A FACT ABOUT LINEAR SPACES ON HYPERSURFACES

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Abstract. A smooth, nondegenerate hypersurface in projective space contains no linear subvarieties of greater than half its dimension. It can contain linear subvarieties of half its dimension. This note proves that a smooth hypersurface of degree $d \geq 3$ contains at most finitely many such subvarieties.

Let k be a field. Let $X \subset \mathbb{P}^n_k$ be a hypersurface of degree $d > 1$. For each integer $m > 0$, denote by $F_m(X)$ the *Fano scheme of m-planes in X*, cf. [1]. There are a number of natural questions about $F_m(X)$: is this scheme nonempty, is this scheme connected, is this scheme irreducible, is this scheme reduced, is this scheme smooth, what is the dimension of this scheme, what is the degree of this scheme, etc.? For each of these questions, the answer is uniform for a *generic hypersurface*. More precisely, there is a non-empty open subset U of the parameter space of hypersurfaces, such that for every point in U the answer to the question is the same. For a generic hypersurface, the answer is often easy to find: The total space of the relative Fano scheme of the universal hypersurface is itself a projective bundle over the Grassmannian $\mathbb{G}(m, n)$, so if the question for a generic hypersurface can be reformulated as a question about the total space of the relative Fano scheme, it is easy to answer the question. However, much less is known if X is assumed to be smooth, but *not* generic.

There are a few easy results, such as the following.

prop-1

Proposition 0.1. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d > 1$ and let m be an integer such that $2m \geq n$. Every m-plane $\Lambda \subset X$ intersects the singular locus of X. In particular, if X is smooth then $F_m(X)$ is empty.*

Proof. Choose a system of homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$ such that Λ is given by $x_{m+1} = \cdots = x_n = 0$. Let F be a defining equation for X. Because $\Lambda \subset X$, $F(x_0, \ldots, x_m, 0, \ldots, 0) = 0$. Also,
$$\frac{\partial F}{\partial x_i}(x_0, \ldots, x_m, 0, \ldots, 0) = 0,$$
for $i = 0, \ldots, m$. Because $d > 1$, for $i = 1, \ldots, n-m$, the homogeneous polynomial on Λ,
$$\frac{\partial F}{\partial x_{m+i}}(x_0, \ldots, x_m, 0, \ldots, 0),$$
is nonconstant. Since $n - m \leq m$, these $n - m$ nonconstant homogeneous polynomials have a common zero in Λ. By the Jacobian criterion, this is a singular point of X. $\square$

Remark 0.2. This also follows easily from the Lefschetz hyperplane theorem.

What happens if $n = 2m+1$? If $m > 1$ or if $m = 1$ and $d > 3$, then a *generic* hypersurface $X \subset \mathbb{P}^n$ contains no m-plane. However there do exist smooth hypersurfaces containing an m-plane. For instance, if $\mathrm{char}(k)$ does not divide d then the Fermat hypersurface $x_0^d + \cdots + x_n^d = 0$ is smooth and contains many m-planes, e.g., $x_0 + x_1 = x_2 + x_3 = \cdots = x_{n-1} + x_n = 0$ when d is odd. However, if $d \geq 3$, a smooth hypersurface cannot contain a positive-dimensional family of m-planes. This was proved independently by Olivier Debarre, using a different argument.

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The setup is as follows. Let $n = 2m+1$. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Let $\Lambda_1, \Lambda_2 \subset X$ be m-planes. Denote by Z the intersection $\Lambda_1 \cap \Lambda_2$. This is either empty or else an r-plane for some integer r. If Z is empty, define r to be -1.

Denote by $X_{\mathrm{sm}} \subset X$ the smooth locus of X, i.e., the maximal open subscheme that is smooth. Denote $\Lambda_{i,\mathrm{sm}} = \Lambda_i \cap X_{\mathrm{sm}}$ for $i = 1, 2$. There are Chow classes,

$$[\Lambda_{i,\mathrm{sm}}] \in A_m(X_{\mathrm{sm}}),\ \ i = 1, 2.$$

Because X_{sm} is smooth, the intersection product $[\Lambda_{1,\mathrm{sm}}] \cdot [\Lambda_{2,\mathrm{sm}}] \in A_0(X_{\mathrm{sm}})$ is defined.

lem-2

Lemma 0.3. *If Z is contained in X_{sm}, then the degree of $[\Lambda_{1,sm}] \cdot [\Lambda_{2,sm}]$ is $(1-(1-d)^{r+1})/d$.*

Proof. If $r = -1$, i.e., if Z is empty, this is obvious. Therefore suppose that Z is an r-plane for some $r \geq 0$. By the excess intersection formula, the class $[\Lambda_{1,\mathrm{sm}}] \cdot [\Lambda_{2,\mathrm{sm}}]$ is the pushforward from Z of the refined intersection product, $(\Lambda_{1,\mathrm{sm}} \cdot \Lambda_{2,\mathrm{sm}})^Z$. And by [2, Prop. 9.1.1], the refined intersection product is,

$$(\Lambda_{1,\mathrm{sm}} \cdot \Lambda_{2,\mathrm{sm}})^Z = \left\{ c(N_{\Lambda_{1,\mathrm{sm}}/X_{\mathrm{sm}}})/c(N_{Z/\Lambda_{2,\mathrm{sm}}}) \cap [Z] \right\}_r.$$

