THE KODAIRA DIMENSION OF SPACES OF RATIONAL CURVES ON LOW DEGREE HYPERSURFACES

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ABSTRACT. For a hypersurface in complex projective space, $X \subset \mathbb{P}^n$, we investigate the singularities and Kodaira dimension of the Kontsevich moduli spaces $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizing rational curves of degree e on X. If $d+e \leq n$ and X is a general hypersurface of degree d, we prove that $\overline{\mathcal{M}}_{0,0}(X,e)$ has only canonical singularities and we conjecture the same is true for the coarse moduli space $\overline{M}_{0,0}(X,e)$. We prove that this conjecture is implied by the "inversion of adjunction" conjecture of Kollár and Shokurov. Also we compute the canonical divisor of $\overline{\mathcal{M}}_{0,0}(X,e)$ and show that for most pairs (d,e) with $n \leq d^2 \leq n^2$, the canonical divisor is a *big* divisor. When combined with the above conjecture, this implies that in many cases $\overline{M}_{0,0}(X,e)$ is a variety of general type. This investigation is motivated by the question of which Fano hypersurfaces are unirational.

1. INTRODUCTION

Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d in complex projective space $\mathbb{P}^n$. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X,e)$ is a proper, Deligne-Mumford stack containing as an open substack the scheme parametrizing smooth, rational curves of degree e on X (c.f. [11]). Except when $d=1$, $d=2$ or $e=1$, very little is known about the singularities and Kodaira dimension of $\overline{\mathcal{M}}_{0,0}(X,e)$. When $d=1$ and $d=2$, the spaces $\overline{\mathcal{M}}_{0,0}(X,e)$ are all smooth [25] and rational [19]. When $e=1$, the space $\overline{\mathcal{M}}_{0,0}(X,e)$ is just the space of lines on X, which is completely understood [20, Thm. V.4.3].

What is known for $d \geq 3$ and $e \geq 2$? If $d < \frac{n+1}{2}$, it is proved in [15] that $\overline{\mathcal{M}}_{0,0}(X,e)$ is integral of the expected dimension and has only local complete intersection singularities. If also $d^2+d+1 < n$, it is proved in [16] that $\overline{\mathcal{M}}_{0,0}(X,e)$ has negative Kodaira dimension; in fact $\overline{\mathcal{M}}_{0,0}(X,e)$ is *rationally connected*. If $d \geq n-1$, the open substack of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizing smooth rational curves is not Zariski dense: the locus of *multiple covers of lines* yields an irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ not contained in the closure of this open set. However all evidence suggests that for $d \leq n-2$ and for $X \subset \mathbb{P}^n$ a general hypersurface of degree d, the stack $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible for all $e \geq 1$.

Question 1.1. For $d \leq n-2$ and $X \subset \mathbb{P}^n$ a general hypersurface of degree d, what type of singularities does $\overline{\mathcal{M}}_{0,0}(X,e)$ have? For which (n,d,e) are the singularities terminal, resp. canonical, log canonical?

Fix $e > 1$ and $d \geq 1$. Let $\mathbb{P}^N$ denote the projective space parametrizing degree d hypersurfaces $X \subset \mathbb{P}^n$. Let $C_d \subset \mathbb{P}^N \times \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ denote the closed substack parametrizing pairs $([X],[f:D\to X])$ of $X \subset \mathbb{P}^n$ a hypersurface and $f: D \to X$ a stable map in $\overline{\mathcal{M}}_{0,0}(X,e)$. The main theorem of this paper is the following.

Theorem 7.5. *If $e \geq 2$ and if $d+e \leq n$, then C_d is an integral, normal, local complete intersection stack of the expected dimension and has at worst canonical singularities.*

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Corollary 7.6. *If $e \geq 2$ and if $d + e \leq n$, then for a general hypersurface $X \subset \mathbb{P}(V)$ of degree d, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X, e)$ is an integral, normal, local complete intersection stack of the expected dimension $(n+1-d)e + (n-3)$ and has at worst canonical singularities.*

It is reasonable to expect that Theorem 7.8 is sharp. For instance the inequality $d + e \leq n$ is consistent with the inequality $d \leq n - 1$ necessary for C_d to be irreducible. On the other hand, Corollary 7.9 is certainly not sharp: it fails to account for the cases $d = 1$ and $d = 2$ where $\overline{\mathcal{M}}_{0,0}(X, e)$ is smooth for every e. For $d \geq 3$ and $e \geq 2$, the space $\overline{\mathcal{M}}_{0,0}(X, e)$ *is singular*, but hopefully it is not too singular.

Question 1.2. For which integers d and n is it true that for a very general hypersurface $X \subset \mathbb{P}^n$ of degree d, X is unirational?

This is the question which motivates this paper, although no new answers are given here. A necessary condition is that $d \leq n$; otherwise the Kodaira dimension of X is nonnegative. For $d = 1, 2, 3$ and $n \geq d$, it is known that a general hypersurface $X \subset \mathbb{P}^n$ of degree d is unirational. For each integer d there is an integer $\phi(d)$ such that for $n \geq \phi(d)$ and $X \subset \mathbb{P}^n$ a general hypersurface, then X is unirational ([12], [23, Chapter 23]). It is conjectured that if $d \leq n$ but large compared to n – e.g. if $d = n$ for $n \geq 4$ – then a general hypersurface $X \subset \mathbb{P}^n$ is *not* unirational. But no example of such a hypersurface has been proved to be non-unirational.

1.1. **Kollár's approach.** The connection between Question 1.2 and this paper comes from a suggestion by Kollár in [22], that a necessary condition for a variety X of dimension at least 2 to be unirational is that for a general point $p \in X$, there exists a rational surface $S \subset X$ containing p.

Definition 1.3. An irreducible, projective variety X over a field k is *swept by rational surfaces* (resp. *separably swept by rational surfaces*) if there exists an irreducible variety Z and a rational transformation $F : Z \times \mathbb{P}^2 \to X$ such that,

- (i) the rational transformation F is dominant (resp. dominant and separable), and
- (ii) the rational transformation $(\mathrm{pr}_Z, F) : Z \times \mathbb{P}^2 \to Z \times X$ is generically finite to its image (resp. generically finite and separable to its image).

Remark 1.4. There are several obvious remarks.

- (i) This definition makes sense for any field k, not necessarily algebraically closed nor of characteristic 0 (although this is the case of interest in the rest of the paper).
- (ii) In the definition above, Z can be replaced by a separable, dominant cover and the conditions will still hold.
- (iii) Let X be a variety that is separably swept by rational surfaces. Let $S \subset Z \times X$ denote the image of (pr_Z, F). By [20, Thm. III.2.4], the base change $S \otimes_{K(Z)} \overline{K}(Z)$ is a rational surface. By [5], in fact there is a separable dominant morphism $Z' \to Z$ and a birational transformation over Z', $G : Z' \times \mathbb{P}^2 \to Z' \times_Z S$. After replacing Z by Z' and replacing F by the composite $Z' \times \mathbb{P}^2 \xrightarrow{G} Z' \times_Z S \xrightarrow{\mathrm{pr}_S} S \xrightarrow{\mathrm{pr}_X} X$, (pr_Z, F) is actually birational to its image.
- (iv) If X is (separably) unirational and $\dim(X) \geq 2$, then X is (separably) swept by rational surfaces.
- (v) If X is swept by rational surfaces (resp. separably swept by rational surfaces), then X is uniruled (resp. separably uniruled).
- (vi) Let $\dim(X) = n$. Definition 1.3 is equivalent to the stronger condition where $\dim(Z) = n-2$. Moreover Z can be required to be smooth over k. In fact, by de Jong's alterations of singularities, up to replacing Z by a generically étale cover, Z can even be required to be smooth and projective.

(vii) The condition that X is swept by rational surfaces (resp. separably swept by rational surfaces) is a birational property that is equivalent to the condition that there exist a finitely-generated field extension L/k of transcendence degree $n-2$ and a finite super-extension (resp. finite separable super-extension) of $K(X)/k$ of the form $L(t_1,t_2)/K(X)$ (resp. such that the compositum $L * K(X)$ equals $L(t_1,t_2)$).

(viii) If X is swept by rational surfaces (resp. separably swept by rational surfaces) and $f : X \to X'$ is a generically finite, dominant rational transformation (resp. generically étale, dominant rational transformation), then also X' is swept by rational surfaces (resp. separably swept by rational surfaces).

(ix) Unlike the analogous situation of rationally connected varieties, given a family of smooth, projective varieties in characteristic zero, it is unclear whether the condition of being swept by rational surfaces is a closed condition, or even an open condition, on fibers of the family.

It is technically more convenient to work with pencils of rational curves than to work with rational surfaces. Replacing $\mathbb{P}^2$ by the birational surface $\mathbb{P}^1 \times \mathbb{P}^1$, Definition 1.3 can be rephrased in terms of pencils of rational curves.

Definition 1.5. Let X be a projective variety and $e \geq 1$ an integer. An integral, closed substack $Y \subset \overline{\mathcal{M}}_{0,0}(X,e)$ is *sweeping* (resp. *separably sweeping*) if,

(i) for a general geometric point of Y, the associated stable map $f : C \to X$ has irreducible domain and is birational to its image, and
(ii) the restriction over Y of the universal morphism, $f : Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C} \to X$ is surjective (resp. surjective and separable).

Remark 1.6. There are several obvious remarks.

(i) This definition makes sense for any field k, not necessarily algebraically closed nor of characteristic 0; although in case of positive characteristic one should keep in mind that $\overline{\mathcal{M}}_{0,0}(X,e)$ may only be an Artin algebraic stack with finite diagonal (not a Deligne-Mumford stack).
(ii) If $Y \subset \overline{\mathcal{M}}_{0,0}(X,e)$ is sweeping, then there is a Zariski dense open substack that is a scheme. In particular, it makes sense to ask whether Y is uniruled (resp. separably uniruled).
(iii) If $Y \subset \overline{\mathcal{M}}_{0,0}(X,e)$ is sweeping (resp. separably sweeping), then for the irreducible component $M \subset \overline{\mathcal{M}}_{0,0}(X,e)$ containing Y, also M is sweeping (resp. separably sweeping).
(iv) If X is uniruled (resp. separably uniruled), then there is an integer e and an irreducible component $M \subset \overline{\mathcal{M}}_{0,0}(X,e)$ that is sweeping (resp. separably sweeping).

Lemma 1.7. *Let X be a projective variety over a field k (not necessarily algebraically closed nor of characteristic 0). The variety X is swept by rational surfaces (resp. separably swept by rational surfaces) iff there exists an integer e and a substack $Y \subset \overline{\mathcal{M}}_{0,0}(X,e)$ that is sweeping (resp. separably sweeping) such that Y is uniruled (resp. separably uniruled).*

Proof. Let X be a variety of dimension n that is swept by rational surfaces (resp. separably swept by rational surfaces). Then there exists a smooth quasi-projective variety Z of dimension $n-2$ and a rational transformation $F : Z \times \mathbb{P}^1 \times \mathbb{P}^1 \to X$ that is dominant (resp. dominant and separable) such that $(\mathrm{pr}_Z, F) : Z \times \mathbb{P}^1 \times \mathbb{P}^1 \to Z \times X$ is generically finite (resp. birational to its image). The indeterminacy locus of F, call it $I \subset Z \times \mathbb{P}^1 \times \mathbb{P}^1$, is a subvariety that has codimension at least 2 at every point. The projection $\mathrm{pr}_{12}(I) \subset Z \times \mathbb{P}^1$ has codimension at least 1 at every point, i.e. it is contained in a divisor $D \subset Z \times \mathbb{P}^1$. Denoting $U = Z \times \mathbb{P}^1 - D$, the rational transformation $F : U \times \mathbb{P}^1 \to X$ is a regular morphism. After shrinking U further, the morphism $(\mathrm{pr}_{12}, F) : U \times \mathbb{P}^1 \to U \times X$ is a finite morphism (resp. a finite, birational morphism). Let $C' \subset U \times X$ denote the image and let $G : C \to U \times X$ denote the normalization of C'. In case X is separably swept by rational surfaces, G is the same as (pr_{12}, F).

After shrinking U further, the morphism $\mathrm{pr}_U : C \to U$ is a smooth, proper morphism of relative dimension 1. Moreover every geometric fiber is dominated by $\mathbb{P}^1$, and therefore the geometric fibers are connected curves that are isomorphic to $\mathbb{P}^1$. So $\mathrm{pr}_U : C \to U$ is a family of genus 0 curves and $G : C \to U \times X$ is a family of stable maps of genus 0 to X of some degree e. There is an induced 1-morphism $\zeta : U \to \overline{\mathcal{M}}_{0,0}(X, e)$. Define $Y \subset \overline{\mathcal{M}}_{0,0}(X, e)$ to be the closed image substack of ζ.

The claim is that Y is sweeping (resp. separably sweeping) and that the coarse moduli space of Y is uniruled (resp. separably uniruled). There is an open substack of Y over which the geometric fibers of the universal curve are irreducible. By construction, this open substack contains the image of ζ. So it is Zariski dense in Y. Similarly, the condition that the stable maps are birational to their images is an open condition on Y. Since this condition holds on the image of ζ, it holds on an open dense subset of Y. Therefore Y satisfies Item (i) of Definition 1.5. Now $F : U \times \mathbb{P}^1 \to X$ is dominant. Therefore also $G : C \to X$ is dominant. So $f : Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C} \to X$ is dominant. Therefore Y satisfies Item (ii) of Definition 1.5, i.e. Y is sweeping. Moreover if X is separably swept by rational surfaces, then F is dominant, generically finite and separable which implies that also $f : Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C} \to X$ is dominant, generically finite and separable (since a sub-extension of a separable field extension is separable), so Y is separably sweeping.

Consider $\zeta : U \to Y$. The claim is that ζ is dominant and generically finite (resp. dominant, generically finite and separable). There is a factorization of F of the form

$$U \times \mathbb{P}^1 \to U \times_Y (Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C}) = C \to Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C} \xrightarrow{f} X. \tag{1}$$

Since F is dominant and generically finite (resp. dominant, generically finite and separable), each of these factors of F is dominant and generically finite (resp. dominant, generically finite and separable). In particular $U \times_Y (Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C}) \to Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C}$ is dominant and generically finite (resp. and separable). But this is just the base-change of $\zeta : U \to Y$ by the smooth surjective morphism $\mathrm{pr}_Y : Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C} \to Y$. Therefore $\zeta : U \to Y$ is dominant and generically finite (resp. dominant, generically finite and separable). Since U is an open subset of $Z \times \mathbb{P}^1$, Y is uniruled (resp. separably uniruled). This proves the forward direction of the lemma.

Conversely, suppose that $Y \subset \overline{\mathcal{M}}_{0,0}(X, e)$ is sweeping (resp. separably sweeping) and Y is uniruled (resp. separably uniruled). Then there exists a rational transformation $\zeta : Z \times \mathbb{P}^1 \to Y$ that is dominant and generically finite (resp. dominant, generically finite and separable).

Let $\pi : C \to Z \times \mathbb{P}^1$ be a projective completion of the pullback of the universal family of curves over $\overline{\mathcal{M}}_{0,0}(X, e)$, and let $H : C \to X$ be the pullback of the universal stable map. After blowing up C, C can be made normal and H can be made a regular morphism. Moreover, over a dense open subset of $Z \times \mathbb{P}^1$, the projection π is a smooth, proper morphism whose geometric fibers are connected curves of genus 0. So the geometric generic fiber of $\mathrm{pr}_Z \circ \pi : C \to Z$ is a conic bundle over $\mathbb{P}^1$. By Tsen's theorem (c.f. [20, Cor. IV.6.6.2]), the base-change $C \otimes_{K(Z)} \overline{K}(Z)$ is a rational surface. By [5], in fact there is a dominant, generically finite and separable morphism $Z' \to Z$ and a birational transformation over Z', $G : Z' \times \mathbb{P}^2 \to Z' \times_Z C$. Observe that the composition $Z' \times \mathbb{P}^1 \to Z' \times \mathbb{P}^1 \to Y$ is still dominant and generically finite (resp. dominant, generically finite and separable). Therefore Z may be replaced by Z' so that C is birational to $Z \times \mathbb{P}^2$ over Z.

