Every rationally connected variety over the function field of a curve has a rational point

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Introduction

Let k be an algebraically closed field and let V be a normal proper variety over k. We say that V is separably rationally connected if there exists a smooth rational curve $C \to V$ mapping to V which avoids the singularities of V such that $T_V|_C$ is ample on C. This definition differs from [8, IV Definition 3.2] in Kollar's book; it agrees with his definition if V has a resolution of singularities, see [8, IV Theorem 3.7]. Here is our main theorem; it is Conjecture IV 6.1.1 of [8] and it was proved by Graber, Harris and Starr in the case that k has characteristic 0, see [5].

Theorem. *Let X be an irreducible nonsingular projective curve. Suppose that $f : P \to X$ is a proper flat morphism whose geometric generic fibre is a normal and separably rationally connected variety. Then f has a section.*

The proof of this theorem is at the end of the paper. We briefly indicate the layout of this paper, and the main ideas.

Graber, Harris and Starr use a topological argument to specialize a family of ramified coverings $Y_t \to X$ to a map $Y_\infty \to X$ so that Y_∞ contains an irreducible component mapping isomorphically onto X. We replace this by an elementary construction which produces, starting with $Y \to X$, a family $Y_t \to X$ specializing in the desired manner so that Y_0 is a nodal curve whose normalization is Y. This is done in Section 1.

We advise that the reader skip Section 2 on a first reading. Here we show that a proper flat family $P \to X$ of varieties over X with reduced general fibre has a normalized pullback all of whose fibres are reduced. The pullback morphism $Y \to X$ can be taken to be finite generically étale. This is used in Section 3 to show that it suffices to produce a section of f after such a normalized base change. In Section 4 we produce the section in this case; here the idea, if not the execution, is similar to that in [5].

Definitions 4.2.1 and 4.2.2 are taken from [5], and so are the main lines of proof, as is the conviction that such a theorem can be proven (in characteristic p).

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1. Making families of curves

Let k be an algebraically closed field. Suppose we are given the following data:

1. A finite morphism $\pi : Y \to X$ of irreducible smooth projective curves over k. We assume that π is generically étale. The degree of this morphism is denoted d.
2. A finite set of points $S \subset X(k)$. We assume that S contains all branch points of the morphism π.

1.1. Proposition. *Given data as above there exist a projective surface W and flat morphisms $f : W \to \mathbb{P}^1$, $g : W \to X$ with the following properties:*
(a) The fibre $W_0 = f^{-1}(0)$ is a nodal curve, whose normalization is the curve Y; the map $Y \to W_0 \xrightarrow{g} X$ is equal to π. The image $g(w)$ of any node $w \in W_0$ is not in S. The surface W is nonsingular along W_0.
(b) The general fibre $W_t = f^{-1}(t)$ is smooth, and the morphism $\pi_t : W_t \to X$ induced by g has degree d.
(c) The formal local structure of the morphism $\pi_t : W_t \to X$ in a neighbourhood of $\pi_t^{-1}(x)$ is isomorphic to the formal local structure of the morphism $\pi : Y \to X$ in a neighbourhood of $\pi^{-1}(x)$ for all $x \in S$ and all $t \in \mathbb{P}^1(k)$, $t \neq \infty$.

(d) There is an irreducible component X' of $W_\infty = f^{-1}(\infty)$ which is mapped birationally to X under g. The surface W is generically smooth along X' and X' has multiplicity 1 in W_∞. (In other words f is generically smooth along X'.)

Proof. We choose an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ of sufficiently high degree. Choose a pair of sections s_0, s_1 of $\mathcal{L}$ which generate $\mathcal{L}$ and let h be the morphism $\varphi_{(s_0,s_1)} : Y \to \mathbb{P}^1$. Let $\overline{Y} \subset X \times \mathbb{P}^1$ be the (reduced) image closed subscheme.

1.2. Lemma. *(a) If* $\deg(\mathcal{L}) > 2g_Y - 2 + 2d$ *then for general choice of* (s_0, s_1) *the curve* $\overline{Y}$ *has at worst ordinary double points, and* $\mathrm{Sing}(\overline{Y}) \cap S \times \mathbb{P}^1 = \emptyset$.
(b) The divisor class of $\overline{Y}$ *on* $X \times \mathbb{P}^1$ *is the divisor class of the invertible sheaf* $pr_1^*(\mathrm{Nm}_\pi(\mathcal{L})) \otimes pr_2^*(\mathcal{O}(d))$.

Proof. For a point $x \in X(k)$ consider the closed subscheme $Y_{2,x} = \pi^{-1}(\mathrm{Spec}\ \mathcal{O}_{X,x}/\mathfrak{m}_x^2)$ of Y. The bound on the degree implies that the map $\Gamma(Y, \mathcal{L}) \to \Gamma(Y_{2,x}, \mathcal{L})$ is surjective. For $x \in S$ we choose the pair (s_0, s_1) such that the associated morphism $Y_{2,x} \to \mathbb{P}^1$ is a closed immmersion. For $x \notin S$ the differential $d\pi$ identifies the tangent spaces $T_y := T_y(Y_{2,x}) = T_yY$ with T_xX (for all $y \in Y_{2,x}(k)$). We choose the pair (s_0, s_1) such that the morphism $Y_{2,x} \to \mathbb{P}^1$ is either a closed immersion, or identifies at worst two points y, y' of $Y_{2,x}$ but does not induce the same map on the tangent spaces $T_y = T_xX = T_{y'}$. The reader checks that these conditions define nonempty open sets U_x (in the space of all pairs), and that the complement of the open U_x for $x \notin S$ has codimension 2. Hence there is a pair satisfying all of these conditions. It is straightforward to check that such a pair works in (a).