Denote $H = c_1(\mathcal{O}_Z(1))$. The normal bundle of Z in $\Lambda_{1,\mathrm{sm}}$ is $\mathcal{O}_Z(1)^{m-r}$. The restriction to Z of the normal bundle of $\Lambda_{2,\mathrm{sm}}$ in $\mathbb{P}^n$ is $\mathcal{O}_Z(1)^{n-m} = \mathcal{O}_Z(1)^{m+1}$. And the restriction to Z of the normal bundle of X in $\mathbb{P}^n$ is $\mathcal{O}_Z(d)$. Therefore,

$$c(N_{\Lambda_{1,\mathrm{sm}}/X_{\mathrm{sm}}})/c(N_{Z/\Lambda_{2,\mathrm{sm}}}) = \frac{(1+H)^{m+1}}{(1+H)^{m-r}(1+dH)} = \frac{(1+H)^{r+1}}{1+dH}.$$

Expanding this out gives,

$$\left(\sum_{i=0}^{r+1} \binom{r+1}{i} H^i \right) \left(\sum_{j=0}^{\infty} (-1)^j d^j H^j \right).$$

In particular, the coefficient of H^r is,

$$\begin{aligned} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} d^{r-i} &= \frac{-1}{d} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r+1-i} d^{r+1-i} \\ &= (1-(1-d)^{r+1})/d. \end{aligned}$$

□

prop-3

Proposition 0.4. *If $d \geq 3$, if Λ_1 and Λ_2 are distinct, and if at least one of Λ_1, Λ_2 is contained in X_{sm}, then $[\Lambda_{1,sm}]$ is not numerically equivalent to $[\Lambda_{2,sm}]$.*

Proof. Let Λ_1 be contained in X_{sm}. By Lemma 0.3,

$$\deg([\Lambda_1] \cdot [\Lambda_1]) = (1-(1-d)^{m+1})/d.$$

Also, $Z = \Lambda_1 \cap \Lambda_2$ is contained in X_{sm}. So by Lemma 0.3,

$$\deg([\Lambda_1] \cdot [\Lambda_{2,\mathrm{sm}}]) = (1-(1-d)^{r+1})/d.$$

Because $d-1 \geq 2$ and $r < m$, we have $(d-1)^{r+1} < (d-1)^{m+1}$. Therefore $(1-(1-d)^{r+1})/d \neq (1-(1-d)^{m+1})/d$, and so $[\Lambda_1]$ is not numerically equivalent to $[\Lambda_{2,\mathrm{sm}}]$. □

cor-4

Corollary 0.5 (Debarre)**.** *There are only finitely many m-planes contained in X_{sm}.*

Proof. By Proposition 0.4, distinct m-planes contained in X_{sm} are not algebraically equivalent. Therefore every irreducible component of $F_m(X)$ that contains a point parametrizing an m-plane in X_{sm} is just a point. Because $F_m(X)$ is quasi-compact, the number of these irreducible components is finite. □

rmk-5

Remark 0.6. Debarre's proof shows more than Corollary 0.5: for any m-plane Λ contained in X_{sm}, $h^0(\Lambda, N_{\Lambda/X}) = 0$. It follows that each such point is a connected component of $F_m(X)$, and that $F_m(X)$ is reduced at this point.

A natural question is, what is the maximal number of m-planes contained in a smooth hypersurface of degree d in $\mathbb{P}^{2m+1}$? There is a naive upper bound that grows as $d^{(m+1)^2}$, but this is too large. The Fermat hypersurface contains $C_m d^{m+1}$ distinct m-planes, where $C_m = (2m+1)(2m-1)\cdot\dots\cdot 3\cdot 1$. Joe Harris points out that for $m=1$, the degree of the *flecnodal curve* gives an upper bound of $11d^2 - 24d$.

More generally, define the *parabolic locus* $P(X) \subset X$ to be the set of points $p \in X$ such that there is a line $L \subset \mathbb{P}^{2m+1}$ that has contact of order $3m+1$ with X at p. This is the pushforward in X of a subscheme in $\mathbb{P}(T_X)$ that is the zero locus of a section of a locally free sheaf. If X is generic, this section is a regular section. Then a Chern class computation gives that the degree of $P(X)$ is a polynomial $p_m(d)$ of degree $m+1$ in d whose leading term is,

$$\left(\frac{(3m+1)!}{2} - 1\right) d^{m+1}.$$

Therefore, for arbitrary X, $\deg(P_m(X)) \leq p_m(d)$, where $P_m(X)$ is the m-cycle of all m-dimensional irreducible components of $P(X)$ (weighted by multiplicity).

Of course every m-plane Λ is contained in $P(X)$. It is not clear that every m-plane is contained in $P_m(X)$, i.e., that Λ is an irreducible component of $P(X)$. And, indeed, this fails if $d \leq 3m$. For $d \geq 3m$, it may be true that every m-plane is an irreducible component of $P(X)$.

In the special case that $m=1$, $P(X)$ is a curve, the *parabolic curve*, for all $d \geq 3$. Therefore, the number of lines in a smooth surface of degree $d \geq 3$ in $\mathbb{P}^3$ is at most $11d^2 - 24d$. Note this gives the correct answer for $d=3$. For $d=4$ this gives the wrong answer; Segre proved the maximal number of lines on a quartic surface is 64, cf. [3]. In fact, by a more involved analysis, Segre proved that the number of lines on a smooth surface of degree $d \geq 3$ is at most $11d^2 - 28d + 12$. The Fermat surface contains $3d^2$ lines. The true maximum is probably strictly between $3d^2$ and $11d^2 - 28d + 12$.

References

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