Denote by F the rational transformation $H \circ G : Z \times \mathbb{P}^2 \to X$. There is a factorization of F of the form

$$Z \times \mathbb{P}^2 \xrightarrow{G} C \to Y \times_{\overline{\mathcal{M}}_{0,0}(X,e)} \mathcal{C} \xrightarrow{f} X. \tag{2}$$

By assumption, each of these factors is dominant (resp. dominant and separable). Therefore F is dominant (resp. dominant and separable). Consider $(\mathrm{pr}_Z \circ \pi, F) \circ G : Z \times \mathbb{P}^2 \to Z \times X$. By the hypotheses on Y, this morphism is generically finite (resp. generically finite and separable to its image). Therefore X is swept by rational surfaces (resp. separably swept by rational surfaces). This finishes the proof of the lemma. ∎

Because of the lemma, it is natural to try to understand the sweeping substacks of $\overline{\mathcal{M}}_{0,0}(X,e)$, and in particular to try to understand the Kodaira dimension of these substacks. Recall that a standard conjecture from the minimal model program predicts that an algebraic variety is uniruled iff its Kodaira dimension is negative. If $X \subset \mathbb{P}^n$ is a hypersurface of degree $d \leq n$, then for $e >> 0$, an irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ will itself be sweeping. So the first step is to determine the Kodaira dimension of each of the irreducible components of $\overline{\mathcal{M}}_{0,0}(X,e)$ that is sweeping.

As mentioned above, for X a general hypersurface of degree $d < \frac{n+1}{2}$ the stacks $\overline{\mathcal{M}}_{0,0}(X,e)$ are irreducible and reduced, and the same is conjectured whenever $d < n-1$. Under the hypothesis that each $\overline{\mathcal{M}}_{0,0}(X,e)$ is an integral, normal stack of the expected dimension, one can compute the canonical divisor of $\overline{\mathcal{M}}_{0,0}(X,e)$, i.e. one can compute the *expected canonical divisor*. This is carried out in Section 10, and is a straightforward extension of the derivation in [28]. The exciting observation is that when d satisfies the inequalities $d \leq n-4$ and $d^2 \geq n+2$, then for almost all values of e, the canonical divisor is *big*. And when $d \leq n-7$ and $d(d+1) \geq 2(n+1)$, then the canonical divisor is big for every $e \geq 1$.

Recall that a sufficient condition for a variety M to be of general type is that the canonical divisor K_M is big and M has only canonical singularities. This raises the hope that in the degree range above the spaces $\overline{M}_{0,0}(X,e)$ are all of general type. The missing ingredient is an analysis of the singularities of $\overline{M}_{0,0}(X,e)$. This paper is the result of an "initial investigation" of the singularities of $\overline{M}_{0,0}(X,e)$. Obviously much work is still needed to prove that $\overline{M}_{0,0}(X,e)$ contains no uniruled sweeping subvariety.

1.2. **Detailed summary.** The proof of Theorem 7.8 is a deformation-and-specialization argument. The stack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ is smooth, therefore the singularities of C_d come from loci in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ over which the fiber dimension of $\pi_d : C_d \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ jumps. This defines a stratification of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$, and the "deepest" stratum corresponds to the locus $Y \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ parametrizing multiple covers of lines. This stratum is a smooth variety, and the normal bundle of $Y \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ is essentially the bundle of all $(e-1)\times(n-1)$ matrices.

The normal cone of $\pi_d^{-1}(Y) \subset C_d$ is a projective cone over the normal bundle of $Y \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$. When $d+e \leq n$ this projective cone is even a *projective Abelian cone* associated to a torsion-free sheaf on the normal bundle of $Y \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$. This torsion-free sheaf is essentially the direct sum of d copies of the quotient of a bundle of rank $(n-1)$ by the universal $(e-1)\times(n-1)$ matrix mentioned above. By an explicit resolution of singularities, this projective Abelian cone is canonical. Then *deformation to the normal cone* produces a family over $\mathbb{P}^1$ whose fibers over $\mathbb{A}^1$ are all isomorphic to C_d and whose fiber over ∞ is the normal cone of $\pi_d^{-1}(Y)$.

Applying *inversion-of-adjunction* results to this family, there exists an open substack $U \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ containing Y such that $\pi_d^{-1}(U)$ is canonical. The action of the group GL_{n+1} on $\mathbb{P}^n$ induces an action of GL_{n+1} on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$. The open substack U is GL_{n+1}-invariant, but also Y intersects the closure of *every* orbit of GL_{n+1} on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$. Therefore U is all of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$, proving that C_d is canonical.

Section 2, Section 3, Section 4 and Section 5 are all of a foundational nature, proving basic results about the singularities of the relative Grassmannian, or *Grassmannian cone*, associated to a torsion-free coherent sheaf $\mathcal{E}$ which has (local) projective dimension 1. The main result of Section 2 is Proposition 2.15: This relates the singularities of a Grassmannian cone C parametrizing rank r locally free quotients of $\mathcal{E}$ to the singularities of the pair $(B, r\cdot B_{g-1})$, where B_{g-1} is the closed subscheme determined by the *Fitting ideal* of $\mathcal{E}$. The main result of Section 3 is Proposition 3.15: This computes the singularities of the Grassmannian cone of a direct sum of a copies of the cokernel of the universal $g\times f$ matrix on the affine space of $g\times f$ matrices. The main result of Section 4 is Corollary 4.13: This applies known results related to inversion-of-adjunction to prove adjunction results for a pair $(B, r\cdot B_{g-1})$ as above. Section 5 is a review of the construction of deformation to the normal cone in preparation for the proof of the main theorem.

In Section 6 and Section 7 the proof of the main theorem is given. Section 6 introduces the closed substack $Y \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ parametrizing multiple covers of lines. The main result is Proposition 6.11, which is an analysis of the coherent sheaves used to define C_d when restricted to a first-order neighborhood of Y in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$. Section 7 gives the proof of Theorem 7.8 along the lines discussed above.

In Section 8, the Reid–Shepherd-Barron–Tai criterion is used to prove that the coarse moduli space $\overline{M}_{0,0}(\mathbb{P}^n, e)$ has only canonical singularities (and in most cases it is even terminal). Combining this analysis with the proof of Theorem 7.8, Section 9 proves that the inversion-of-adjunction conjecture of Kollár and Shokurov implies that the coarse moduli space of the stack C_d has only canonical singularities when either $e \geq 3$ and $d + e \leq n$ or $e = 2$ and $d + 3 \leq n$.

Finally, in Section 10, the expected canonical divisor of the stack $\overline{\mathcal{M}}_{0,0}(X, e)$ is computed – this is the same as the canonical divisor on the coarse moduli space $\overline{M}_{0,0}(X, e)$ in most cases. When $n + 1 < d^2 < (n - 3)^2$, for most choices of e the expected canonical divisor of $\overline{M}_{0,0}(X, e)$ is big.

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2. Discrepancies of a Grassmannian cone

Let B be a Noetherian scheme that is connected, normal, and $\mathbb{Q}$-Gorenstein of pure dimension b. Let $\phi : \mathcal{G} \to \mathcal{F}$ be a morphism of locally free $\mathcal{O}_B$-modules of rank g and f respectively such that the cokernel $\mathcal{E} = \mathrm{Coker}(\phi)$ has generic rank $e = f - g$. In this section all results are of a local nature on B. So the results apply equally well to a coherent sheaf $\mathcal{E}$ that has local projective dimension 1 (in the sense of [27, p. 280]).

Notation 2.1. Denote by $\det(\mathcal{E})$ the invertible sheaf $\det(\mathcal{F}) \otimes_{\mathcal{O}_B} \det(\mathcal{G})^\vee$. Let r be an integer, $1 \leq r \leq e$, and denote by the pair $(\pi : C \to B, \alpha : \pi^*\mathcal{E} \to \mathcal{Q})$ the relative Grassmannian cone over B parametrizing rank r locally free quotients of $\mathcal{E}$. Denote by $\mathcal{O}_C(1)$ the invertible sheaf on C, $\det(\mathcal{Q})$. Denote by the pair $(\rho : C' \to B, \beta : \rho^*\mathcal{F} \to \mathcal{Q}')$ the relative Grassmannian bundle over B parametrizing rank r locally free quotients of $\mathcal{F}$. Denote by $\mathcal{O}_{C'}(1)$ the invertible sheaf on C', $\det(\mathcal{Q}')$. The surjection $\pi^*\mathcal{F} \to \pi^*\mathcal{E} \xrightarrow{\alpha} \mathcal{Q}$ induces a morphism of B-schemes which we denote $h : C \to C'$.

Comparing universal properties, h is a closed immersion whose ideal sheaf is the image of the composite morphism

$$\rho^*\mathcal{G} \otimes (Q')^\vee \xrightarrow{\phi \otimes \beta^\dagger} \pi^*\mathcal{F} \otimes_{\mathcal{O}_{C'}} \rho^*\mathcal{F}^\vee \xrightarrow{\mathrm{Trace}} \mathcal{O}_{C'}. \tag{3}$$

This morphism is denoted by γ.

If $\mathcal{E}$ is locally free, then $\pi : C \to B$ is Zariski locally a Grassmannian bundle. But, in general, the fiber dimension is not necessarily constant. The results of the next few sections hold for arbitrary integer r; although the case of interest in the remainder of the paper is $r = 1$. For $r = 1$, $\pi : C \to B$ is a *projective Abelian cone* (paraphrasing notation of [9] and [3]). We begin with an obvious criterion for C to be irreducible.

Notation 2.2. Let k be an integer $k = 0, \ldots, g$. Denote by $B_k \subset B$ the closed subscheme whose ideal sheaf is generated by the $(k+1) \times (k+1)$-minors of ϕ, i.e. B_k is the locus where ϕ has rank at most k.

Lemma 2.3. (i) *The coherent sheaf $\mathcal{E}$ is torsion-free iff $\mathrm{codim}_B(B_{g-1}) \geq 2$.*

(ii) *The scheme C is irreducible iff $\mathrm{codim}_B(B_k - B_{k-1}) \geq r(g - k) + 1$ for all $k = 0, \ldots, g - 1$, in which case C has dimension $c = b + r(e - r)$.*

(iii) *Furthermore, the scheme C is regular in codimension 1 points if $codim_B(B_k - B_{k-1}) \geq r(g-k)+2$ for all $k = 0, \ldots, g-1$.*

If $g = 0$, all of these conditions are vacuously satisfied.

Proof. **(i)**: Torsion sections of the sheaf $\mathcal{E}$ correspond locally on B to sections of $\mathcal{F}$ that are generically in the image of $\mathcal{G}$. Since B is normal and since $\mathcal{G}$ is locally free, the image of $\mathcal{G}$ in $\mathcal{F}$ equals the intersection of its localization at all codimension 1 points of S. Hence a section of $\mathcal{F}$ corresponds to a torsion (resp. torsion-free) section of $\mathcal{E}$ iff its localization at all codimension 1 points of B is torsion (resp. torsion-free) in $\mathcal{E}$. Therefore $\mathcal{E}$ is torsion-free iff $\mathrm{codim}_B(B_{g-1}) \geq 2$.

(ii): Of course C' has pure dimension $b + r(f-r)$ at every point. Consider the morphism γ. Since the rank of $\rho^*\mathcal{G} \otimes (\mathcal{Q}')^\vee$ is $g \cdot r$, the dimension of C at every point is at least $c := b + r(f-r) - g \cdot r = b + r(e-r)$.

The restriction of π over the locally closed subscheme $B_k - B_{k-1}$ is proper and smooth of relative dimension $r\,((f-k)-r)$ and has geometrically irreducible (and nonempty) fibers. In particular, the preimage of $B - B_{g-1}$ is normal and irreducible of dimension $c = b + r(e-r)$. Therefore, to prove that C is irreducible, it suffices to prove that for each $k = 0, \ldots, g-1$, the dimension of $\pi^{-1}(B_k - B_{k-1})$ is at most $c-1$. Conversely, if any of these sets has dimension c or greater, then it is not in the closure of $\pi^{-1}(B - B_{g-1})$, and therefore C is reducible. Hence C is irreducible iff $\dim \pi^{-1}(B_k - B_{k-1}) \leq c-1$. The dimension of this set is clearly $c - [\mathrm{codim}_B\,(B_k - B_{k-1}) - r(g-k)]$. Therefore C is irreducible iff $\mathrm{codim}_B\,(B_k - B_{k-1}) \geq r(g-k)+1$ for all $k = 0, \ldots, g-1$.

(iii): Now suppose that in fact $\mathrm{codim}_B\,(B_k - B_{k-1}) \geq r(g-k)+2$ for all $k = 0, \ldots, g-1$. The preimage $\pi^{-1}(B - B_{g-1})$ is normal. So to prove that C is regular in codimension 1 points, it suffices to prove that C is regular in codimension 1 points that are contained in one of the subsets $\pi^{-1}(B_k - B_{k-1})$ for $k = 0, \ldots, g-1$. The inequality guarantees that each of the sets $\pi^{-1}(B_k - B_{k-1})$ has codimension at least 2 in C, therefore there are no such codimension 1 points. ☐

Remark 2.4. It can happen that ϕ satisfies the inequality of (ii) so that C is irreducible, and yet C is not regular in codimension 1. For example, let $B = \mathbb{A}^2$, let $r = 1$, and let ϕ be the morphism $\phi : \mathcal{O}_{\mathbb{A}^2} \to \mathcal{O}_{\mathbb{A}^2}^{\oplus 2}$ with matrix $(x^2, y^2)^\dagger$. On the other hand, the third inequality is not a necessary condition, cf. Proposition 3.15.

Hypothesis 2.5. Unless stated otherwise, the coherent sheaf $\mathcal{E}$ is torsion-free, i.e. $\mathrm{codim}_B(B_{g-1}) \geq 2$.

Lemma 2.6. *Suppose that C has pure dimension $c = b + r(e-r)$.*

(i) *If B is Cohen-Macaulay, then C is Cohen-Macaulay. If also C is regular in codimension 1, then C is normal.*

(ii) *If B is Gorenstein, then C is Gorenstein.*

(iii) *The morphism π admits a relative dualizing complex of the form $\omega_\pi[r(e-r)]$ where ω_π is the invertible sheaf $\pi^* \det(\mathcal{E})^{\otimes r} \otimes_{\mathcal{O}_C} \mathcal{O}_C(-e)$.*

(iv) *If C is normal, then C is $\mathbb{Q}$-Gorenstein and the $\mathbb{Q}$-Cartier divisor class K_C equals $\pi^* K_B + K_\pi$ where K_π is the divisor class of ω_π.*

Proof. **(i)**: By assumption $h(C)$ has pure codimension $g \cdot r$ in C, which equals the rank of $\mathcal{G} \otimes_{\mathcal{O}_{C'}} (Q')^\vee$. By [27, Thm. 17.3 and Thm. 17.4], if B is Cohen-Macaulay then $h(C)$ is Cohen-Macaulay. It follows from Serre's criterion, [27, Thm. 23.8], that if C is regular in codimension 1, then C is normal.

(ii): Using [27, Exer. 18.1], if B is Gorenstein then $h(C)$ is Gorenstein.

(iii): The morphism ρ is smooth and has a relative dualizing complex $\omega_\rho[r(f-r)]$ where ω_ρ is the invertible sheaf $\rho^* \det(\mathcal{F})^{\otimes r} \otimes_{\mathcal{O}_{C'}} \mathcal{O}_{C'}(-f)$. The morphism h is a regular embedding and has a relative dualizing complex $\omega_h[-rg]$ where ω_h is the pullback of the invertible sheaf $\mathcal{E}xt^{rg}_{\mathcal{O}_{C'}}(h_*\mathcal{O}_C, \mathcal{O}_{C'})$.

Forming the Koszul complex associated to the sheaf map γ, ω_h is isomorphic to $\pi^*\det(\mathcal{G}^\vee)^{\otimes r} \otimes_{\mathcal{O}_C} \mathcal{O}_C(g)$. Therefore the composite $\pi = \rho \circ h$ has a relative dualizing complex $\omega_\pi[r(e-r)]$ where ω_π is an invertible sheaf isomorphic to $\pi^*\det(\mathcal{E})^{\otimes r} \otimes \mathcal{O}_C(-e)$.