To verify (b) we have to compute $\overline{Y} \cap \{x\} \times \mathbb{P}^1$ as a divisor on $\mathbb{P}^1$ and $\overline{Y} \cap X \times \{t\}$ as a divisor on X. The first divisor clearly has degree d as desired. The second is the image in X of a fibre of the map $\varphi : Y \to \mathbb{P}^1$. The divisor class of the fibre is $c_1(\mathcal{L})$ and the divisor class of the image is $c_1(\mathrm{Nm}_\pi(\mathcal{L}))$. $\square$

Let $D \subset X$ be the divisor $D = \sum_{x \in S} x$. Pick a large number N divisible by d and let $\mathcal{L} = \mathcal{O}_Y(\frac{N}{d}\pi^{-1}(D))$. Then $\mathrm{Nm}_\pi(\mathcal{L}) \cong \mathcal{O}_X(ND)$. By the lemma we have following rational equivalence of divisors on $X \times \mathbb{P}^1$:

$$[\overline{Y}] \equiv \sum_{j=1}^{d} [X \times \{\sigma_j\}] + \sum_{x \in S} N[\{x\} \times \mathbb{P}^1].$$

Here $\sigma_1, \dots, \sigma_d \in \mathbb{P}^1(k)$ are pairwise distinct points chosen such that $X \times \{\sigma_j\} \cap \mathrm{Sing}(\overline{Y}) = \emptyset$ for all $j = 1, \dots, d$.

The above rational equivalence implies that there exists a pencil of curves D_t on $X \times \mathbb{P}^1$ with the following properties: (i) $D_0 = \overline{Y}$ has at worst nodes as singularities, (ii) $D_\infty = \sum X \times \{\sigma_j\} + \sum_{x \in S} N\{x\} \times \mathbb{P}^1$ misses the nodes of D_0. Let W be the blow up of the surface $X \times \mathbb{P}^1$ in the zero dimensional closed subscheme $D_0 \cap D_\infty$. Then we obtain the right half of the following diagram (the left half will be explained below)

$$\begin{tikzcd}
\coprod_{x \in S} Z_x \times \mathbb{P}^1 \arrow[r] \arrow[d, "\pi|_{Z_x}"'] & W \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{P}^1 \\
X \arrow[r] & X &
\end{tikzcd}$$

such that $D_t = f^{-1}(t)$ (scheme theoretically) and such that g is the composition $W \to X \times \mathbb{P}^1 \to X$. Properties (i), (ii) imply that the general fibre of f is smooth and that W is smooth along $D_0 = f^{-1}(0) = \overline{Y}$. Further, the morphism f is also smooth generically along the irreducible components $X \times \{\sigma_j\}$ of the fibre D_∞. Note that these components are mapped isomorphically to X under the morphism g. Thus (a), (b) and (d) are satisfied; it remains to show (c).

For $x \in S$ we let $Z_x \subset Y$ be the scheme

$$Z_x = \pi^{-1}(\text{Spec } \mathcal{O}_{X,x}/\mathfrak{m}_x^N) = Y \times_X \text{Spec}\,(\mathcal{O}_{X,x}/\mathfrak{m}_x^N).$$

The morphism $(\pi \times h) : Y \to \overline{Y}$ induces a closed immersion of Z_x into $\overline{Y}$. Our choice of $D_\infty = \sum X \times \{\sigma_j\} + \sum_{x \in S} N\{x\} \times \mathbb{P}^1$ insures that $(\pi \times h)(Z_x) \subset D_\infty$ and hence we may think of Z_x as a subscheme of D_t for all $t \in \mathbb{P}^1(k)$. We conclude there exists a morphism $Z_x \times \mathbb{P}^1 \to W$ which for every $t \in \mathbb{P}^1$ induces the closed immersion $Z_x \to D_t$ we just described. It fits into the commutative diagram displayed above.

Note that for $t \neq \infty$ the morphism $\pi_t = g|_{D_t} : D_t \to X$ is finite. Upon comparing lengths we see that

$$Z_x = \pi_t^{-1}(\text{Spec } \mathcal{O}_{X,x}/\mathfrak{m}_x^N).$$

In particular this implies that D_t is smooth along Z_x (by looking at tangent spaces). Thus Lemma 1.4 (a) below applies and we deduce that the formal local structure of the morphism $\pi_t : D_t \to X$ in a neighbourhood of $\pi_t^{-1}(x)$ is isomorphic to the formal local structure of the morphism $Y \to X$ in a neighbourhood of $\pi^{-1}(x)$ for all $x \in S$ and all $t \in \mathbb{P}^1(k)$, $t \neq \infty$. $\square$

1.3. Corollary. *Suppose that $\pi : Y \to X$ is a finite generically étale morphism of projective nonsingular curves over k and that $S \subset X(k)$ is a finite set of points. There exists a projective surface W and morphisms $f : W \to C$, $g : W \to X$ such that the following conditions hold:*
(a) f is semi-stable family of curves with smooth general fibre.
(b) There is a point $0 \in C(k)$ such that $W_0 = f^{-1}(0)$ has the following description: $W_0 = Y \cup \bigcup L_i$, where $g|_Y = \pi$, $g(L_i)$ is a point of X not in S, $L_i \cong \mathbb{P}^1$ and $Y \cap L_i = \{y_{i1}, y_{i2}\}$ where (of course) $\pi(y_{i1}) = \pi(y_{i2})$ in X.
(c) There is a point $\infty \in C(k)$ and an irreducible component X' of the fibre W_∞ which is mapped isomorphically to X under g.

