0}^{b} (i_0 + 1)(3i_0 - 2b).$$

After some tedious calculation, this comes out as 0 (since it is a cubic polynomial in b, it suffices to consider $b = 0, 1, 2$). If a is even, say $a = 2b$, then $\lfloor \frac{a}{2} \rfloor = b$ and $\lfloor \frac{a-1}{2} \rfloor = b - 1$. So the difference of dimensions is,

$$\dim_{\mathbb{Q}} M_{0,0} - \dim_{\mathbb{Q}} M_{-1,1} = (b+1)^2 + \sum_{i_0=0}^{b-1} (3i_0^2 - (2b-4)i_0 - (2b-1)).$$

After more tedious calculation, this comes out as,

$$\dim_{\mathbb{Q}} M_{0,0} - \dim_{\mathbb{Q}} M_{-1,1} = \frac{1}{2}(b+2)(b+1),$$

(since it is a cubic polynomial in b, again it suffices to consider $b = 0, 1, 2$). This is precisely the dimension of $\mathrm{Sym}^b(\mathrm{Sym}^2 E)$, so the computation agrees with the result of Section 3.

7.2. **The case $a = 1$.** The case $a = 1$ is also simple. The elements of $K_{e,1}$ are $s^j = (\underline{i}^j, (j))$, $1 \leq j \leq e-1$, where

$$i_k^j = \begin{cases} 1, & k = j \\ 0, & \text{otherwise} \end{cases}$$

The elements of $K'_{e,1}$ are $t^j = (\underline{i}^j, (j-1))$, $2 \leq j \leq e$. In particular $K_{e,1}$ and $K'_{e,1}$ have the same number of elements. For each element, there are 2 choices for $\underline{h}$. Thus $K_{e,1,u}$ and $K'_{e,1,u}$ have the same number of elements for $u = 0, 1$. Therefore,

$$H^0(\overline{\mathcal{M}}_{0,0}(\mathbb{P}E/B, e), \mathcal{R}^\vee) = (0).$$

subsec-a1

References

[1] A. J. de Jong and J. Starr. Cubic fourfolds and spaces of rational curves. submitted to Illinois Journal of Mathematics, preprint available at `http://www-math.mit.edu/~jstarr/papers`.

[2] B. Fantechi and R. Pandharipande. Stable maps and branch divisors. preprint arXiv: math.AG/9905104.

[3] J. Starr. Kodaira dimensions of spaces of rational curves on low degree hypersurfaces. in preparation.

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