(iv): Suppose that C is normal. Let $U \subset C$ be the smooth locus of C and let $V \subset C'$ denote the smooth locus of C'. Since h is a regular embedding, $U \subset h^{-1}(V)$. Also, since ρ is smooth, V is just $\rho^{-1}(W)$, where $W \subset B$ is the smooth locus. Now $\omega_B|_W$ is isomorphic to $\mathcal{O}_B(K_B)|_W$ and $\omega_{C'}|_{\rho^{-1}(W)}$ is isomorphic to $\rho^*\mathcal{O}_B(K_B) \otimes_{\mathcal{O}_{C'}} \omega_\rho$. By the same reasoning as above, $\omega_C|_V \cong h^*\rho^*\mathcal{O}_B(K_B) \otimes \omega_\pi$. But of course $\omega_C|_V$ is isomorphic to $\mathcal{O}_C(K_C)|_V$. Therefore the $\mathbb{Q}$-Weil divisor class K_C is equal to $\rho^*K_C + K_\pi$ where K_π is the divisor class of ω_π. Since this is a $\mathbb{Q}$-Cartier divisor class, C is $\mathbb{Q}$-Gorenstein. ∎

Corollary 2.7. *Let $Y \subset B$ be a closed subscheme that is a regular embedding of pure codimension $codim_B(Y)$, i.e. for every closed point $p \in Y$, the ideal sheaf $\mathcal{I}_p \subset \mathcal{O}_{B,p}$ is generated by a regular sequence of length $codim_B(Y)$.*

(i) *If $C \times_B Y$ has the expected dimension $b - codim_B(Y) + r(e-r)$, then there exists an open subset $U \subset B$ containing Y such that $C \times_B U$ has the expected dimension $b + r(e-r)$.*
(ii) *If also $C \times_B Y$ is irreducible, then U can be chosen so that $C \times_B U$ is irreducible.*
(iii) *If also $C \times_B Y$ is normal and B is Cohen-Macaulay, then U can be chosen so that $C \times_B U$ is normal.*

Proof. **(i)**: Let $C_i \subset C$ be an irreducible component of C that has nonempty intersection with $C \times_B Y$. The dimension of C is *at least* $b + r(e-r)$; the claim is that it is exactly $b+r(e-r)$. Since $Y \subset B$ is a regular embedding locally defined by a regular sequence of length $\operatorname{codim}_B(Y)$, it follows by Krull's Hauptidealsatz that $\dim(C_i \times_B Y) \geq \dim(C_i) - \operatorname{codim}_B(Y)$. On the other hand, since $C_i \times_B Y$ is a closed subscheme of $C \times_B Y$, $\dim(C_i \times_B Y) \leq \dim(C \times_B Y) = b + r(e-r) - \operatorname{codim}_B(Y)$. Therefore $\dim(C_i) = b + r(e-r)$. So for any irreducible component $C_i \subset C$ whose dimension is larger than $b + r(e-r)$, $\pi(C_i) \cap Y = \emptyset$. Define U to be the complement of the finitely many closed sets $\pi(C_i)$ as above. Then U satisfies (i).

(ii): Suppose that also $C \times_B Y$ is irreducible. By Lemma 2.3, for each $k = 0, \ldots, g-1$, $\operatorname{codim}_Y(Y_k - Y_{k-1}) \geq r(g-k)+1$. Now $Y_k = B_k \cap Y$. So again by Krull's Hauptidealsatz, for every irreducible component $(B_k)_i$ of B_k that intersects Y, $\dim((B_k)_i) \leq \dim(Y_k) + \operatorname{codim}_B(Y)$, i.e. $\operatorname{codim}_B((B_k)_i) \geq \operatorname{codim}_Y(Y_k)$. Now shrink the U from the last paragraph by taking the complement of the finitely many irreducible components $(B_k)_i$ that do not intersect Y and have the wrong codimension. Then for every $(B_k)_i$ that intersects U, $\operatorname{codim}_B((B_k)_i) \geq \operatorname{codim}_Y(Y_k) \geq r(g-k)+1$. Therefore, by Lemma 2.3, $C \times_B U$ is irreducible.

(iii): Finally, suppose that also $C \times_B Y$ is normal and B is Cohen-Macaulay. By Item (i) of Lemma 2.6, to prove that $C \times_B U$ is normal, it suffices to prove that $C \times_B U$ is regular in codimension one. Let $(C \times_B U)_{\text{sing}}$ be the singular locus of $C \times_B U$. Since $C \times_B Y \subset C \times_B U$ is a Cartier divisor, every regular point of $C \times_B Y$ is also a regular point of $C \times_B U$. Therefore the intersection of $(C \times_B U)_{\text{sing}}$ with $C \times_B Y$ is contained in $(C \times_B Y)_{\text{sing}}$. Since $C \times_B Y$ is normal, $(C \times_B Y)_{\text{sing}}$ has codimension at least 2 in $C \times_B Y$. So, again by Krull's Hauptidealsatz, every irreducible component of $(C \times_B U)_{\text{sing}}$ that intersects $C \times_B Y$ has codimension at least 2 in $C \times_B U$. After shrinking U more, $C \times_B U$ is normal. ∎

Hypothesis 2.8. Unless stated otherwise, C is irreducible of the expected dimension $c = b + r(e-r)$.

Definition 2.9. A morphism of schemes $u : \widetilde{B} \to B$ is a *resolution of* $\mathcal{E}$ if,

(i) u is a birational, proper morphism,
(ii) $\widetilde{B}$ is smooth,
(iii) the exceptional locus of u is a simple normal crossings divisor $E_1 \cup \cdots \cup E_k$, and

(iv) the coherent sheaf $\widetilde{\mathcal{E}} := u^*\mathcal{E}/\text{torsion}$ is a locally free $\mathcal{O}_{\widetilde{B}}$-module of rank e.

Notation 2.10. Let $u : \widetilde{B} \to B$ be a resolution of $\mathcal{E}$. Denote by $\widetilde{\mathcal{G}}$ the kernel of the induced surjective sheaf map $u^*\mathcal{F} \to \widetilde{\mathcal{E}}$ and denote by $\widetilde{\phi} : \widetilde{\mathcal{G}} \to u^*\mathcal{F}$ the induced injection, i.e. $\widetilde{\mathcal{E}}$ is the cokernel of $\widetilde{\phi}$. Denote by the pair $(\text{pr}_1 : \widetilde{B} \times_B C' \to \widetilde{B}, \text{pr}_2^*\beta : \text{pr}_2^*\rho^*\mathcal{F} \to \text{pr}_2^*\mathcal{Q}')$ the base-change of $(\rho : C' \to B, \beta)$. Denote by the pair $(\widetilde{\pi} : \widetilde{C} \to \widetilde{B}, \widetilde{\alpha} : \widetilde{\pi}^*\widetilde{\mathcal{E}} \to \widetilde{Q})$ the Grassmannian bundle parametrizing rank r locally free quotients of $\widetilde{\mathcal{E}}$. Denote by $\mathcal{O}_{\widetilde{C}}(1)$ the invertible sheaf on $\widetilde{C}$, $\det(\widetilde{Q})$. The surjection $u^*\mathcal{F} \to \widetilde{\mathcal{E}}$ induces a closed immersion that is denoted $\widetilde{h} : \widetilde{C} \to \widetilde{B} \times_B C'$.

Because the morphism $u^*\mathcal{F} \to \widetilde{\mathcal{E}}$ factors through the pullback $u^*\mathcal{F} \to u^*\mathcal{E}$, the composition $\text{pr}_2 \circ h : \widetilde{C} \to C'$ factors through h, i.e. there is an induced morphism $v : \widetilde{C} \to C$. Of course $v^*\mathcal{Q} \cong \widetilde{\mathcal{Q}}$ and $\pi \circ v = u \circ \widetilde{\pi}$.

Lemma 2.11. *The morphism $v : \widetilde{C} \to C$ is a weak resolution of singularities, that is*

(i) *v is a proper, birational morphism, and*

(ii) *$\widetilde{C}$ is nonsingular.*

Moreover, the exceptional locus of v is contained in the divisor $\widetilde{\pi}^{-1}(E_1 \cup \cdots \cup E_k)$.

Proof. This is obvious. □

Remark 2.12. If B is a finite type scheme over an algebraically closed field of characteristic 0, then a resolution of $\mathcal{E}$ exists. Form the Grassmannian bundle $(\pi_e : G \to B, \alpha_e : \pi_e^*\mathcal{E} \to Q)$ parametrizing rank e locally free quotients of $\mathcal{E}$. Since $\mathcal{E}$ generically has rank e, there is an irreducible component G_0 of G such that $\pi_e : G_0 \to B$ is birational. Let $Z \subset G_0$ denote the fundamental locus of π_e^{-1}. By [17], there exists a log resolution $\widetilde{B} \to B$ of the pair (G_0, Z), and the induced morphism $u : \widetilde{B} \to B$ will be a resolution of $\mathcal{E}$.

Lemma 2.13. *The morphism $u : \widetilde{B} \to B$ is a log resolution of the pair (B, B_{g-1}).*

Proof. The morphism $\phi : \mathcal{G} \to \mathcal{F}$ induces an element $\bigwedge^g \phi \in \mathit{Hom}_{\mathcal{O}_B}(\bigwedge^g \mathcal{G}, \bigwedge^g \mathcal{F})$, i.e. an element in $\bigwedge^g \mathcal{F} \otimes_{\mathcal{O}_B} (\bigwedge^g \mathcal{G})^\vee$. There is an induced map

$$\text{Id} \otimes \bigwedge^g \phi : \bigwedge^e \mathcal{F} \to \bigwedge^e \mathcal{F} \otimes_{\mathcal{O}_B} \bigwedge^g \mathcal{F} \otimes_{\mathcal{O}_B} \left(\bigwedge^g \mathcal{G}\right)^\vee. \tag{4}$$

Compose this map with the wedge product map on $\mathcal{F}$ to get a map $\bigwedge^e \mathcal{F} \to \det(\mathcal{E})$. Consider the restriction of this map to the torsion-free subsheaf $\mathcal{G} \cdot \bigwedge^{e-1} \mathcal{F} \subset \bigwedge^e \mathcal{F}$. On the generic point of B, it is clear that this map is the zero map. A morphism between torsion-free sheaves on B that is zero at the generic point of B is the zero map. So the restriction of the map to $\mathcal{G} \cdot \bigwedge^{e-1} \mathcal{F}$ is zero, proving that the map factors through a map $\psi : \bigwedge^e \mathcal{E} \to \det(\mathcal{E})$.

Since $\mathcal{E}$ is torsion-free, B_{g-1} has codimension at least 2 in B. So ψ is an isomorphism in codimension 1. Denote by $\mathcal{I} \subset \mathcal{O}_B$ the unique ideal sheaf so that $\text{Image}(\psi)$ equals $\mathcal{I} \cdot \det(\mathcal{E})$. In other words, $\mathcal{I}$ is the e^{th} *Fitting ideal* of ϕ – the ideal sheaf generated by the $g \times g$ minors of ϕ. This is precisely the ideal sheaf of the subscheme B_{g-1}. It follows that $(u^* \bigwedge^e \mathcal{E})/\text{torsion}$ is just $u^{-1}\mathcal{I} \cdot u^*\det(\mathcal{E})$.

The surjection $u^*\mathcal{E} \to \widetilde{\mathcal{E}}$ induces a surjection $(u^* \bigwedge^e \mathcal{E})/\text{torsion} \to \bigwedge^e \widetilde{\mathcal{E}}$. It is easy to see that this surjection is in fact an isomorphism. Hence there is a canonical isomorphism $u^{-1}\mathcal{I} \cdot u^*\det(\mathcal{E}) \cong \det(\widetilde{\mathcal{E}})$. In particular, the pullback ideal sheaf $u^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{B}}$ is a Cartier divisor. Moreover, this divisor is a subdivisor of the simple normal crossings divisor $E_1 \cup \cdots \cup E_k$, and so it is also a simple normal crossings divisor. □

By definition, the log discrepancies $a(E_i; B, r\cdot B_{g-1})$ of $(B, r\cdot B_{g-1})$ along the divisors $E_1,\dots,E_k \subset \widetilde{B}$ are defined by

$$K_{\widetilde{B}} - u^*K_B - r\cdot u^{-1}(B_{g-1}) = \sum_{i=1}^{k} (a(E_i; B, r\cdot B_{g-1}) - 1)\, E_i \tag{5}$$

where $u^{-1}(B_{g-1}) \subset \widetilde{B}$ is defined to be the closed subscheme corresponding to $u^{-1}\mathcal{I}\cdot\mathcal{O}_{\widetilde{B}}$.

Lemma 2.14. *The relative canonical divisor of $v : \widetilde{C} \to C$ is equal to the following divisor*

$$K_{\widetilde{C}} - v^*K_C = \sum_{i=1}^{k} (a(E_i; B, r\cdot B_{g-1}) - 1)\, \widetilde{\pi}^*E_i. \tag{6}$$

Proof. By the isomorphism in the proof of Lemma 2.13, $u^{-1}\mathcal{I} \cong \det(\widetilde{\mathcal{E}}) \otimes_{\mathcal{O}_{\widetilde{B}}} u^*\det(\mathcal{E})^\vee$. On the other hand, applying Lemma 2.6 to both C and $\widetilde{C}$,

$$K_{\widetilde{C}} - v^*K_C = \widetilde{\pi}^*\left(K_{\widetilde{B}} - u^*K_B + r\cdot C_1(\det(\widetilde{\mathcal{E}})) - r\cdot u^*C_1(\det(\mathcal{E}))\right). \tag{7}$$

□

Proposition 2.15. *The pair $(C,\emptyset)$ is log canonical (resp. Kawamata log terminal, canonical) iff the pair $(B, r\cdot B_{g-1})$ is log canonical (resp. Kawamata log terminal, canonical).*

Proof. Of course the total discrepancy of $(C,\emptyset)$, totaldiscrep$(C,\emptyset)$, is the minimum of 0 and the discrepancy of C, discrep$(C,\emptyset)$. So $(C,\emptyset)$ is log canonical (resp. Kawamata log terminal, canonical) iff totaldiscrep$(C,\emptyset)$ is $=0$ (resp. >-1, ≥ -1). By a standard argument, cf. [24, Cor. 2.32] and [7, Prop. 1.3(iv)], totaldiscrep$(C,\emptyset)$ = totaldiscrep$(\widetilde{C}, -K_{\widetilde{C}/C})$. Since $\widetilde{C}$ is smooth and $\widetilde{\pi}^*(E_1\cup\dots\cup E_k)$ is a simple normal crossings divisor, the total discrepancy is given by a combinatorial formula in the coefficients $a(\widetilde{\pi}^*(E_i); \widetilde{C}, -K_{\widetilde{C}/C})$, cf. [24, Defn. 2.28] and [24, Defn. 2.34]. But these are the same as the coefficients $a(E_i; B, r\cdot B_{g-1})$. So the total discrepancy of $(C,\emptyset)$ equals the minimum of 0 and the integers $a(E_i; B, r\cdot B_{g-1}) - 1$. Moreover, by assumption all of the divisors E_i are exceptional for u. Therefore the minimum of the integers $a(E_i; B, r\cdot B_{g-1}) - 1$ is the discrepancy of $(B, r\cdot B_{g-1})$. Therefore $(C,\emptyset)$ is log canonical, etc. iff $(B, r\cdot B_{g-1})$ is log canonical, etc. □

Remark 2.16. If each log discrepancy $a(E_i; B, r\cdot B_{g-1})$ is different than 1, then the exceptional locus of $v : \widetilde{C} \to C$ is all of $\widetilde{\pi}^{-1}(E_1\cup\dots\cup E_k)$ and v is a strong resolution of singularities. In this case C is terminal iff $(B, r\cdot B_{g-1})$ is terminal. But if any log discrepancy equals 1, this can fail; e.g. if $B = \mathbb{A}^2$ and $\phi : \mathcal{O}_{\mathbb{A}^2} \to \mathcal{O}_{\mathbb{A}^2}^{\oplus 2}$ is the map with matrix $(x,y)^\dagger$, then for $r=1$ the cone C is the blowing up of $\mathbb{A}^2$ in the origin. So C is smooth, and thus terminal. But the pair $(\mathbb{A}^2, \{0\})$ is only canonical.