Proof. Enlarge S so that it contains the branch points of the morphism π and apply the proposition. Apply base change by the map $\delta : \mathbb{P}^1 \to \mathbb{P}^1$, $s \mapsto t = s^2/(s+1)$ and blow up the resulting ordinary double points of $W \times_{\mathbb{P}^1, \delta} \mathbb{P}^1$ above $s = 0$ to obtain the lines L_i. Finally, apply the semi-stable reduction theorem to make f semistable. Details left to the reader. $\square$

We end this section by stating the lemma that was used above. The lemma is basically a restatement of Krasner's lemma, see [3, Section 3.4.2].

1.4. Lemma. *(a) Let $\pi : Y \to X$ be a finite morphism of smooth curves and assume that π is generically étale. Let $x \in X$. There exists an integer N_0 such that for every $N \geq N_0$ we have the following: If $Y' \to X$ is another finite morphism of smooth curves and if there exists an isomorphism*

$$Y \times_X \text{Spec } \mathcal{O}_{X,x}/\mathfrak{m}_x^N \cong Y' \times_X \text{Spec } \mathcal{O}_{X,x}/\mathfrak{m}_x^N$$

over $\mathcal{O}_{X,x}$ then there exists an isomorphism

$$Y \times_X \text{Spec } \widehat{\mathcal{O}}_{X,x} \cong Y' \times_X \text{Spec } \widehat{\mathcal{O}}_{X,x}$$

over $\mathcal{O}_{X,x}$.
(b) Let X be a smooth curve over k and let $\widehat{\mathcal{O}}_{X,x} \subset R$ be a finite generically étale extension of complete discrete valuation rings. There exists an integer N_0 such that for every $N \geq N_0$ we have: If $Y \to X$ is a finite morphism of smooth curves, and $y \in Y(k)$ is over x such that

$$\mathcal{O}_{Y,y}/\mathfrak{m}_x^N \mathcal{O}_{Y,y} \cong R/\mathfrak{m}_x^N$$

over $\mathcal{O}_{X,x}$ then $\widehat{\mathcal{O}}_{Y,y} \cong R$ over $\mathcal{O}_{X,x}$.

Proof. Part (a) follows from part (b) by writing $Y' \times_X \mathrm{Spec}\ \widehat{\mathcal{O}}_{X,x}$ as a disjoint union of branches. Part (b) follows from Krasner's Lemma in the following way. We can write $R = \widehat{\mathcal{O}}_{X,x}[T]/(f(T))$. For any pair (y, Y') as in (b) we can choose an isomorphism $\widehat{\mathcal{O}}_{Y',y} = \widehat{\mathcal{O}}_{X,x}[T']/(f'(T'))$ such that $f'(T) - f(T) \in \mathfrak{m}_x^N$. In other words, the class of T' in $\widehat{\mathcal{O}}_{Y',y}$ is a solution of $f(X) = 0$ up to order N. Krasner's Lemma says that, provided N_0 is large enough, this implies that there is a root of $f(X) = 0$ in $\widehat{\mathcal{O}}_{Y',y}$. The corresponding map of R to $\widehat{\mathcal{O}}_{Y',y}$ is the desired isomorphism. □

2. Obtaining reduced fibres after normalized base change

Suppose that $f : P \to X$ is a flat morphism of a normal variety P to an irreducible smooth projective curve X over k. We will assume that the following equivalent conditions hold: (a) the geometric generic fibre is reduced, (b) the general fibre is reduced, and (c) the smooth locus of the morphism f is dense in P.

We note that for $x \in X(k)$ the fibre $P_x = f^{-1}(x)$ is a scheme satisfying S_1 (every local ring of dimension ≥ 1 had depth ≥ 1). Thus P_x is reduced if and only if P_x is generically reduced, i.e., all local rings of P_x at generic points are fields.

Let $\pi : Y \to X$ be a finite morphism of another irreducible smooth curve Y to X. Set

$$P_Y = (P \times_X Y)^{normalized}.$$

In this section we present a technical result that is similar to Lemma 2.3 in [7] and which follows in a straightforward manner from the results in [4]. Namely we prove there is at least one finite generically étale morphism $Y \to X$ such that $P_Y \to Y$ has reduced fibres, i.e., such that the smooth locus of $P_Y \to Y$ is dense in every fibre.

2.1. Proposition. *With the assumptions and notations as above. There exists a $\pi : Y \to X$ such that all fibres of $P_Y \to Y$ are reduced and such that π is generically étale.*

Proof. First we examine this question when $X = \mathrm{Spec}\ R$ is the spectrum of a complete discrete valuation ring R with algebraically closed residue field k. Indeed, let P be an irreducible scheme, and let $P \to \mathrm{Spec}\ R$ be a flat morphism of finite type with geometrically reduced generic fibre. It follows by unwinding [4, Theorem 2.1', page 368] that there exists a finite generically étale extension $R \subset R'$ and a commutative diagram

$$\begin{array}{ccccc} P' & \longrightarrow & P \otimes_R R' & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\ R' & \longrightarrow & \mathrm{Spec}\ R' & \longrightarrow & \mathrm{Spec}\ R. \end{array}$$

where $P' \to P \otimes_R R'$ is finite and an isomorphism over the generic point of Spec R', and where the morphism $P' \to \mathrm{Spec}\ R'$ is flat with reduced geometric fibres. This implies that the normalization $(P \otimes_R R')^{normalized}$ of $P \otimes_R R'$ dominates P'. Since the special fibre of $P' \to \mathrm{Spec}\ R'$ is reduced we see that $(P \otimes_R R')^{normalized} \to P'$ is an isomorphism above the generic points of the special fibre of P' and hence we see that $(P \otimes_R R')^{normalized}$ has reduced special fibre by the discussion before the proposition as desired. We note that any further extension $R \subset R' \subset R''$ (assumed generically étale) will do the job as well.