2.1. **Further manipulations.** The results of this subsection are straightforward and are not actually used in the rest of the paper, but it is natural to state them here. In this subsection the Grassmannian cone C is denoted C_r to emphasize the integer r. Proofs are left to the reader.

Lemma 2.17. *Let $1 \leq s < r \leq e$. If C_r is normal of pure dimension $b + r(e-r)$, then also C_s is normal of pure dimension $b + s(e-s)$. Moreover, if C_r is log canonical (resp. Kawamata log terminal, canonical, terminal) then also C_s is log canonical (resp. Kawamata log terminal, canonical, terminal).*

Let $\phi^{(i)} : \mathcal{G}^{(i)} \to \mathcal{F}^{(i)}$, $i = 1, \dots, N$ be a sequence of injective morphisms of locally free sheaves such that for each $i = 1, \dots, N$ the cokernel $\mathcal{E}^{(i)}$ has generic rank $e^{(i)} = f^{(i)} - g^{(i)}$. Let $r^{(1)}, \dots, r^{(N)}$ be a sequence of integers with $1 \leq r^{(i)} \leq e^{(i)}$. Let $\pi^{(i)} : C^{(i)} \to B$ denote the Grassmannian cone of rank $r^{(i)}$ locally free quotients of $\mathcal{E}^{(i)}$ and let $\pi : C \to B$ denote the fiber product $C^{(1)} \times_B \cdots \times_B C^{(N)}$.

Lemma 2.18. *Suppose that C has pure dimension $c = b + \sum_i r^{(i)}(e^{(i)} - r^{(i)})$. Then Lemma 2.6 applies to $\pi : C \to B$ where now the dualizing complex is $\omega_\pi[\sum_i r^{(i)}(e^{(i)} - r^{(i)})]$ with ω_π equal to the tensor product of $\pi^*\left[det(E^{(1)})^{\otimes r^{(1)}} \otimes \cdots \otimes det(E^{(N)})^{\otimes r^{(N)}}\right]$ with $\mathcal{O}_{C^{(1)}}(-e^{(1)}) \otimes \cdots \otimes \mathcal{O}_{C^{(N)}}(-e^{(N)})$.*

Lemma 2.19. *Suppose that C is normal of pure dimension c. For each $i = 1, \dots, N$ let $Z^{(i)}$ denote the closed subscheme associated to the $e^{(i)}$ Fitting ideal of $\phi^{(i)}$.*

(i) *For each divisor E of $K(B)$, the log discrepancy $a(\pi^{-1}(E); C, \emptyset)$ equals the log discrepancy $a(E; B, \sum_i r^{(i)} Z^{(i)})$.*

(ii) *The pair $(C, \emptyset)$ is log canonical (resp. Kawamata log terminal, canonical) iff $(B, \sum_i r^{(i)} Z^{(i)})$ is log canonical (resp. Kawamata log terminal, canonical).*

(iii) *For every subset $I \subset \{1, \dots, N\}$, the fiber product $C^{(I)} = \prod_B (C^{(i)} | i \in I)$ is normal of pure dimension $b + \sum_{i \in I} r^{(i)} Z^{(i)}$.*

(iv) *If C is log canonical (resp. Kawamata log terminal, canonical), then $C^{(I)}$ is log canonical (resp. Kawamata log terminal, canonical).*

Define $\mathcal{G} := \oplus_i \mathcal{G}_i$, define $\mathcal{F} := \oplus_i \mathcal{F}_i$, define $\phi : \mathcal{G} \to \mathcal{F}$ to be the direct sum over $i = 1, \dots, N$ of $\phi^{(i)}$ and define $\mathcal{E}$ to be the cokernel of ϕ, i.e. $\mathcal{E} \cong \oplus_i \mathcal{E}^{(i)}$. Denote by e the sum $\sum_i e^{(i)}$ and let r be an integer $1 \leq r \leq e$. Define $\pi' : C' \to B$ to be the Grassmannian cone parametrizing rank r locally free quotients of $\mathcal{E}$.

Lemma 2.20. (i) *The e^{th} Fitting ideal of ϕ is the product $\mathcal{I}^{(1)} \cdot \dots \cdot \mathcal{I}^{(N)}$.*

(ii) *If C' is normal of pure dimension $b + r(e - r)$, then for every subset $I \subset \{1, \dots, N\}$, the Grassmannian bundle $C'_I \to B$ parametrizing rank r locally free quotients of $\oplus_{i \in I} \mathcal{E}^{(i)}$ is normal of pure dimension $b + r(\sum_{i \in I} e^{(i)} - r)$.*

(iii) *If, moreover, C' is log canonical (resp. Kawamta log terminal, canonical), then also C'_I is log canonical (resp. Kawamata log terminal, canonical).*

3. Log discrepancies of generic determinantal varieties

Let K be a field, not necessarily algebraically closed nor of characteristic zero. In this section all schemes are K-schemes. The interested reader will see how to prove analogous results over Spec $(\mathbb{Z})$, and thus over an arbitrary base scheme.

Let S be a K-scheme and let $\mathcal{G}$, $\mathcal{F}$ be locally free $\mathcal{O}_S$-modules of finite rank g and f respectively with $g \leq f$.

Notation 3.1. Denote by $\pi^{(0)} : M^{(0)}(S, \mathcal{G}, \mathcal{F}) \to S$ the affine bundle $\underline{\text{Spec}}_S \text{Sym}^* \left(Hom_{\mathcal{O}_S}(\mathcal{G}, \mathcal{F})^\vee \right)$. When there is no risk of confusion, denote $M^{(0)}(S, \mathcal{G}, \mathcal{F})$ by $M^{(0)}$. There is a tautological sheaf map, denoted

$$\phi : \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}} \to \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}}. \tag{8}$$

Definition 3.2. For $k = 0, \dots, g$, the k^{th} *generic determinantal variety* $M_k^{(0)} \subset M^{(0)}$ is defined to be the closed subscheme of $M^{(0)}$ whose ideal sheaf is generated by the $(k+1) \times (k+1)$-minors of ϕ just as in Notation 2.2 (c.f. also [1, Sec. II.2]). For technical reasons, denote by $M_{-1}^{(0)}$ the empty set. In particular, $M_0^{(0)}$ is just the zero section of $\pi^{(0)} : M^{(0)} \to S$.

In this section the log discrepancies of the pair $(M^{(0)}, M_k^{(0)})$ are computed. This is straightforward once a log resolution is constructed. The log resolution is constructed by first blowing up $M_0^{(0)}$, then blowing up the strict transform of $M_1^{(0)}$, the blowing up the strict transform of $M_2^{(0)}$, etc.

3.1. **The log resolution.** The log resolution of $(M^{(0)}, M_k^{(0)})$ used here is the obvious one: successively blow up the strict transforms of the schemes $M_0^{(0)}, M_1^{(0)}, \dots, M_k^{(0)}$. Using the action of the group $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G})$, it is easy to prove this does give a log resolution. For completeness, we go through the proof in (somewhat tedious) detail.

Remark 3.3. The set of subschemes $M_k^{(0)} \subset M^{(0)}$ form a stratification, and in case the ground field is $\mathbb{C}$, this is a *conical stratification* in the sense of [26] and the blowing up we construct coincides with the *minimal wonderful compactification*. We choose not to follow [26] for two reasons: First of all, the log resolution is quite general and is valid over an arbitrary ground field K, not just over $\mathbb{C}$ (and in fact over Spec $(\mathbb{Z})$, though we don't prove this). More importantly, in computing log discrepancies later, it is *crucial* that the log resolution has the additional property that for each $M_k^{(0)}$, the inverse image of the ideal sheaf of $M_k^{(0)}$ is an invertible sheaf, i.e., it contains no embedded points. This typically fails for the minimal wonderful compactification associated to a conical stratification. For example, consider the nodal plane cubic $p \in C$ sitting in $\mathbb{P}^2$ sitting as a linear subvariety of $\mathbb{P}^3$. Then $(\{p\}, C - \{p\}, \mathbb{P}^3 - C)$ is a conical stratification of $\mathbb{P}^3$. And the inverse image of the ideal sheaf of C in the minimal wonderful compactification has an "embedded line" on the exceptional divisor over p. It would be interesting to know if there are extra hypotheses of a general nature that can be added to the definition of a conical stratification so that the minimal wonderful compactification has the additional property.

In the case that $f = g$, the log resolution in this section is identical to that in [18]. Moreover in [18] it is proved that the inverse image of the ideal sheaf of $M_k^{(0)}$ is an invertible sheaf. However the case of most interest here is $f \neq g$. Thus the full description of the log resolution and proofs of the basic properties of the log resolution are given. The next lemma gives a precise definition of the sequence of blowing ups mentioned above.

Lemma 3.4. *There exists a sequence of schemes $M^{(r)}$ for $r = 0, \dots, g$ and morphisms $u^{(s,r)} : M^{(r)} \to M^{(s)}$ for each $0 \leq s \leq r \leq g$ with the following properties*

(i) *For $0 \leq t \leq s \leq r \leq g$, we have $u^{(t,s)} \circ u^{(s,r)} = u^{(t,r)}$.*

(ii) *For $r = 0, \dots, g$, the morphism $u^{(0,r)} : M^{(r)} \to M^{(0)}$ is an isomorphism over the open subscheme $M^{(0)} - M_{r-1}^{(0)}$.*

(iii) *For each $0 \leq r \leq k \leq g$, define $M_k^{(r)} \subset M^{(r)}$ to be the closure of the pullback by $u^{(0,r)}$ of $M_r^{(0)} - M_{k-1}^{(0)}$. Then, for $r = 0, \dots, g-1$ the morphism $u^{(r,r+1)} : M^{(r+1)} \to M^{(r)}$ is the blowing up of $M^{(r)}$ along $M_r^{(r)}$.*

Proof. This is almost tautological. The one thing that needs to be checked is that, defining $M^{(r+1)}$ to be the blowing up of $M^{(r)}$ along $M_r^{(r)}$, the induced map $u^{0,r+1} : M^{(r+1)} \to M^{(0)}$ is an isomorphism over $M^{(0)} - M_r^{(0)}$. But this follows immediately from the two facts: $u^{r,0}$ is an isomorphism over $M^{(0)} - M_{r-1}^{(0)}$ and $u^{r,r+1}$ is an isomorphism over the preimage under $u^{r,0}$ of $M^{(0)} - M_r^{(0)}$. □

Notation 3.5. Let $(S, \mathcal{G}, \mathcal{F})$ be a datum with $\mathrm{rank}(\mathcal{G}) = g$, $\mathrm{rank}(\mathcal{F}) = f$ (and of course $g \leq f$). For each $r = 1, \dots, k$, denote by $E_{r-1}^{(r)}(S, \mathcal{G}, \mathcal{F}) \subset M^{(r)}(S, \mathcal{G}, \mathcal{F})$ the exceptional divisor of the blowing up $u^{r-1,r}$. For $r = 2, \dots, k$, and for $k = 0, \dots, r-2$, denote by $E_k^{(r)} \subset M^{(r)}$ the strict transform of $E_k^{(k+1)}$ under the morphism $u^{k+1,r} : M^{(r)} \to M^{(k+1)}$. Clearly for each $0 \leq r < s \leq g$, the exceptional locus of $u^{r,s} : M^{(s)} \to M^{(r)}$ is $E_r^{(s)} \cup \dots \cup E_{s-1}^{(s)}$.

Definition 3.6. Let $(S, \mathcal{G}', \mathcal{F}')$ and $(S, \mathcal{G}, \mathcal{F})$ be data of pairs of locally free sheaves on S with $g' \leq f'$ and $g \leq f$. A *morphism* between the data is a triple $\zeta = (p, q, T)$ where

(i) $p : \mathcal{G} \to \mathcal{G}'$ is a surjective morphism of $\mathcal{O}_S$-modules,

(ii) $q : \mathcal{F}' \to \mathcal{F}$ is an injective morphism of $\mathcal{O}_S$-modules whose cokernel is locally free, and

(iii) $T : \mathcal{G} \to \mathcal{F}$ is a morphism of $\mathcal{O}_S$-modules, and such that there are direct sum decompositions

$$\mathcal{G} = \mathrm{Ker}(T) \oplus \mathrm{Ker}(p), \quad \mathcal{F} = \mathrm{Image}(T) \oplus \mathrm{Image}(q). \tag{9}$$

In particular, $g - g' = f - f'$. The *rank* of ζ is the common integer $g - g' = f - f'$.

Lemma 3.7. *Let $\zeta = (p, q, T)$ be a morphism $(S, \mathcal{G}', \mathcal{F}') \to (S, \mathcal{G}, \mathcal{F})$ of rank l. For $r = 0, \ldots, g'$ there exist morphisms of S-schemes*

$$\tau^r(\zeta) : GL(\mathcal{F}) \times_S GL(\mathcal{G}) \times_S M^{(r)}(S, \mathcal{G}', \mathcal{F}') \to M^{(r+l)}(S, \mathcal{G}, \mathcal{F}) \tag{10}$$

satisfying the following conditions.

(i) *The image of the composite morphism $u^{0,l} \circ \tau^0(\zeta)$ is contained in $M^{(0)}(S, \mathcal{G}, \mathcal{F}) - M^{(0)}_{l-1}(S, \mathcal{G}, \mathcal{F})$.*

(ii) *The morphism $\tau^0(\zeta)$ is the unique morphism such that $u^{0,l} \circ \tau^0(\zeta)$ is the morphism whose restriction to the fiber over a point $x \in S$ maps a triple (α, β, L) in $GL(\mathcal{F}_x) \times GL(\mathcal{G}_x) \times M^{(0)}(x, \mathcal{G}'_x, \mathcal{F}'_x)$ to the element $\alpha \circ (T_x + q_x \circ L \circ p_x) \circ \beta^{-1}$.*

(iii) *For $0 \leq r \leq s \leq g'$, $u^{l+r, l+s} \circ \tau^s(\zeta)$ equals $\tau^r(\zeta) \circ (Id \times Id \times u^{r,s})$; moreover, the corresponding commutative diagram is Cartesian.*

(iv) *For each $0 \leq r \leq g'$, the morphism $\tau^r(\zeta)$ is quasi-compact, separated and smooth, and $Image(\tau^r(\zeta)) = (u^{0,l+r})^{-1}(M^{(0)} - M^{(0)}_{l-1})$.*

(v) *For each $0 \leq r \leq k \leq g'$,*

$$(\tau^r(\zeta))^{-1}(M^{(r+1)}_{k+1}(S, \mathcal{G}, \mathcal{F})) = GL(\mathcal{F}) \times_S GL(\mathcal{G}) \times_S M^{(r)}_k(S, \mathcal{G}', \mathcal{F}').$$

(vi) *For each $0 \leq r \leq g'$ and each $0 \leq i \leq r - 1$,*

$$(\tau^r(\zeta))^{-1}(E^{(r+l)}_{i+l}(S, \mathcal{G}, \mathcal{F}) = GL(\mathcal{F}) \times_S GL(\mathcal{G}) \times_S E^{(r)}_i(S, \mathcal{G}', \mathcal{F}').$$

In particular, this is a Cartier divisor.

Proof. The claim is that for each $0 \leq r \leq g'$, there exists a sequence of morphisms $\tau^0(\zeta), \ldots, \tau^r(\zeta)$ satisfying (i)–(vi). The claim is proved by induction on r. First consider the case $r = 0$. The morphism in (ii),

$$\tau(\zeta) : \mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(0)}(S, \mathcal{G}', \mathcal{F}') \to M^{(0)}(S, \mathcal{G}, \mathcal{F}), \tag{11}$$

is defined more precisely by giving a natural transformation of the obvious functors represented by the two schemes and invoking Yoneda's lemma.