Let $x_1, \ldots, x_r \in X(k)$ be the points of X such that $P_{x_i} = f^{-1}(x_i)$ is not reduced. Let R_i be the complete local ring of x_i on X. Choose $R_i \subset R_i'$ as above. By the remark at the end of the last paragraph we way choose the extensions $R_i \subset R_i'$ such that the field extensions $K_i = f.f.(R_i) \subset f.f.(R_i') = L_i$ have a degree d independent of i.

Let $K = k(X)$. We claim that there exists a separable field extension $K \subset L$ of degree d such that $L \otimes_K K_i$ is isomorphic to L_i. To see this we write $L_i = K_i(\alpha_i)$ and we let $f_i \in K_i[X]$ be the minimal polynomial of α_i over K_i. We choose a polynomial $f(X) \in K[X]$ which approximates f_i simultaneously for all i and we set $L = K[X]/(f)$. By Krasner's lemma we'll have $L \otimes_K K_i \cong L_i$ if the approximation is good enough, see our Lemma 1.4 (b). □

2.2. Remark. In the preprint version of this paper an alternative proof of this result was given using Lemma 2.3 in [7], avoiding the reference [4]. The strategy there was to show that the result with π possibly inseparable actually implies the a priori stronger result with π generically separable. We refer the interested reader to the web page of the first author for a web-based version of this argument.

3. The reduction to the case of reduced fibres

Suppose that $f : P \to X$ is a proper flat morphism of a normal scheme P to an irreducible smooth projective curve X over k. We will assume that the following equivalent conditions hold: (a) the geometric generic fibre is reduced, (b) the general fibre is reduced, and (c) the smooth locus of the morphism f is dense in P. Consider the question: Does f have a section? In this section we will show that the answer is affirmative if and only if f has a section after normalized base change P_Y to those $Y \to X$ such that P_Y has reduced fibres over Y.

3.1. Theorem. *Assumptions and notations as above.*
(i) In case k uncountable. Assume that $P_Y \to Y$ has a section whenever $\pi : Y \to X$ has the following two properties: (a) π is generically étale, and (b) $P_Y \to Y$ has reduced fibres. Then $P \to X$ has a section.
(ii) In case k countable. Suppose that for some uncountable algebraically closed extension K of k the assumptions of (i) hold for $P_K \to X_K$ over K. Then $P \to X$ has a section.

Proof. Case (ii) can be deduced from (i) by a specialization argument. Namely, by (i) we obtain a section $\sigma_K : X_K \to P_K$ over some (uncountable) extension $k \subset K$. Clearly σ_K can be defined over a finitely generated extension of k, i.e., over the function field of some variety T over k. By shrinking T we may assume we have a section of $P \times T \to X \times T$ and by choosing a k-rational point in T we obtain our section of $P \to X$.

Next we prove (i). By Proposition 2.1 there is at least one $\pi : Y \to X$ generically étale such that $P_Y \to Y$ has reduced fibres. Let $\pi_t : W_t \to X$ be the family of morphisms of curves constructed in Proposition 1.1 using as $S = \{x_1, \dots, x_r\}$ the set of branch points of the morphism $Y \to X$. By construction, for $t \neq \infty$ the formal local structure of the morphism $\pi_t : W_t \to X$ in a neighbourhood of $\pi_t^{-1}(x_i)$ is isomorphic to the formal local structure of the morphism $\pi : Y \to X$ in a neighbourhood of $\pi^{-1}(x_i)$. This clearly implies that the normalization P_t of $P \times_X W_t$ has reduced fibres.

By assumption each of the morphisms $P_t \to W_t$ has a section σ_t, and so $P \times_X W_t \to W_t$ has a section σ_t. The relative Hilbert scheme Hilb of $P \times_X W$ over $\mathbb{P}^1$ is locally of finite type over k and has a countable number of irreducible components. Since k is uncountable, we see that infinitely many of the points $\sigma_t(W_t)$ of Hilb will be in the same irreducible component $Z \subset$ Hilb which dominates $\mathbb{P}^1$. We deduce that there is a section σ of $P \times_X W_L \to W_L$ where $k(t) \subset L$ is a finite extension (by taking an L-valued point of Z). Let C be the smooth projective curve over k whose function field is L and let $c \in C(k)$ be a point which lies over $t = \infty$ (under the map $C \to \mathbb{P}^1$ coming from $k(t) \subset L$). Note that there is a component X' of the fibre of

$$W \times_{f,\mathbb{P}^1} C \longrightarrow C$$

over c which is mapped birationally to X under the composite $W \times_{\mathbb{P}^1} C \to W \to X$, and along which $W \times_{\mathbb{P}^1} C$ is generically smooth (this corresponds to the component X' of Proposition 1.1 (d)). Think of σ

as a rational map fitting into the following commutative diagram:

$$\begin{tikzcd} C \arrow[d] & W \times_{\mathbb{P}^1} C \arrow[l] \arrow[r, "\sigma"] \arrow[d] & P \arrow[d] \\ \mathbb{P}^1 & W \arrow[l, "f"'] \arrow[r, "g"] & X \end{tikzcd}$$

By construction σ extends to a neighbourhood of the generic point of X' and this gives a $k(X') = k(X)$-valued point of P over X as desired. ☐

4. The case of reduced fibres

Let X be an irreducible nonsingular projective curve over k, and let $P \to X$ be a flat morphism. We will assume the following assumptions hold:

4.0.1. P is a normal projective variety.
4.0.2. All fibres of f are reduced and connected.
4.0.3. The following equivalent conditions hold: (a) the geometric generic fibre of f is normal, (b) there is a nonempty Zariski open set $U \subset X$ such that for each $x \in U(k)$ the fibre $P_x = f^{-1}(x)$ is normal, and (c) there is a nonempty Zariski open set $U \subset X$ such that the nonsmooth locus of $P_U \to U$ has codimension ≥ 2 in the fibres.
4.0.4. There exists a nonempty Zariski dense open set $V \subset P$ contained in the smooth locus of f with the following property: for every point $x \in X(k)$ and every finite set of closed points $p_1, \ldots, p_n \in V \cap f^{-1}(x)$, there exists a smooth rational curve $\mathbb{P}^1 \cong C \to V \cap f^{-1}(x) = V_x$ mapping to V_x containing each p_i and such that $T_{P/X}|_C$ is an ample vector bundle.

This last condition means that the general fibre P_x of f is rationally connected. The version of this we give above is equivalent to the following apparently weaker condition:

4.0.4'. There exists a closed point $x \in X$ and a smooth rational curve $\mathbb{P}^1 \to P_x$ mapping into the smooth locus of $P_x = f^{-1}(x)$ such that $T_{P/X}|_C$ is an ample vector bundle.

It follows from the proof of [8, Theorem IV.3.9.4] that, under the hypothesis 4.0.4', a Zariski dense open subset $V \subset P$ as in 4.0.4 exists. Thus 4.0.4 and 4.0.4' are equivalent conditions, but 4.0.4 is the form which is more useful to us. Namely, it will be used in the proof of the following theorem.

4.1. Theorem. *In the situation above f has a section.*

We do not immediately start the proof of the theorem. Instead we first give a definition. Note that the open subset $V \subset P$ of condition 4 will be fixed throughout the discussion.

4.2. Definition. Let Y be an irreducible nonsingular projective curve over k.
4.2.1. A morphism $\Phi : Y \to P$ is *pre-flexible* (relative to f) if $\Phi(Y)$ is contained in the smooth locus of f, $\Phi(Y) \cap V \neq \emptyset$, and $f \circ \Phi : Y \to X$ is generically étale.
4.2.2. A pre-flexible curve $\Phi : Y \to P$ is *flexible* if $H^1(Y, \Phi^* T_{P/X}) = 0$.
4.2.3. A *flexible curve with links* is a morphism $\Phi : Y \to P$ from a connected, projective, nodal curve $Y = Y_0 \cup L_1 \cup \ldots \cup L_r$ such that $\Phi|_{Y_0}$ is flexible, such that each link L_i is a smooth rational curve, the links L_i are pairwise disjoint, and each L_i intersects Y_0 in two distinct smooth points $p_i, q_i \in Y_0$, and such that $\Phi|_{L_i}$ is a closed immersion of L_i as a rational curve in a fibre of $P \to X$ such that $T_{P/X}|_{L_i}$ is ample.

4.3. Lemma. *There exists a pre-flexible curve $\Phi : Y \to P$.*

Proof. Conditions 2 and 3 imply that the complement of the smooth locus of f has codimension ≥ 2 in P. Let $p \in V$ be a point. Let Y be a connected component of the intersection of $\dim P - 1$ general hyperplanes in P which pass through p. By Bertini's theorem and the codimension estimate Y is a smooth projective curve contained in the smooth locus of f. It clearly meets V and the generic étaleness follows by choosing the hyperplanes suitably. □

4.4. Lemma. *There exists a flexible curve $\Phi : Y \to P$.*

Proof. By the lemma above we know there exists a pre-flexible curve $\Phi_0 : Y_0 \to P$. Let $n_1 \geq 0$ be an integer such that for every invertible sheaf $\mathcal{L}$ on Y_0 with $\deg(\mathcal{L}) \geq n_1$, we have $H^1(Y, \Phi_0^* T_{P/X} \otimes \mathcal{L}) = 0$. Let $n_2 = h^1(Y, \Phi_0^* T_{P/X})$. Choose an integer $n \geq n_1 + n_2$ and choose distinct points $p_1, \dots, p_n \in \Phi_0^{-1}(V)$. Let $C_1, \dots, C_n \subset V$ be curves as in condition (4) which contain $p_1, \dots, p_n$. Form the comb $Y_0 \cup C_1 \cup \dots \cup C_n$ where each tooth C_i is connected to the handle Y_0 at the point p_i, see [8, II Definition 7.7] for the definition of the concept "comb". Define $\Psi : (Y_0 \cup C_1 \cup \dots \cup C_n) \to P$ by $\Psi|_{Y_0} = \Phi_0$ and $\Psi|_{C_i}$ is the inclusion map. By [8, III Theorem 7.9], there exists a subcomb, say $Y_0 \cup C_1 \cup \dots C_m$ and a smoothing of the restriction of Ψ such that $m \geq n - n_2 \geq n_1$. The condition that a curve be contained in the nonsingular locus of f and intersect V is open in the parameter space, so for a general member $\Phi : Y \to P$ of the 1-parameter smoothing of $\Psi : Y_0 \cup C_1 \cup \dots \cup C_m \to P$, Φ is pre-flexible. By [8, III Lemma 7.16.1], if Φ is a general deformation, then $H^1(Y, \Phi^* T_{P/X}) = 0$. Thus $\Phi : Y \to P$ is a flexible curve. □