Let T be a k-scheme. By the universal properties of the three factors, a morphism from T to $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(0)}(S, \mathcal{G}', \mathcal{F}')$ is equivalent to a morphism $f : T \to S$ together with a triple (α, β, L) consisting of $\alpha : f^*\mathcal{F} \to f^*\mathcal{F}$ an automorphism of $\mathcal{O}_T$-modules, $\beta : f^*\mathcal{G} \to f^*\mathcal{G}$ is an automorphism of $\mathcal{O}_T$-modules, and $L : f^*\mathcal{G}' \to f^*\mathcal{F}'$ a morphism of $\mathcal{O}_T$-modules. There is an associated morphism of $\mathcal{O}_T$-modules $f^*\mathcal{G} \to f^*\mathcal{F}$ by $L' = \alpha \circ (f^*T + f^*q \circ Lf^*p) \circ \beta^{-1}$. By the universal property of $M^{(0)}(S, \mathcal{G}, \mathcal{F})$ the pair (f, L') is equivalent to a morphism $T \to M^{(0)}(S, \mathcal{G}, \mathcal{F})$. The association $(f, \alpha, \beta, L) \mapsto (f, L')$ is a natural transformation of Yoneda functors and so determines a morphism of S-schemes τ.

(i) and (ii): Using the direct sum decompositions of $\mathcal{G}$ and $\mathcal{F}$ induced by (p, q, T), for any point $x \in T$ the rank of L'_x equals $\mathrm{rank}(T_{f(x)}) + \mathrm{rank}(L_x)$, i.e. $l + \mathrm{rank}(L_x)$. This is bigger than $l - 1$, hence the image of τ is contained in the complement of $M^{(0)}_{l-1}(S, \mathcal{G}, \mathcal{F})$. Since $u^{0,l} : M^{(l)}(S, \mathcal{G}, \mathcal{F}) \to M^{(0)}(S, \mathcal{G}, \mathcal{F})$ is an isomorphism over the complement of $M^{(0)}_{l-1}(S, \mathcal{G}, \mathcal{F})$, there is a unique morphism $\tau^0(\zeta)$ such that $u^{0,l} \circ \tau^0(\zeta)$ equals τ.

(iv); r=0: This is equivalent to the claim that τ is quasi-compact, smooth and separated. It is clear that both $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(0)}(S, \mathcal{G}', \mathcal{F}')$ and $M^{(0)}(S, \mathcal{G}, \mathcal{F})$ are quasi-compact, smooth and separated over S. Therefore τ is quasi-compact, finitely-presented and separated. To show that τ is smooth it suffices to check the Jacobian criterion for fibers of τ over geometric points of S.

Let $x \in S$ be a geometric point and let (α, β, L) be a closed point of the fiber of $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(0)}(S, \mathcal{G}', \mathcal{F}')$ over x. The smooth group scheme $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G})$ acts on both the domain and target of τ and τ is equivariant for this action. Therefore it suffices to check the Jacobian criterion at one representative point of every orbit. Hence it suffices to consider the special case $\alpha = \mathrm{id}_{\mathcal{F}}$ and $\beta = \mathrm{id}_{\mathcal{G}}$. The Zariski tangent space to the fiber of $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(0)}(S, \mathcal{G}', \mathcal{F}')$ over x at any closed point is canonically identified with the $\kappa(x)$-vector space $\mathrm{Hom}(\mathcal{F}_x, \mathcal{F}_x) \times \mathrm{Hom}(\mathcal{G}_x, \mathcal{G}_x) \times \mathrm{Hom}(\mathcal{G}'_x, \mathcal{F}'_x)$. Similarly the Zariski tangent space to the fiber of $M^{(0)}(S, \mathcal{G}, \mathcal{F})$ is canonically identified with $\mathrm{Hom}(\mathcal{G}_x, \mathcal{F}_x)$. And $d\tau$ maps a triple (α_1, β_1, L_1) to the element $\alpha_1 \circ (T_x + q_x \circ L \circ p_x) + q_x \circ L_1 \circ p_x + (T_x + q_x \circ L \circ p_x) \circ \beta_1$.

Let $d = \mathrm{rank}(L)$. There exists an ordered bases for $\mathcal{G}_x$ and $\mathcal{F}_x$ with respect to which T_x has the matrix representation,

$$\left[\begin{array}{c|c|c} I_{l,l} & 0_{l,d} & 0_{l,g'-d} \\ \hline 0_{d,l} & 0_{d,d} & 0_{d,g'-d} \\ \hline 0_{f'-d,l} & 0_{f'-d,d} & 0_{f'-d,g'-d} \end{array}\right], \tag{12}$$

and with respect to which $q_x \circ L \circ p_x$ has the matrix representation,

$$\left[\begin{array}{c|c|c} 0_{l,l} & 0_{l,d} & 0_{l,g'-d} \\ \hline 0_{d,l} & I_{d,d} & 0_{d,g'-d} \\ \hline 0_{f'-d,l} & 0_{f'-d,d} & 0_{f'-d,g'-d} \end{array}\right]. \tag{13}$$

For any linear operator $L' \in \mathrm{Hom}(\mathcal{G}_x, \mathcal{F}_x)$ consider the matrix representation of L' with respect to the ordered bases above,

$$\left[\begin{array}{c|c|c} L'_1 & L'_2 & L'_3 \\ \hline L'_4 & L'_5 & L'_6 \\ \hline L'_7 & L'_8 & L'_9 \end{array}\right], \tag{14}$$

where the block submatrices L'_i have the same dimensions as the blocks in the matrices of T and L. Then, denoting,

$$\alpha_1 = \left[\begin{array}{c|c|c} L'_1 & L'_2 & 0 \\ \hline L'_4 & L'_5 & 0 \\ \hline L'_7 & L'_8 & 0 \end{array}\right], \tag{15}$$

$$\beta_1 = \left[\begin{array}{c|c|c} 0_{l,l} & 0_{l,d} & L'_3 \\ \hline 0_{d,l} & 0_{d,d} & L'_6 \\ \hline 0_{f'-d,l} & 0_{f'-d,d} & 0_{f'-d,g'-d} \end{array}\right], \tag{16}$$

$$q_x \circ L_1 \circ p_x = \left[\begin{array}{c|c|c} 0_{l,l} & 0_{l,d} & 0_{l,g'-d} \\ \hline 0_{d,l} & 0_{d,d} & 0_{d,g'-d} \\ \hline 0_{f'-d,l} & 0_{f'-d,d} & L'_9 \end{array}\right], \tag{17}$$

the pair (α_1, β_1, L_1) maps to L' under $d\tau$. Hence τ satisfies the Jacobian criterion and τ and $\tau^0(\zeta)$ are smooth.

(iii), (v), and (vi); r=0: For $L' = \tau(\alpha, \beta, L)$, $\mathrm{rank}(L') = l + \mathrm{rank}(L)$. Hence the preimage under τ of $M^{(0)}_{l+k}(S, \mathcal{G}, \mathcal{F})$ is the closed subscheme $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(0)}_k(S, \mathcal{G}', \mathcal{F}')$; i.e., (v) holds. Finally (iii) and (vi) are vacuous for the "sequence" of morphisms $\tau^0(\zeta)$. This finishes the base case $r = 0$.

(iii) and (iv); induction step: Now comes the induction step. Let $r = 1, \dots, g'$. By way of induction, suppose that morphisms $\tau^0(\zeta), \dots, \tau^{r-1}(\zeta)$ have been constructed satisfying (i)–(vi). Since $\tau^{r-1}(\zeta)$ is smooth, the fiber product of $\tau^{r-1}(\zeta)$ with the blowing up $u^{l+r-1,l+r}$ is canonically isomorphic to the blowing up of $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$ along the preimage of $M^{(l+r-1)}_{l+r-1}(S, \mathcal{G}, \mathcal{F})$. By the induction hypothesis, the preimage is precisely $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(r-1)}_{r-1}(S, \mathcal{G}', \mathcal{F}')$. So the base-change of $u^{l+r-1,l+r}$ by $\tau^{r-1}(\zeta)$ is just $\mathrm{Id} \times \mathrm{Id} \times u^{r-1,r}$. Define $\tau^r(\zeta) : \mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(r)}(S, \mathcal{G}', \mathcal{F}') \to M^{(l+r)}(S, \mathcal{G}, \mathcal{F})$ to be the base-change of $\tau^{r-1}(\zeta)$ by $u^{l+r-1,l+r}$.

By construction of $\tau^r(\zeta)$ and the induction hypothesis, $\tau^0(\zeta),\dots,\tau^r(\zeta)$ satisfies (iii). Since $\tau^r(\zeta)$ is the base-change of a quasi-compact, separated and smooth morphism, $\tau^r(\zeta)$ is also quasi-compact, separated and smooth, i.e. (iv) holds.

(v); induction step: Since $\tau^r(\zeta)$ and $\tau^{r-1}(\zeta)$ are smooth, and by (iii), the process of forming the strict transform by $u^{l+r,l+r-1}$ of a closed subscheme and then forming the preimage under $\tau^r(\zeta)$ is the same as the process of first forming the preimage under $\tau^{r-1}(\zeta)$ and then forming the strict transform under $\mathrm{Id}\times\mathrm{Id}\times u^{r,r-1}$. By the induction hypothesis and (v), the preimage under $\tau^{r-1}(\zeta)$ of $M_k^{(r+l-1)}(S,\mathcal{G},\mathcal{F})$ equals $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M_k^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$. The strict transform of this subscheme under $\mathrm{Id}\times\mathrm{Id}\times u^{r-1,r}$ is $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M_k^{(r)}(S,\mathcal{G}',\mathcal{F}')$. Therefore (v) is satisfied for $\tau^r(\zeta)$.

(vi); induction step: As above, for $i=0,\dots,r-2$ the pullback by $\tau^r(\zeta)$ of the strict transform by $u^{l+r,l+r-1}$ of $E_{i+l}^{(l+r-1)}(S,\mathcal{G},\mathcal{F})$ equals the strict transform of the pullback by $\tau^{r-1}(\zeta)$. By the induction hypothesis and (vi), this pullback is $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S E_i^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$. The strict transform of this subscheme under $\mathrm{Id}\times\mathrm{Id}\times u^{r,r-1}$ is $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S E_i^{(r)}(S,\mathcal{G}',\mathcal{F}')$. Finally, using (iii), the preimage under $\tau^r(\zeta)$ of the exceptional divisor of $u^{r+l,r+l-1}$, i.e. of $E_{r+l-1}^{(r+l)}(S,\mathcal{G},\mathcal{F})$, equals the exceptional divisor of $\mathrm{Id}\times\mathrm{Id}\times u^{r,r-1}$, i.e. $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S E_{r-1}^{(r)}(S,\mathcal{G}',\mathcal{F}')$. This proves (vi) and finishes the induction step. The lemma is proved by induction on r. $\square$

Lemma 3.8. *Let V be a smooth K-scheme and let $D\subset V$ be a simple normal crossings divisor. The K-scheme $\mathbb{A}^1\times V$ is smooth and the divisor $D'=(\mathbb{A}^1\times D)\cup(\{0\}\times V)$ is a simple normal crossings divisor in $\mathbb{A}^1\times V$.*

Proof. This follows immediately from the definition of *simple normal crossings divisor.* $\square$

Notation 3.9. Denote by $U^{(0)}\subset M^{(0)}(S,\mathcal{G},\mathcal{F})$ the complement of the zero section $M_0^{(0)}(S,\mathcal{G},\mathcal{F})$. For each $r=0,\dots,g$ denote by $U^{(r)}\subset M^{(r)}$ the open subscheme $U^{(r)}=\left(u^{0,r}\right)^{-1}(U^{(0)})$ and denote by $v^{r,s}:U^{(s)}\to U^{(r)}$ the restriction of $u^{r,s}$ to $U^{(s)}$. For each $r=0,\dots,g$ and each $i=r,\dots,g$, denote by $\mathcal{I}_k^{(r)}$ the ideal sheaf of the closed subscheme $M_k^{(r)}\subset M^{(r)}$. Finally denote by $f:\mathbb{A}^1\times U^{(0)}\to M^{(0)}$ the morphism sending a pair (λ,L) in $\mathbb{A}^1\times U^{(0)}$ to the point $\lambda\cdot L\in M^{(0)}$.

The preimage under f of $M_0^{(0)}$ is precisely $\{0\}\times U^{(0)}$, which is a Cartier divisor in $\mathbb{A}^1\times U^{(0)}$. Therefore, by the universal property of blowing up, there is a unique morphism $f^1:\mathbb{A}^1\times U^{(0)}\to M^{(1)}$ such that $u^{0,1}\circ f^1=f$. It is easy to check that $f^1:\mathbb{A}^1\times U^{(1)}\to M^{(1)}$ is a $\mathbb{G}_m$-torsor where $\mathbb{G}_m$ acts on $\mathbb{A}^1\times U^{(1)}$ by $\mu\cdot(\lambda,L)=(\mu\cdot\lambda,\mu^{-1}\cdot L)$. In particular, f^1 is smooth and surjective. The preimage under f^1 of $E_0^{(1)}$ is $\{0\}\times U^{(1)}$. The next lemma proves that for $k=1,\dots,g-1$, the preimage under f^1 of $M_k^{(1)}$ is $\mathbb{A}^1\times\left(U^{(1)}\cap M_k^{(1)}\right)$.

Lemma 3.10. *Define $\mathcal{G}^{(1)}$ to be the locally free sheaf $(u^{0,1})^*\mathcal{G}(E_0^{(1)})$ and let $(u^{0,1})^*\mathcal{G}\to\mathcal{G}^{(1)}$ denote the canonical sheaf map.*

(i) *There is a factorization $\phi^{(1)}:\mathcal{G}^{(1)}\to(u^{0,1})^*\mathcal{F}$ of $(u^{0,1})^*\phi$.*
(ii) *The pullback of $\phi^{(1)}$ by f^1 is canonically isomorphic to the pullback $pr_2^*\phi$ of the restriction of ϕ to $U^{(0)}=U^{(1)}$.*
(iii) *For every geometric point x of $M^{(1)}$, rank$(\phi^{(1)}|_x)\geq 1$.*
(iv) *For each $k=1,\dots,g-1$, the inverse image ideal sheaf $(u^{0,1})^{-1}(\mathcal{I}_k^{(0)})$ equals the ideal sheaf $\mathcal{I}_k^{(1)}\cdot\mathcal{O}_{M^{(1)}}(-(k+1)E_0^{(1)})$.*
(v) *For each $k=1,\dots,g-1$, the preimage under $f^{(1)}$ of $M_k^{(1)}$ equals $\mathbb{A}^1\times(U^{(1)}\cap M_k^{(1)})$.*

Proof. **(i)**: The restriction of $(u^{0,1})^*\phi$ to $E_0^{(1)}$ is the zero map. Therefore it factors through the elementary-transform-up of $(u^{0,1})^*\mathcal{G}$, i.e. it factors through a morphism $\phi^{(1)} : \mathcal{G}^{(1)} \to (u^{0,1})^*\mathcal{F}$. This proves (i).

(iii): The morphism $(f^1)^*\mathcal{G}^{(1)}$ equals $\mathrm{pr}_2^*\mathcal{G}(0 \times U^{(1)})$. This is canonically isomorphic to $\mathrm{pr}_2^*\mathcal{G}$. Via this isomorphism, the pullback $(f^1)^*\phi^{(1)}$ equals $\mathrm{pr}_2^*\phi$. Since $\phi|_{U^{(0)}}$ has rank at least 1 at all geometric points, the same is true of $\mathrm{pr}_2^*\phi$. Therefore $\phi^{(1)}$ has rank at least 1 at all geometric points.

(iv) and (v): One can check that two ideal sheaves are equal after a faithfully flat base change. Since f^1 is faithfully flat, to prove both (iv) and (v), it suffices to prove that the inverse image ideal sheaf of $\mathcal{I}_k^{(0)}$ in $\mathbb{A}^1 \times U^{(1)}$ equals

$$\mathrm{pr}_2^{-1}(\mathcal{I}^{(0)}) \cdot \mathcal{O}_{\mathbb{A}^1 \times U^{(1)}}(-(k+1)\{0\} \times U^{(1)}). \tag{18}$$

Let t denote the coordinate on $\mathbb{A}^1$. Then the preimage under f^1 of $(u^{0,1})^*\phi$ is precisely the matrix $t \cdot \mathrm{pr}_2^*\phi$. Therefore the ideal sheaf generated by the $(k+1) \times (k+1)$-minors of this matrix is just t^{k+1} times the ideal generated by $(k+1) \times (k+1)$-minors of $\mathrm{pr}_2^*\phi$, i.e. the inverse image ideal sheaf of $\mathcal{I}_k^{(0)}$ under $u^{0,1} \circ f^1$ is as above. □

Lemma 3.11. *For $r = 1, \dots, g$ there exist morphisms of S-schemes*

$$f^r : \mathbb{A}^1 \times U^{(r)} \to M^{(r)}$$

satisfying the following conditions.