4.5. Definition. Suppose that $\pi : Y \to X$ is a morphism from a connected, projective, nodal curve to X. A *W-diagram* is a diagram:

$$\begin{array}{ccccc} X_\infty, Y & \longrightarrow & W & \xrightarrow{g} & X \\ \downarrow & & \downarrow{\scriptstyle h} & & \\ b_\infty, b & \longrightarrow & B & & \end{array}$$

where B is a smooth curve, h is a proper, flat family of nodal curves with smooth general fibre such that $h^{-1}(b) \cong Y$ for some $b \in B(k)$, $g|_Y = \pi$, and there is a point $b_\infty \in B(k)$ and an irreducible component $X_\infty \subset h^{-1}(b_\infty)$ with $g|_{X_\infty} : X_\infty \to X$ an isomorphism.

4.6. Lemma. *There exists a morphism $\Phi : Y \to P$ which is a flexible curve with links and such that there exists a W-diagram for $\pi = f \circ \Phi$.*

Proof. By Lemma 4.4, we know that there exists a flexible curve $\Phi_0 : Y_0 \to P$. By Proposition 1.1, we know that there exists a curve with links $Y = Y_0 \cup L_1 \cup \dots \cup L_r$ such that there exists a W-diagram for $\pi : Y \to X$ where $\pi|_{Y_0} = f \circ \Phi_0$ and where each $\pi|_{L_i}$ is a constant morphism. Moreover, we can choose Y so that the sets $L_i \cap Y_0$ all miss the finitely many points of $Y_0 - \Phi_0^{-1}(V)$. By condition (4), we can find embeddings $\Phi_i : L_i \to P$ as in condition (4) passing through $\Phi_0(p_i)$ and $\Phi_0(q_i)$. Define $\Phi : Y \to P$ by $\Phi|_{Y_0} = \Phi_0$ and each $\Phi|_{L_i} = \Phi_i$. □

4.7. Lemma. *Suppose C is a smooth curve, $\alpha : A \to C, \beta : B \to C$ are projective, flat morphisms and $\gamma : A \to B$ is a morphism of C-schemes which is an isomorphism over some point $c \in C$. Then there exists a Zariski open set $c \in U \subset C$ such that $\gamma : \alpha^{-1}(U) \to \beta^{-1}(U)$ is an isomorphism.*

Proof. Even though this result is well-known, we could not find a reference. Therefore we will include a proof. Without loss of generality we may assume that C is affine. Choose an ample invertible sheaf $\mathcal{L}$ on B. The pullback $\gamma^*\mathcal{L}$ is ample on $\beta^{-1}(c)$. By [EGA, III, part 1, Theorem 4.7.1], $\gamma^*\mathcal{L}$ is ample when we shrink C. There exists an integer N such that for $n \geq N$, we have that both $\mathcal{L}^n$ and $\gamma^*\mathcal{L}^n$ are very ample and have no higher cohomologies. By cohomology and basechange, the graded algebras $S(A, \mathcal{L}^n)$ and $S(B, \gamma^*\mathcal{L}^n)$ are flat $\mathcal{O}_B$-algebras. Moreover the pullback map $\gamma^* : S(B, \mathcal{L}) \to S(A, \gamma^*\mathcal{L}^n)$ is an isomorphism when we tensor with $\mathcal{O}_C / m_c \mathcal{O}_C$. Thus by Nakayama's lemma applied to the finitely-generated graded pieces (and using the fact that both algebras are generated and presented in finite degrees), we can find an open subset $U \subset C$ such that γ^* is an isomorphism over U. So γ is an isomorphism over U. $\square$

Proof of Theorem 4.1. By Lemma 4.6, we know there exists a $\Phi : Y \to P$ which is a flexible curve with links such that there is a W-diagram for $\pi : Y \to X$. We use the notation of Lemma 4.6 for Y and the notation of Definition 4.5 for the W-diagram. Let $Z \subset W \times P$ be the scheme theoretic preimage of the graph $\Gamma_g \subset W \times X$ under $(1, f) : W \times P \to W \times X$. We think of Z as a B-scheme using $h \circ \mathrm{pr}_1 : Z \to B$. Then the graph $\Gamma_\Phi \subset Z_b$ can be considered as a point in the relative Hilbert scheme $[\Gamma_\Phi] \in \mathrm{Hilb}(Z/B)(k)$ over the point $b \in B(k)$. Since $f : P \to X$ is smooth along $\Phi(Y)$, we see that $(1, f) : W \times P \to W \times X$ is smooth along Γ_Φ, and that $Z_b \to (\Gamma_g)_b = \Gamma_\pi$ is smooth along Γ_Φ. Since Γ_Φ is a section of the last mentionned morphism, it is a locally complete intersection. By [8, I Lemma 2.12.1], $\Gamma_\Phi \subset Z_b$ is locally unobstructed. Here is a diagram to illustrate the above; all squares but the right one are cartesian.

$$\begin{tikzcd}
\Gamma_\Phi \arrow[r] \arrow[d, "\cong"'] & Z_b \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & W \times P \arrow[r] \arrow[d] & P \arrow[d, "f"'] \\
\Gamma_\pi \arrow[r, "\cong"] & (\Gamma_g)_b \arrow[r] & \Gamma_g \arrow[r] & W \times X \arrow[r] & X
\end{tikzcd}$$

By [8, I Proposition 2.14.2], the obstruction group of $\Gamma_\Phi \subset Z_b$ is $H^1(Y, \underline{\mathrm{Hom}}_Y(I/I^2, \mathcal{O}_Y))$ where we identify Γ_Φ with Y and where I/I^2 is the conormal bundle of $\Gamma_\Phi \subset Z_b$. From the diagram we infer that $\underline{\mathrm{Hom}}_Y(I/I^2, O_Y)$ is isomorphic to $\Phi^* T_{P/X}$. Consider the short exact sequence

$$0 \longrightarrow \oplus_{i=1}^r \Phi^* T_{P/X}|_{L_i}(-p_i - q_i) \longrightarrow \Phi^* T_{P/X} \longrightarrow \Phi_0^*(T_{P/X}) \longrightarrow 0.$$

Because $\Phi_0 : Y_0 \to P$ is flexible, $H^1(Y_0, \Phi_0^* T_{P/X}) = 0$. Because each $\Phi^* T_{P/X}$ is ample on L_i, every line bundle summand has degree ≥ 1, so that every line bundle summand of $\Phi^* T_{P/X}(-p_i - q_i)$ has degree ≥ -1 and $H^1(L_i, \Phi_i^* T_{P/X}(-p_i - q_i)) = 0$. Thus by the long exact sequence in cohomology associated to the short exact sequence above, we conclude that the obstruction space of $\Gamma_\Phi \subset Z_b$ is zero. Therefore the morphism $\mathrm{Hilb}(Z/B) \to B$ is smooth at $[\Gamma_\Phi]$, see [8, I Theorem 2.10]. Therefore we can find a map $D \to \mathrm{Hilb}(Z/B)$ of a smooth, connected curve D into the Hilbert scheme and a point $d \in D$ mapping to $[\Gamma_\Phi]$ such that the composite map $D \to B$ is étale at d. Since the Hilbert scheme satisfies the valuative criterion of properness over B we may also assume that $D \to B$ is finite, hence there is a point $d_\infty \in D(k)$ which maps to b_∞ in B. The base change by $(D, d, d_\infty) \to (B, b, b_\infty)$ of the W-diagram of $\pi : Y \to X$ leads to a W-diagram. Thus, by replacing the W-diagram of $\pi : Y \to X$ by the base-change, we may assume that $\mathrm{Hilb}(Z/B) \to B$ has a section passing through $[\Gamma_\Phi]$, i.e., there is a closed subscheme $\Gamma \subset Z$ flat over B with $\Gamma_b = \Gamma_\Phi$. By Lemma 4.6, the composition $\Gamma \hookrightarrow Z \to W$ is an isomorphism over an open subset U, $b \in U \subset B$, i.e. $\Gamma \to W$ is birational. Let $\Gamma' \to \Gamma$ be the normalization, so $\Gamma' \to W$ is still birational. And the indeterminacy locus of a birational morphism of normal varieties has codimension ≥ 2. Therefore, X_∞ is not contained in the indeterminacy locus, i.e. there exists a curve $X' \subset \Gamma'$ which maps isomorphically to X_∞. By construction, the composite map $\Gamma' \to Z \to P \to X$ equals the composite map $\Gamma' \to W \to X$. Thus the image of X' in P is a curve which maps isomorphically to X, i.e. $f : P \to X$ has a section. This ends the proof of Theorem 4.1. $\square$

Proof of the main Theorem. Assumption and notations as in the theorem. First we normalize P to get a normal variety. So Theorem 3.1 applies. Hence it suffices to prove the theorem in those case where all fibres of $P \to X$ are reduced. (This reduction changes the field of definition, but no matter.) If the original P was projective then conditions 4.0.1, 4.0.2 and 4.0.3 above are satisfied. Condition 4.0.4' follows from the existence of the rational curve in the geometric generic fibre. Hence Theorem 4.1 applies and we're done.

The nonprojective case. The main theorem claims the same result for any proper $P \to X$ and the argument above only works if P is projective. Why do we not apply Chow's lemma and reduce to the projective case?

Here is one reason: Suppose that $P' \to P$ is a projective birational morphism such that P' is projective and normal. To apply the previous argument we need the geometric generic fibre $P'_{\bar{\eta}}$ of $P' \to X$ to be normal. Unfortunately we do not know how to produce $P' \to P$ with this property.

Finally, here is our proof in the general proper case circumventing this difficulty. As before we first normalize P to get a normal variety. So Theorem 3.1 applies. Hence we may assume all fibres of $P \to X$ are reduced. Note that 4.0.1, 4.0.2, 4.0.3 and 4.0.4 hold except for the projectivity in 4.0.1. The only place this was used in the proof of Theorem 4.1 is in the proof of Lemma 4.3. Thus we give an additional argument to prove the existence of a pre-flexible curve in the proper nonprojective case. The idea is to use 4.0.4 to construct a surface $\Sigma \to X$ with rational fibres, a morphism $\Sigma \to P$ over X, and to construct the pre-flexible curve as a divisor on Σ. We will use the following notations: $V \subset P$ denotes the open mentioned in 4.0.4, $\mathrm{Sm}(P/X) \subset P$ denotes the open subset of points of P where the morphism $f : P \to X$ is smooth. Note that $\mathrm{Sm}(P/X)$ is dense in every fibre of f.

We claim that there exists a 1-dimensional closed subscheme $T \subset P$ with the following properties: (a) T is the scheme theoretic union of irreducible reduced curves flat and generically étale over X (in other words, $T \to X$ is finite, flat and generically étale), (b) each irreducible component of T meets V, and (c) for every $x \in X(k)$, the intersection $T_x \cap \mathrm{Sm}(P/X)$ is not empty. We leave it to the reader to construct T. (Hints: Start with some irreducible curve meeting V, generically étale over X, and not contained in a fibre of $P \to X$. Then (a) and (b) hold and (c) holds except for a finite number of points $x_i \in X(k)$. For each i add an irreducible component T_i passing through a nonsingular point of the fibre P_{x_i}, generically étale over X with $T_i \cap V \neq \emptyset$.)