(i) *The image of the composite morphism $u^{0,1} \circ f^1$ is contained in $U^{(0)}$.*
(ii) *The morphism f^1 is the unique morphism such that $u^{1,0} \circ f^1 = f$.*
(iii) *For $1 \le r \le s \le g$, $u^{r,s} \circ f^s$ equals $f^r \circ (Id \times v^{r,s})$; moreover, the corresponding commutative diagram is Cartesian.*
(iv) *For each $1 \le r \le g$, the morphism f^r is a $\mathbb{G}_m$-torsor, in particular it is surjective and smooth.*
(v) *For each $1 \le r \le k \le g$,*
$$(f^r)^{-1}(M_k^{(r)}) = \mathbb{A}^1 \times (U^{(r)} \cap M_k^{(r)}).$$
(vi) *For each $2 \le r \le g$ and $1 \le k \le r-1$,*
$$(f^r)^{-1}(E_k^{(r)}) = \mathbb{A}^1 \times (U^{(r)} \cap E_k^{(r)}).$$
And for each $1 \le r \le g$, the preimage under f^r of $E_0^{(r)}$ is $\{0\} \times U^{(r)}$.

Proof. **(i), (ii), and** $r = 1$: Item (i) is trivial and is only included to maintain symmetry with Lemma 3.7. Item (ii) follows from the construction of $f^{(1)}$. The claim is that for each $1 \le r \le g$, there exists a sequence of morphisms $f^1, \dots, f^r$ satisfying (i)–(vi). The claim is proved by induction on r. For $r = 1$ this has already been established; in particular, (v) follows from (v) of Lemma 3.10.

(iii) and (iv); induction step: Now comes the induction step. Let $r = 2, \dots, g$. By way of induction, suppose that morphisms $f^1, \dots, f^{r-1}$ have been constructed satisfying (i)–(vi). Since f^{r-1} is smooth, the fiber product of f^{r-1} with the blowing up $u^{r-1,r}$ is canonically isomorphic to the blowing up of $\mathbb{A}^1 \times U^{(r-1)}$ along the preimage of $M_{r-1}^{(r-1)}$. By the induction hypothesis and (v), the preimage is precisely $\mathbb{A}^1 \times (U^{(r-1)} \cap M_{r-1}^{(r-1)})$. So the base-change of $u^{r-1,r}$ by f^{r-1} is just $\mathrm{Id} \times v^{r-1,r}$. Define $f^r : \mathbb{A}^1 \times U^{(r)} \to M^{(r)}$ to be the base-change of f^{r-1} by $u^{r-1,r}$.

By construction of f^r and the induction hypothesis, $f^1, \dots, f^r$ satisfies (iii). Since f^r is the base-change of the $\mathbb{G}_m$-torsor f^{r-1}, also f^r is a $\mathbb{G}_m$-torsor, i.e. (iv) holds.

(v); induction step: Using the Cartesian property of (iii) and since f^{r-1} and f^r are smooth, the process of forming the strict transform under $u^{r-1,r}$ and then forming the preimage under f^r is the same as the process of first forming the preimage under f^{r-1} and then forming the strict transform under $\mathrm{Id} \times v^{r-1,r}$. By the induction hypothesis and (v), the preimage under f^{r-1} of

$M_k^{(r-1)}$ equals $\mathbb{A}^1 \times (U^{(r-1)} \cap M_k^{(r-1)})$. The strict transform of this subscheme under $\mathrm{Id} \times v^{r-1,r}$ is $\mathbb{A}^1 \times (U^{(r)} \cap M_k^{(r)})$.

(vi); induction step: As above, for $k = 0, \ldots, r-2$ the pullback by f^r of the strict transform of $E_k^{(r-1)}$ equals the strict transform of the pullback by f^{r-1}. By the induction hypothesis and (vi), the pullback of $E_0^{(r-1)}$ equals $\{0\} \times U^{(r-1)}$ and the pullback of $E_k^{(r-1)}$ equals $\mathbb{A}^1 \times (U^{(r-1)} \cap E_k^{(r-1)})$ for $k = 1, \ldots, r-2$. The strict transforms of these subschemes are $\{0\} \times U^{(r)}$ and $\mathbb{A}^1 \times (U^{(r)} \cap E_k^{(r)})$ respectively. Finally, using (iii), the preimage under f^r of the exceptional divisor of $u^{r-1,r}$, i.e. of $E_{r-1}^{(r)}$, equals the exceptional divisor of $\mathrm{Id} \times v^{r-1,r}$, i.e. $\mathbb{A}^1 \times (U^{(r)} \cap E_{r-1}^{(r)})$. This proves (vi) and finishes the induction step. The lemma by induction on r. $\square$

The next proposition is the main result of this section.

Proposition 3.12. (i) *For each $r = 0, \ldots, g$, the scheme $M^{(r)}$ is smooth over S.*

(ii) *For each $r = 1, \ldots, g$, the closed subscheme $E_0^{(r)} \cup \cdots \cup E_{r-1}^{(r)}$ is a simple normal crossings divisor in $M^{(r)}$; moreover the intersection with every geometric fiber over S is a simple normal crossings divisor.*

(iii) *For each $r = 1, \ldots, g$, the scheme $M_r^{(r)}$ is smooth over S, and therefore $M_r^{(r)} \to M^{(r)}$ is a regular embedding.*

(iv) *For $r = 0, \ldots, g$ there exist locally free sheaves of rank g on $M^{(r)}$, $\mathcal{G}^{(r)}$, and morphisms of sheaves $\phi^{(r)} : \mathcal{G}^{(r)} \to (u^{0,r})^*\mathcal{F}$, such that $\mathcal{G}^{(0)} = \mathcal{G}$ and $\phi^{(0)} = \phi$, such that $\mathcal{G}^{(1)} = (u^{0,1})^*\mathcal{G}^{(0)}(E_0^{(1)})$ and $\phi^{(1)}$ is as in Lemma 3.10, and that satisfy the following condition: for each $r = 1, \ldots, g$ there is a factorization $(u^{r-1,r})^*\mathcal{G}^{(r-1)} \xrightarrow{\psi^{(r)}} \mathcal{G}^{(r)} \xrightarrow{\phi^{(r)}} (u^{0,r})^*\mathcal{F}$ such that the cokernel of $\psi^{(r)}$ is the pushforward from $E_{r-1}^{(r)}$ of a locally free sheaf of rank $g+1-r$ and such that $\phi^{(r)}$ has rank at least r at all geometric points, and has rank g at the generic point of $E_{r-1}^{(r)}$.*

(v) *The morphisms $\psi^{(r)}$ and $\phi^{(r)}$ above are unique up to unique isomorphism (considering the sheaves $\mathcal{G}^{(r)}$ as subsheaves of $\mathcal{G} \otimes_{\mathcal{O}_{M^{(0)}}} K(M^{(0)})$, then the morphisms are honestly unique). Moreover, they are equivariant for the obvious action of the group scheme $GL(\mathcal{F}) \times GL(\mathcal{G})$.*

(vi) *For each $0 \le r < s \le g$ and each $k \ge s$, the inverse image ideal sheaf $(u^{r,s})^*\mathcal{I}_k^{(r)}$ equals*

$$\mathcal{I}_k^{(s)} \cdot \mathcal{O}_{M^{(s)}}(-(\sum_{i=r}^{s-1}(k+1-i)E_i^{(s)})). \tag{19}$$

Proof. **g=0**: The proposition can be checked Zariski locally over S. And Zariski locally over S, the locally free sheaves $\mathcal{G}$ and $\mathcal{F}$ are free. Hence it suffices to consider the case that $S = \mathrm{Spec}\ (K)$. The result is proved by induction on g. For $g = 0$, there is nothing to prove. Therefore, by way of induction, we may suppose that $g > 0$ and the result has been proved whenever $\mathrm{rank}(\mathcal{G}) < g$.

The idea of the induction step is described in the next 2 paragraphs. First of all, for $M^{(0)}$ (i)–(vi) are obvious. Thus it suffices to consider $M^{(r)}$ with $r = 1, \ldots, g$. The next step is to restrict over $U^{(1)}, \ldots, U^{(g)}$ and check (i)–(vi) through when restricted over these open sets. This can be checked after a smooth surjective base-change on the sets $U^{(r)}$. Lemma 3.7 gives a sequence of such base-changes $\tau^{r-1}(\zeta)$ for any rank 1 morphism $\zeta : (S, \mathcal{G}', \mathcal{F}') \to (S, \mathcal{G}, \mathcal{F})$. After base-change by $\tau^{r-1}(\zeta)$, (i)–(vi) over $U^{(r)}$ reduce to (i)–(vi) over $M^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$. Since $\mathrm{rank}(\mathcal{G}') < \mathrm{rank}(\mathcal{G})$, (i)–(vi) over $M^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$ hold by the induction assumption.

Next, to establish the proposition over all of $M^{(r)}$, consider the sequence of morphisms $f^r : \mathbb{A}^1 \times U^{(r)} \to M^{(r)}$ from Lemma 3.11. These morphisms are smooth and surjective, so (i)–(vi) may be checked after base-change by f^r. And these reduce to (i)–(vi) over $U^{(r)}$. Since (i)–(vi) have been proved over $U^{(r)}$, this finishes the induction step. The proof follows by induction.

(i); $U^{(r)}$: First (i)–(vi) are proved when restricted over $U^{(1)},\dots,U^{(g)}$. Let $T\in U^{(0)}$ be a geometric point with $\mathrm{rank}(T)=1$. There exists $\mathcal{G}'$ of rank $g-1$, $\mathcal{F}'$ of rank $f-1$, $p:\mathcal{G}\to\mathcal{G}'$, and $q:\mathcal{F}'\to\mathcal{F}$ so that $\zeta=(p,q,T)$ is a morphism $(\mathcal{G}',\mathcal{F}')\to(\mathcal{G},\mathcal{F})$. By Lemma 3.7, for each $r=0,\dots,g-1$, there exists a quasi-compact, separated, smooth morphism $\tau^r(\zeta):\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M^{(r)}(S,\mathcal{G}',\mathcal{F}')\to M^{(r+1)}(S,\mathcal{G},\mathcal{F})$ whose image is $U^{(r+1)}$.

Since $\tau^{r-1}(\zeta):\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M^{(r-1)}(S,\mathcal{G}',\mathcal{F}')\to U^{(r)}$ is smooth and surjective, to prove that $U^{(r)}$ is smooth over S it suffices to prove that $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$ is smooth over S. By the induction hypothesis, $M^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$ is smooth over S. And $\mathrm{GL}(\mathcal{F})$ and $\mathrm{GL}(\mathcal{G})$ are obviously smooth over S. Therefore the fiber product is smooth over S. This proves (i) over $U^{(r)}$.

(ii); $U^{(r)}$: Similarly, to prove that $U^{(r)}\cap\left(E_0^{(r)}\cup\cdots\cup E_{r-1}^{(r)}\right)$ is a simple normal crossings divisor, it suffices to prove that the pullback under $\tau^{r-1}(\zeta)$ is a simple normal crossings divisor. By (vi) of Lemma 3.7, this pullback is of the form $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S E$ where $E=E_0^{(r-1)}\cup\dots E_{r-2}^{(r-1)}$ (the "missing" divisor is due to the fact that the pullback of $E_0^{(r)}$ is the empty set). By the induction hypothesis, E is a simple normal crossings divisor in $M^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$. Therefore $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S E$ is a simple normal crossings divisor in $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$. Hence $U^{(r)}\cap\left(E_0^{(r)}\cup\cdots\cup E_{r-1}^{(r)}\right)$ is a simple normal crossings divisor in $U^{(r)}$. This proves (ii) over $U^{(r)}$.

(iii); $U^{(r)}$: To prove that $U^{(r)}\cap M_r^{(r)}$ is smooth over S, it suffices to prove that the pullback under $\tau^{r-1}(\zeta)$ is smooth over S. By (v) of Lemma 3.7, the pullback is $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M_{r-1}^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$. By the induction hypothesis $M_{r-1}^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$ is smooth over S, so also $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M_{r-1}^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$ is smooth over S. It follows that $U^{(r)}\cap M_r^{(r)}$ is smooth over S. This establishes (iii) over $U^{(r)}$.

(v); $U^{(r)}$: Item (iv) is quite a bit more involved. By the induction hypothesis, the maps $(\psi')^{(r)}:(u^{r-1,r})^*(\mathcal{G}')^{(r-1)}\to(\mathcal{G}')^{(r)}$ and the maps $(\phi')^{(r)}:(\mathcal{G}')^{(r)}\to(u^{0,r})^*\mathcal{F}'$ on $M^{(r)}(S,\mathcal{G}',\mathcal{F}')$ are all defined and satisfy the conditions in (iv) and (v). First (v) is proved on $U^{(r)}$. Observe that if the sequence of maps $\psi^{(s)},\phi^{(s)}$ exists for $s=0,\dots,r-1$, then there is *at most one pair* $\psi^{(r)},\phi^{(r)}$ satisfying the hypotheses. This is because the restriction of $(u^{r-1,r})^*\phi^{(r-1)}$ to $E_{r-1}^{(r)}$ has rank at least $r-1$. So the kernel has rank at least $g+1-r$. If there exists a pair $\psi^{(r)},\phi^{(r)}$, then the restriction of $(u^{r-1,r})^*\phi^{(r-1)}$ to $E_{r-1}^{(r)}$ has constant rank $r-1$ (i.e. the cokernel is locally free of rank $g+1-r$) and $\psi^{(r)}:(u^{r-1,r})^*\mathcal{G}^{(r-1)}\to\mathcal{G}^{(r)}$ must be the elementary-transform-up along $E_{r-1}^{(r)}$ whose kernel equals the kernel of $(u^{r-1,r})^*\phi^{(r-1)}$. And then $\phi^{(r)}$ is the unique morphism through which $(u^{r-1,r})^*\phi^{(r-1)}$ factors. This proves (v) (in fact, without restricting over $U^{(r)}$). Equivariance with respect to $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})$ follows by induction on r and the uniqueness just mentioned.

(iv); $U^{(r)}$: Next the existence of $\psi^{(r)},\phi^{(r)}$ is proved when restricted over $U^{(r)}$. This is proved by faithfully flat (in fact smooth) descent, i.e. a descent datum is constructed for the faithfully flat cover $\tau^{r-1}(\zeta):\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})\times_S M^{(r-1)}(S,\mathcal{G}',\mathcal{F}')\to U^{(r)}$. The uniqueness in (iv) and equivariance with respect to $\mathrm{GL}(\mathcal{F})\times_S\mathrm{GL}(\mathcal{G})$ will give the cocycle condition.

For each $r=1,\dots,g$ define $\mathcal{G}_{\mathrm{pre}}^{(r)}$ on $M^{(r-1)}(S,\mathcal{G}',\mathcal{F}')$ to be the direct sum of $(\mathcal{G}')^{(r-1)}$ and $\mathrm{Ker}(p)\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(r-1)}}$. In particular, $\mathcal{G}^{(1)}$ is simply $\mathcal{G}\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(0)}}$. For each $r=2,\dots,g$ define $\psi_{\mathrm{pre}}^{(r)}:(u^{r-2,r-1})^*\mathcal{G}_{\mathrm{pre}}^{(r-1)}\to\mathcal{G}_{\mathrm{pre}}^{(r)}$ to be the direct sum of $(\psi')^{(r-1)}:(u^{r-2,r-1})^*(\mathcal{G}')^{(r-2)}\to(\mathcal{G}')^{(r-1)}$ and the identity map on $\mathrm{Ker}(p)\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(r-1)}}$. For each $r=1,\dots,g$ define $\phi_{\mathrm{pre}}^{(r)}:\mathcal{G}_{\mathrm{pre}}^{(r)}\to\mathcal{F}\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(r-1)}}$ to be the sum of the map

$$q\circ(\phi')^{(r-1)}:(\mathcal{G}')^{(r-1)}\to\mathcal{F}'\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(r-1)}}\to\mathcal{F}\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(r-1)}}\tag{20}$$

with the map $T:\mathrm{Ker}(p)\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(r-1)}}\to\mathcal{F}\otimes_{\mathcal{O}_S}\mathcal{O}_{M^{(r-1)}}$.