Pick a point $x \in X(k)$ such that $T_x \subset V$, and such that T_x is a reduced set of points (i.e., such that $T \to X$ is étale at all points of T_x). Pick a smooth rational curve $C \subset V_x$ such that $T_x \subset C$ and such that $T_{P/X}|_C$ is ample. Think of T_x as a divisor on C. We claim we may assume that $T_{P/X}|_C \otimes \mathcal{O}_C(-T_x)$ is ample. The proof of [8, IV Theorem 3.9.4] shows that $T_{P/X}|_C \otimes \mathcal{O}_C(-N)$ can be made ample for any N. (Note that the open V corresponds to the open X^0 in the statement of [8, IV 3.9.4].) Here is a direct argument. Pick general points c_i, $i = 1, \ldots, m$, $m >> 0$ on C, and pick a rational curve $C_i \to P_x$ passing through c_i such that $T_{P/X}|_{C_i}$ is ample. The union $C \cup \bigcup C_i$ is a comb. The results of [8, Section II 7] say that a subcomb can be smoothed fixing T_x and free over T_x union two auxiliary points. This freeness implies the claim (see [8, II 3.1]).

Let $\mathrm{Hilb}_{P/X}$ be the relative Hilbert scheme of P over X. This is in general just an algebraic space and not a scheme, see [2]. For a scheme Y over X a Y-valued point of $\mathrm{Hilb}_{P/X}$ corresponds to a closed subscheme $Z \subset P \times_X Y$ flat and of finite presentation over Y. There is a closed algebraic subspace $\mathrm{Hilb}_{T \subset P/X}$ which parametrizes only those closed subschemes $Z \subset P \times_X Y$ which contain the closed subscheme $T \times_X Y$. The curve C we constructed above defines a point $[C] \in \mathrm{Hilb}_{T \subset P/X}(k)$ lying over $x \in X(k)$.

The ampleness above means that the morphism $\mathrm{Hilb}_{T \subset P/X} \to X$ is smooth at the point $[C]$. Namely, the obstruction space for the corresponding deformation problem is equal to $H^1(C, N_C P_x \otimes \mathcal{O}_C(-T_x))$. This is a quotient of the cohomology group $H^1(C, T_{P/X}|_C \otimes \mathcal{O}(-T_x))$ which is zero.

Thus we can find an (irreducible) étale neighbourhood (Y, y) of (X, x) and a map $\psi : Y \to \mathrm{Hilb}_{T \subset P/X}$ such that $\psi(y) = [C]$. However, since the Hilbert scheme satisfies the valuative criterion of properness, we can extend ψ to a morphism $\bar{\psi} : \overline{Y} \to \mathrm{Hilb}_{T \subset P/X}$ on a nonsingular projective completion $\overline{Y}$ of Y. This means we have a closed subscheme $Z \subset P \times_X \overline{Y}$ fitting into a commutative diagram:

$$\begin{array}{ccccccc}
T \times_X \overline{Y} & \longrightarrow & Z & \longrightarrow & P \times_X \overline{Y} & \longrightarrow & P \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{Y} & \longrightarrow & \overline{Y} & \longrightarrow & \overline{Y} & \xrightarrow{\pi} & X
\end{array}$$

The morphism π is finite and generically étale. The morphism $Z \to \overline{Y}$ is flat and proper and the fibre of $Z \to \overline{Y}$ over y is C.

We choose a resolution of singularities $\Sigma \to Z$; then Σ is a proper smooth surface over k hence projective (see [6, II 4.2]). Clearly, $\Sigma \to \overline{Y}$ is a ruled surface; its fibre over the point y is the curve C. Let $U \subset \Sigma$ be the inverse image of $\mathrm{Sm}(P/X)$. By our choice of C we have: (a) the general fibre of $\Sigma \to \overline{Y}$ is contained in U. Namely, the fibre $\Sigma_y = C$ is contained in U. By our choice of T we have: (b) all fibres of $U \to \overline{Y}$ are nonempty. Namely, for each $\overline{y} \in \overline{Y}(k)$ there is some $t \in T(k)$ over $\pi(\overline{y})$ which is in $\mathrm{Sm}(P/X)$.

We claim that there exists a finite generically étale morphism of irreducible nonsingular projective curves $\widetilde{Y} \to \overline{Y}$ and an $\overline{Y}$-morphism $\widetilde{Y} \to U \subset \Sigma$. This is a generality on ruled surfaces and open subsets satisfying (a), (b) which follows easily from the lemma below. Granted this generality, we see that the composition $\widetilde{Y} \to \Sigma \to P$ is the pre-flexible curve as desired. □

4.8. Lemma. *Let $\Sigma \to X$ be a nonsingular ruled surface over an irreducible nonsingular projective curve. Assume given irreducible components C_i, $i = 1, \dots, r$ of the singular fibres such that no fibre is covered completely by the C_i. Then there exists a* **projective** *surface Σ' over X and a birational morphism $\Sigma \to \Sigma'$ over X which contracts all the curves C_i.*

Proof. We leave it to the reader that condition (b) of [1, Theorem 2.3] holds, and then [1, Theorem 2.3] implies the lemma. If the characteristic of k is $p > 0$ (which is the case of interest here) then one can also apply [1, Theorem 2.9] directly. □

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