On $\mathrm{GL}(\mathcal{F})$ there is a universal automorphism $\alpha:\mathcal{F}\otimes_{\mathcal{O}_S}\mathcal{O}\to\mathcal{F}\otimes_{\mathcal{O}_S}\mathcal{O}$ and on $\mathrm{GL}(\mathcal{F})$ there is a universal automorphism $\beta:\mathcal{G}\otimes_{\mathcal{O}_S}\mathcal{O}\to\mathcal{G}\otimes_{\mathcal{O}_S}\mathcal{O}$. By slight abuse of notation, also denote by α and

β the pullbacks of these automorphisms to $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$. By definition, the pullback by $\tau^{r-1}(\zeta)$ of $(u^{0,r})^*\phi$ equals $\alpha \circ (u^{0,r-1})^* \mathrm{pr}_3^* \phi_{\mathrm{pre}}^{(1)} \circ \beta^{-1}$. Next the part of the descent datum on $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$ defining $\mathcal{G}^{(r)}$, $\psi^{(r)}$ and $\phi^{(r)}$ is specified. For each $r = 1, \dots, g$ define $\tau^r(\zeta)^* \mathcal{G}^{(r)}$ to be $\mathrm{pr}_3^* \mathcal{G}_{\mathrm{pre}}^{(r)}$. Define $\tau^0(\zeta)^* \psi^{(1)}$ to be β^{-1}, where the domain of β^{-1} is identified with $\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O} \cong \tau^0(\zeta)^*(u^{0,1})^* \mathcal{G}^{(0)}$ and where the range of β^{-1} is identified with $\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O} \cong \mathrm{pr}_3^* \mathcal{G}_{\mathrm{pre}}^{(1)}$. For $r = 2, \dots, g$, define $\tau^{r-1}(\zeta)^* \psi^{(r)}$ to be $\mathrm{pr}_3^* \psi_{\mathrm{pre}}^{(r)}$. For all $r = 1, \dots, g$, define $\tau^{r-1}(\zeta)^* \phi^{(r)}$ to be $\alpha \circ \mathrm{pr}_3^* \phi_{\mathrm{pre}}^{(r)}$.

Now to finish specifying the descent data, patching morphisms on the fiber product of $\tau^{r-1}(\zeta)$ with itself are needed. There are canonical descent data associated to the sheaves $\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(r)}}$ and $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(r)}}$ on $M^{(r)}(S, \mathcal{G}, \mathcal{F})$. And, up to unique isomorphism, there is *at most one way* of extending the descent data for $\mathcal{G}^{(r)}$, $\psi^{(r)}$ and $\phi^{(r)}$ so that the descent data giving $\psi^{(r)}$ and $\phi^{(r)}$ are morphisms from the descent datum for $\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(r)}}$ to the descent datum for $\mathcal{G}^{(r)}$ and from the descent datum for $\mathcal{G}^{(r)}$ to the descent datum for $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(r)}}$ respectively. This doesn't prove that such descent data exists! Proving existence is an exercise in the compatibilities of all the sheaves and morphisms defined so far, and is left to the reader. The key point, as always, is that on $M^{(r)}(S, \mathcal{G}, \mathcal{F})$, on the base-change by $\tau^{r-1}(\zeta)$, and on the double base-change by $\tau^{r-1}(\zeta)$, the morphism $\psi^{(r)}$ is, up to unique isomorphism, the elementary-transform-up determined by the kernel of $(u^{r-1,r})^* \phi^{(r-1)}$ restricted to $E_{r-1}^{(r)}$.

The upshot is that the sheaves $\mathcal{G}^{(r)}$ and sheaf maps $\psi^{(r)}$, $\phi^{(r)}$ exist when restricted over $U^{(r)}$. Checking the properties in (iv), i.e. that the cokernel of $\psi^{(r)}$ is as specified and that the rank of $\phi^{(r)}$ is as specified, can be done after base-changing by $\tau^{r-1}(\zeta)$. And then it follows from the construction of the descent datum, and by the induction hypothesis applied to $M^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$. Again, the details are left to the interested reader.

(vi); $U^{(r)}$: Next (vi) is proved when restricted over $U^{(r)}$, or rather over $U^{(s)}$. The map $v^{1,0} : U^{(1)} \to U^{(0)}$ is an isomorphism, so it suffices to consider the cases $1 \le r < s \le g$. To check that two ideal sheaves are equal, it suffices to check after faithfully flat base-change. So it suffices to check after base-change by $\tau^{s-1}(\zeta)$. By (iii) of Lemma 3.7, the inverse image under $\tau^{s-1}(\zeta)$ of the inverse image under $u^{r,s}$ equals the inverse image under $\mathrm{Id} \times \mathrm{Id} \times u^{r-1,s-1}$ of $\tau^{r-1}(\zeta)$. By (v) of Lemma 3.7, the inverse image under $\tau^{r-1}(\zeta)$ of $\mathcal{I}_k^{(r)}$ equals the ideal sheaf of the closed subscheme $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M_{k-1}^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$ of $\mathrm{GL}(\mathcal{F}) \times_S \mathrm{GL}(\mathcal{G}) \times_S M^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$. By the induction hypothesis and (v), the inverse image of $\mathcal{I}_{k-1}^{(r-1)}(S, \mathcal{G}', \mathcal{F}')$ under $u^{r-1,s-1}$ is the product of $\mathcal{I}_{k-1}^{(s-1)}(S, \mathcal{G}', \mathcal{F}')$ with the invertible ideal sheaf associated to the Cartier divisor $\sum_{j=r-1}^{s-2}((k-1)+1-j)E_j^{(s-1)}$. Making the substitution $i = j+1$, the Cartier divisor is $\sum_{i=r}^{s-1}(k+1-i)E_{i-1}^{(s-1)}$. Taking the inverse image of this ideal sheaf under pr_3, and using (v) and (vi) of Lemma 3.7, this gives the same ideal sheaf as the inverse image under $\tau^{s-1}(\zeta)$ of the ideal sheaf in (vi) above. This establishes (vi) over $U^{(r)}$. This finishes the proof of the proposition "over $U^{(r)}$".

(i), $M^{(r)}$: To finish the induction step, (i)–(vi) have to be proved over all of $M^{(r)}(S, \mathcal{G}, \mathcal{F})$. This is done using the morphisms $f^r : \mathbb{A}^1 \times U^{(r)} \to M^{(r)}$ from Lemma 3.11. By (iv) of Lemma 3.11, the morphism f^r is smooth and surjective, so to check the target of f^r is smooth over S, it suffices to check the domain of f^r is smooth over S. As established above, $U^{(r)}$ is smooth over S so that $\mathbb{A}^1 \times U^{(r)}$ is smooth over S. Therefore $M^{(r)}$ is smooth over S. This proves (i) over all of $M^{(r)}$.

(ii), $M^{(r)}$: Similarly, to show that $E_0^{(r)} \cup \dots \cup E_{r-1}^{(r)}$ is a simple normal crossings divisor, it suffices to prove that the pullback by f^r is a simple normal crossings divisor. By (vi) of Lemma 3.11, the preimage of $E_0^{(r)}$ is $\{0\} \times U^{(r)}$ and the preimage of $E_1^{(r)} \cup \dots \cup E_{r-1}^{(r)}$ is $\mathbb{A}^1 \times \left(U^{(r)} \cap \left(E_1^{(r)} \cup \dots \cup E_{r-1}^{(r)}\right)\right)$. As proved above, $U^{(r)} \cap \left(E_1^{(r)} \cup \dots \cup E_{r-1}^{(r)}\right)$ is a simple normal crossings divisor in $U^{(r)}$. So by Lemma 3.8, the divisor $(f^r)^{-1}\left(E_0^{(r)} \cup \dots \cup E_{r-1}^{(r)}\right)$ is a simple normal crossings divisor. This proves (ii) over all of $M^{(r)}$.

(iii), $M^{(r)}$: To show that $M_r^{(r)}$ is smooth over S, it suffices to show that the preimage under f^r is smooth over S. By (v) of Lemma 3.11, the preimage of $M_r^{(r)}$ is $\mathbb{A}^1 \times \left(U^{(r)} \cap M_r^{(r)}\right)$. As proved above, $U^{(r)} \cap M_r^{(r)}$ is smooth. So $\mathbb{A}^1 \times \left(U^{(r)} \cap M_r^{(r)}\right)$ is smooth, and therefore $M_r^{(r)}$ is smooth. This proves (iii) over all of $M^{(r)}$.

(iv) and (v); $M^{(r)}$: As over $U^{(r)}$, (iv) and (v) are a bit more involved (although most of the work is already done). As before, Item (5) is automatic, once the existence of $\mathcal{G}^{(r)}$, $\psi^{(r)}$ and $\phi^{(r)}$ satisfying (iv) is proved. Existence is proved by faithfully flat descent with respect to the faithfully flat (in fact smooth) morphism $f^r : \mathbb{A}^1 \times U^{(r)} \to M^{(r)}$. Next the part of the descent data on $\mathbb{A}^1 \times U^{(r)}$ defining $\mathcal{G}^{(r)}$, $\psi^{(r)}$ and $\phi^{(r)}$ is specified. For each $r = 1, \dots, g$, define $(f^r)^*\mathcal{G}^{(r)}$ to be $\mathrm{pr}_2^*(\mathcal{G}^{(r)}|_{U^{(r)}}) \otimes \mathcal{O}_{\mathbb{A}^1 \times U^{(r)}}(\{0\} \times U^{(r)})$ (of course this definition looks circular, but recall that $\mathcal{G}^{(r)}|_{U^{(r)}}$, $\psi^{(r)}|_{U^{(r)}}$ and $\phi^{(r)}|_{U^{(r)}}$ were already constructed above). Let t be the coordinate on $\mathbb{A}^1$ considered as a global section of the invertible sheaf $\mathcal{O}_{\mathbb{A}^1 \times U^{(r)}}(\{0\} \times U^{(r)})$ whose vanishing locus is precisely $\{0\} \times U^{(r)}$ (this conflicts with the usual terminology that calls this section "1"). The point is that there is a canonical everywhere nonzero global section of this invertible sheaf (which in the usual terminology is denoted by "$\frac{1}{t}$", but here is denoted by "1"), and with respect to this trivialization the regular function t corresponds to a section whose vanishing locus is $\{0\} \times U^{(r)}$.

Define $(f^1)^*(\psi^{(1)})$ to be the map

$$\mathrm{Id} \otimes t : \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathbb{A}^1 \times U^{(1)}} \to \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathbb{A}^1 \times U^{(1)}}(\{0\} \times U^{(1)}). \tag{21}$$

Of course the domain of this map is identified with the pullback by f^1 of $\mathcal{G}^{(0)} = \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(1)}}$ and the target is identified with $(f^1)^*\mathcal{G}^{(1)}$ defined above. For $r = 2, \dots, g$, define $(f^r)^*(\psi^{(r)})$ to be the map $\mathrm{pr}_2^*(\psi^{(r)}|_{U^{(r)}}) \otimes \mathrm{Id}$. Define $(f^1)^*(\phi^{(1)})$ to be the composition of the canonical isomorphism $\mathrm{Id} \otimes 1 : \mathrm{pr}_2^*\mathcal{G}^{(r)} \otimes \mathcal{O}_{\mathbb{A}^1 \times U^{(r)}}(\{0\} \times U^{(r)}) \to \mathrm{pr}_2^*\mathcal{G}^{(r)}$ with $\mathrm{pr}_2^*(\phi^{(r)}|_{U^{(r)}})$. Observe that $(f^1)^*\phi^{(1)}$ is the same map constructed in Lemma 3.10.

Next it is proved these definitions of $(f^r)^*\psi^{(r)}$ and $(f^r)^*\phi^{(r)}$ have the properties from (iv). For $r = 1$, this is precisely Lemma 3.10. Suppose that $r \geq 2$. As established above, $(\phi^{(r)}|_{U^{(r)}}) \circ (\psi^{(r)}|_{U^{(r)}})$ equals $(v^{r-1,r})^*(\phi^{(r-1)}|_{U^{(r-1)}})$. Pulling back by pr_2^* and using (iii) from Lemma 3.11, $(f^r)^*\phi^{(r)} \circ (f^r)^*\psi^{(r)}$ equals $(\mathrm{Id} \times v^{r-1,r})^*(f^{r-1})^*\phi^{(r-1)} = (f^r)^*(u^{r-1,r})^*\phi^{(r-1)}$. This is the first necessary property. As established above, the cokernel of $\psi^{(r)}|_{U^{(r)}}$ is the push forward of a locally free sheaf of rank $g+1-r$ from the divisor $U^{(r)} \cap E_{r-1}^{(r)}$ in $U^{(r)}$. Therefore the cokernel of $(f^r)^*\psi^{(r)}$, i.e. the cokernel of $\mathrm{pr}_2^*(\psi^{(r)}|_{U^{(r)}})$, is the push forward of a locally free sheaf of rank $g+1-r$ from the divisor $\mathbb{A}^1 \times (U^{(r)} \cap E_{r-1}^{(r)})$, i.e. the divisor $(f^r)^{-1}(E_{r-1}^{(r)})$. This is the second necessary property. And as established above, $\phi^{(r)}|_{U^{(r)}}$ has rank at least r at all geometric points. Therefore $(f^r)^*\phi^{(r)}$, i.e. $\mathrm{pr}_2^*(\phi^{(r)}|_{U^{(r)}})$, has rank at least r at all geometric points. This is the last necessary property.

To finish the proof of (iv), the rest of the descent data must be specified, i.e. the patching morphisms on the fiber product of f^r with itself are needed. As over $U^{(r)}$, there is at most one way of completing the descent data so that $\psi^{(r)}$ and $\phi^{(r)}$ give morphisms with the given descent data for $\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(r)}}$ and $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(r)}}$. Proving that one can complete the descent data is an exercise left to the reader. The key point (as always!) is that $\psi^{(r)}$ is the elementary-transform-up determined by the kernel of $(u^{r-1,r})^*\phi^{(r-1)}$ restricted to $E_{r-1}^{(r)}$. This proves (iv) and (v) on all of $M^{(r)}$.

(vi); $M^{(r)}$: First suppose that $r = 0$. By (iv) of Lemma 3.10, $(u^{0,1})^{-1}(\mathcal{I}_k^{(0)})$ equals $\mathcal{I}_k^{(1)} \cdot \mathcal{O}_{M^{(1)}}(-(k+1)E_0^{(1)})$. By (v) of Lemma 3.11, $(f^1)^{-1}(\mathcal{I}_k^{(1)})$ is the ideal sheaf $\mathrm{pr}_2^{-1}(\mathcal{I}_k^{(1)}|_{U^{(1)}})$. By (vi) of Lemma 3.11, $(f^1)^{-1}\mathcal{O}_{M^{(1)}}(-E_0^{(1)})$ is the ideal sheaf of $\{0\} \times U^{(1)}$. The process of forming the preimage of an ideal sheaf by pr_2 and then forming the preimage of that ideal sheaf by $\mathrm{Id} \times v^{1,s}$ is the same as the process of first forming the preimage of the ideal sheaf by $v^{1,s}$ and then forming the preimage by pr_2. As established above, $(v^{1,s})^{-1}(\mathcal{I}_k^{(1)}|_{U^{(1)}})$ equals the restriction to $U^{(s)}$ of the ideal sheaf

$$\mathcal{I}_k^{(s)} \cdot \mathcal{O}_{U^{(s)}}(-(\sum_{i=1}^{s-1}(k+1-i)E_i^{(s)})). \tag{22}$$

And the inverse image under $\mathrm{Id}\times v^{1,s}$ of the ideal sheaf of $\{0\}\times U^{(1)}$ is the ideal sheaf of $\{0\}\times U^{(s)}$. Putting the pieces together, the inverse image under $\mathrm{Id}\times v^{1,s}$ of the inverse image under f^1 of the inverse image under $u^{0,1}$ of $\mathcal{I}_k^{(0)}$ equals the ideal sheaf

$$\mathrm{pr}_2^{-1}(\mathcal{I}_k^{(s)}|_{U^{(s)}})\cdot\mathcal{O}_{\mathbb{A}^1\times U^{(s)}}\left(-(k+1)\{0\}\times U^{(s)}\right)\otimes\mathcal{O}_{\mathbb{A}^1\times U^{(s)}}\left(-\left(\sum_{i=1}^{s-1}(k+1-i)\mathbb{A}^1\times\left(U^{(s)}\cap E_i^{(s)}\right)\right)\right). \tag{23}$$

By (v) and (vi) of Lemma 3.11, this is precisely the inverse image under f^s of the ideal sheaf from (vi). Using (iii) of Lemma 3.11 one last time, and using that one can check equality of ideal sheaves after faithfully flat base-change, (vi) holds when $r=1$.

Checking (vi) for $r>1$ is even easier and follows by the same sort of argument as above; the details are left to the reader. This proves that (vi) holds over all of $M^{(r)}$, and thus finishes the proof that the proposition holds over all of $M^{(r)}$. The proposition is proved by induction on the rank of $\mathcal{G}$. □

3.2. **Computation of the log discrepancies.** In this section the log resolution from the last section is applied to compute the log discrepancies of the pair $(M^{(0)}, M_k^{(0)})$. In the application, it is also necessary to compute the log discrepancies of the pair $(M^{(0)}, q\cdot M_k^{(0)})$. Then, combined with the results of Section 2, these computations are used to find the log discrepancies of some projective cones.

Lemma 3.13. *Let $(S,\mathcal{G},\mathcal{F})$ be a datum with rank$(\mathcal{G})=g$ and rank$(\mathcal{F})=f$. For each $0\le r<s\le g$, the relative canonical divisor of $u^{r,s}: M^{(s)}\to M^{(r)}$ equals*

$$K_{M^{(s)}}-(u^{r,s})^*K_{M^{(r)}}=\sum_{i=r}^{s-1}((f-i)(g-i)-1)\,E_i^{(s)}. \tag{24}$$

For $r<s\le k<g$ and each positive integer q, the inverse image under $u^{r,s}$ of the ideal sheaf $\left(\mathcal{I}_k^{(r)}\right)^q$ equals

$$\left(\mathcal{I}_k^{(s)}\right)^q\cdot\mathcal{O}_{M^{(s)}}\left(-q\left(\sum_{i=r}^{s-1}(k+1-i)E_i^{(s)}\right)\right). \tag{25}$$

and the associated cycle is

$$q\left[M_k^{(s)}\right]+\sum_{i=r}^{s-1}q(k+1-i)E_i^{(s)} \tag{26}$$

Finally, for $k<s\le g$ and each positive integer q, the inverse image under $u^{r,s}$ of the ideal sheaf $\left(\mathcal{I}_k^{(r)}\right)^q$ is an invertible ideal sheaf defining the Cartier divisor

$$\sum_{i=r}^{k}q(k+1-i)E_i^{(s)} \tag{27}$$

Proof. By Proposition 3.12, $M_r^{(r)}\subset M^{(r)}$ is a regular embedding. And by [1, Prop., p. 67], the codimension of $M_r^{(r)}$, which equals the codimension of $M_r^{(0)}$ in $M^{(0)}$, equals $(f-r)(g-r)$. Therefore the relative canonical divisor of the blowing up $u^{r,r+1}: M^{(r+1)}\to M^{(r)}$ is $(f-r)(g-r)E_r^{(r+1)}$. The first formula follows since the relative canonical divisor of a composition of birational morphisms is the sum of the relative canonical divisors of the separate morphisms.

The second formula follows from (vi) of Proposition 3.12. The final formula follows from the second formula and that fact that $(u^{r,r-1})^{-1}\mathcal{I}_{r-1}^{(r-1)}$ equals the invertible ideal sheaf $\mathcal{O}_{M^{(r)}}(-E_{r-1}^{(r)})$. □

Corollary 3.14. *Let $(S,\mathcal{G},\mathcal{F})$ be a datum with rank$(\mathcal{G}) = g$ and rank$(\mathcal{F}) = f$. Suppose that S is smooth, that $g \geq 1$, that $f \geq 2$ and that $0 \leq k \leq g-1$. Consider the pair $(M^{(r)}(S,\mathcal{G},\mathcal{F}), q \cdot M_k^{(r)}(S,\mathcal{G},\mathcal{F}))$ where $q \geq 0$. For $i = r,\dots,k$, the log discrepancies are*

$$a(E_i^{(g)}; M^{(r)}, q \cdot M_k^{(r)}) = (f-i)(g-i) - q(k+1-i). \tag{28}$$

Define $a = min\{(f-i)(g-i) - q(k+1-i) | i = r,\dots,k\}$. Then $(M^{(r)}, q \cdot M_k^{(r)})$ is log canonical iff $a \geq 0$, in which case the minimal log discrepancy, $mld(M^{(r)}; M^{(r)}, q \cdot M_k^{(r)})$, equals $min(1,a)$. In particular, if $q \leq f-g+1$, then the pair $(M^{(r)}, q \cdot M_{g-1}^{(r)})$ is log canonical and the minimal log discrepancy equals $min(1, f-g+1-q)$.

Proof. The corollary follows from the computations in Lemma 3.13 and the definition of the log discrepancies and minimal log discrepancies of a pair (c.f. [7, Defn. 1.1]). □

Let $(S,\mathcal{G},\mathcal{F})$ be a datum with rank$(\mathcal{G}) = g$ and rank$(\mathcal{F}) = f$. Let $\mathcal{A}$ and $\mathcal{A}'$ be locally free sheaves on S with rank$(\mathcal{A}) = a$ and rank$(\mathcal{A}') = a'$ with $a > 0$. Define $\mathcal{E}$ to be the cokernel of the following sheaf map on $M^{(0)}$,

$$\mathrm{Id} \otimes \phi : \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}} \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}} \oplus \mathcal{A}' \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}}. \tag{29}$$

The sheaf map is zero on the summand $\mathcal{A}' \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}}$, so this locally free sheaf will be a direct summand of $\mathcal{E}$. Denote by the pair $(\pi : C \to M^{(0)}, \alpha : \pi^*\mathcal{E} \to \mathcal{Q})$ the relative Grassmannian cone parametrizing rank r locally free quotients of $\mathcal{E}$.

Proposition 3.15. *If $a \cdot r \leq f - g$ and if S is smooth and geometrically connected, then C has pure dimension equal to the expected dimension $d = dim(C) = dim(S) + f \cdot g + r\left((a(f-g) + a') - r\right)$ and C is a normal, integral, local complete intersection scheme that has, at worst, canonical singularities.*

Proof. Denote $g' = a \cdot g = \mathrm{rank}(\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{G})$. For each $k = 0,\dots,g-1$ and $l = 0,\dots,a-1$, $B_{a\cdot k+l} = B_{a \cdot k} = M_k^{(0)}$. And this has codimension $(f-k)(g-k) = (f-g)(g-k) + (g-k)^2$. Also $r(g' - (a\cdot k + l)) + 1 = a \cdot r(g-k) + 1 - rl$. By assumption, $f - g \geq a \cdot r$, and for $k \leq g-1$, $(g-k)^2 \geq 1$. Therefore $(f-g)(g-k) + (g-k)^2 \geq a \cdot r(g-k) + 1 - rl$ for all $k = 0,\dots,g-1$ and $l = 0,\dots,a-1$. Therefore, by Lemma 2.3, C is irreducible of the expected dimension. And by the proof of Lemma 2.6, C is a local complete intersection scheme. In particular it is Cohen-Macaulay. Moreover, $\pi : C \to M^{(0)}$ is smooth over $M^{(0)} - M_{g-1}^{(0)}$. So C is generically reduced. Since C is Cohen-Macaulay, it follows that C is everywhere reduced. A reduced Cohen-Macauly scheme satisfies Serre's condition $S2$ for normality. Thus to prove that C is normal, it suffices to prove that C is regular in codimension 1.

If $a \cdot r < f - g$, then the same parameter count as above shows that for all $k = 0,\dots,g-1$ and $l = 0,\dots,a-1$ we have that $(f-k)(g-k) \geq r(g' - (a \cdot k + l)) + 2$. By Lemma 2.3 it follows that C is regular in codimension 1 so that C is normal. Therefore assume that $a \cdot r = f - g$. Of course C is regular on the dense open subset $\pi^{-1}(M^{(0)} - M_{g-1}^{(0)})$. The only codimension one point not contained in this locus is the generic point of $\pi^{-1}(M_{g-1}^{(0)} - M_{g-2}^{(0)})$.

As in the proof of Lemma 2.3, denote by $(\rho : C' \to M^{(0)}, \beta : \rho^*\mathcal{F} \to \mathcal{Q}')$ the Grassmannian bundle parametrizing rank r locally free quotients of $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}} \oplus \mathcal{A}' \otimes_{\mathcal{O}_S} \mathcal{O}_{M^{(0)}}$. There is a natural closed immersion $h : C \to C'$ compatible with projection to $M^{(0)}$.

Observe that $C' = P \times_S M^{(0)}$ where $\sigma : P \to S$ is the Grassmannian bundle parametrizing rank r locally free quotients of $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F} \oplus \mathcal{A}'$. Therefore there is a projection $\mathrm{pr}_2 : C \to P$ compatible with projection to S. The question is local, so it suffices to base-change to an open subset of S over which $\mathcal{A}$ is trivial. Choose an ordered basis for $\mathcal{A}$ so that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}$ is just $\mathcal{F}^{\oplus a}$. Let $W \subset P$ denote the dense open set over which the sheaf map $(\pi')^*(\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}) \to \mathcal{Q}'$ is surjective. On W there is a smooth, surjective morphism to the Grassmannian P' parametrizing rank r locally free quotients of $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}$. Any rank r quotient space of $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}$, i.e. of $\mathcal{F}^{\oplus a}$, is represented as the image of a matrix $\mathcal{F}^{\oplus a} \to \mathcal{O}^{\oplus r}$ of the form

$$M = \begin{bmatrix} v_{1,1} & v_{1,2} & \dots & v_{1,a} \\ \vdots & \vdots & \ddots & \vdots \\ v_{r,1} & v_{r,2} & \dots & v_{r,a} \end{bmatrix} \tag{30}$$

where the $v_{i,j}$ are sections of $\mathcal{F}^\vee$. This matrix determines a point x' in P'. Let $x \in W$ be any point mapping to x'. Thinking of the fiber of $\mathrm{pr}_2 : C \to P'$ over x as a subscheme of $M^{(0)}$, it is just the subscheme of matrices $L : \mathcal{G} \to \mathcal{F}$ such that $M \circ L^{\oplus a}$ is the zero matrix, i.e. it is the set of matrices L such that the kernel of the transpose matrix $L^\dagger$ contains the subspace

$$K = \mathrm{span}\{v_{i,j} | 1 \leq i \leq r, 1 \leq j \leq a\}. \tag{31}$$

This is the space of matrices $L^\dagger$ from $\mathcal{F}^\vee/K$ to $\mathcal{G}^\vee$. Let $V' \subset P'$ be the dense open subset parametrizing quotients where $\dim(K) = a \cdot r$ and let $V \subset W$ be the preimage of V. Over V, $\mathrm{pr}_2 : C \to P$ is a vector bundle of rank $(f - a \cdot r)g$. In particular, the preimage is a nonempty, smooth scheme, i.e. $\mathrm{pr}_2^{-1}(V)$ is contained in the smooth locus C_{smooth}. But of course, the map $L^\dagger$ may still have any rank between 0 and g (recall that $f - a \cdot r \geq g$, so that the dimension of $\mathcal{F}^\vee/K$ is greater than the dimension of $\mathcal{G}^\vee$). Therefore this open set intersects the preimage of every strata $M^{(0)}_k - M^{(0)}_{k-1}$. Combined with an obvious homogeneity argument, the open set intersects every fiber of π. So the smooth locus C_{smooth} intersects every fiber of π and C is regular in codimension 1 points. Therefore, in every case, C is normal.

The $(a \cdot (f - g) + a')^{\mathrm{th}}$ Fitting ideal $\mathcal{J}$ of $\mathrm{Id} \otimes \phi$, i.e. the ideal generated by the maximal minors of the matrix of $\mathrm{Id} \otimes \phi$, is easily seen to be $\mathcal{I}^a$ where $\mathcal{I}$ is the $(f-g)^{\mathrm{th}}$ Fitting ideal of $\mathcal{I}$, i.e. the ideal sheaf of $M^{(0)}_{g-1}$. By Proposition 2.15, the cone $(C, \emptyset)$ is canonical iff the pair $(M^{(0)}, a \cdot r \cdot M^{(0)}_{g-1})$ is canonical. And by Corollary 3.14, $(M^{(0)}, a \cdot r \cdot M^{(0)}_{g-1})$ is canonical. Therefore $(C, \emptyset)$ is canonical, which finishes the proof. □

Remark 3.16. When $a \cdot r > f - g$, the cone C has more than one irreducible component. It would be interesting to determine the minimal log discrepancies of the different irreducible components of C, in particular of the unique irreducible component which dominates $M^{(0)}$. The first case is when $a \cdot r = f - g + 1$. In this case it follows from the proof above that the second irreducible component is the closure of the preimage of $M^{(0)}_{g-1} - M^{(0)}_{g-2}$, and that the restriction of pr_2 to this irreducible component is birationally a vector bundle of rank $(f - a \cdot r)g$ over P. It may be possible to use this structure to compute the minimal log discrepancies of the two irreducible components.

4. Adjunction for (B, B_{g-1})

Let K be a field, not necessarily algebraically closed nor of characteristic zero. In this section all schemes are K-schemes. K-schemes. We use the results of the last section. The interested reader will see how to prove analogous results over an arbitrary base scheme.

In Section 2 the log discrepancies of a Grassmannian cone $C \to B$ were related to the log discrepancies of the pair (B, B_{g-1}). In this section the following question is considered: Let $S \subset B$ be a Cartier divisor. If (S, S_{g-1}) is log canonical (resp. Kawamata log terminal, canonical), is (B, B_{g-1}) log canonical (resp. Kawamata log terminal, canonical) on a Zariski open set containing S? This question is a version of "inversion of adjunction". Combining theorems by Kollár and Shokurov about inversion of adjunction with some new constructions gives some answers to this question.

Hypothesis 4.1. Throughout this section $\mathcal{E}$ is torsion-free of rank $e = f - g > 0$.

Let the pair $(\rho : C' \to B, \beta : \rho^*\mathcal{F} \to \mathcal{Q}')$ denote the Grassmannian bundle of rank g locally free quotients of $\mathcal{F}$. The first construction equates the log discrepancies of the pair (B, B_{g-1}) with the log discrepancies of a pair $(C', \mathcal{D}_\phi)$ for a Cartier divisor $\mathcal{D}_\phi \subset C'$. Morally, the construction is a version of the following well-known principle (pointed-out to me by Kollár).

Principle 4.2. A pair (B, Z) is log canonical (resp. Kawamata log terminal, canonical) if for a general hypersurface $H \subset B$ containing Z, the pair (B, H) is log canonical (resp. Kawamata log terminal, canonical).

Indeed, if we locally trivialize $\mathcal{F}$ so that $C' \cong B \times \text{Grass}(g, f)$, then the fibers of $\mathcal{D}_\phi \to \text{Grass}(g, f)$ considered as subvarieties of B are hypersurfaces containing B_{g-1}. Therefore one expects that the general fiber of the pair $(C', \mathcal{D}_\phi)$ is log canonical, etc. iff (B, B_{g-1}) is log canonical, etc. In this case, it is even true that $(C', \mathcal{D}_\phi)$ is log canonical, etc. iff (B, B_{g-1}) is log canonical, etc.

Notation 4.3. Denote by $\delta : \rho^*\mathcal{G} \to \mathcal{Q}'$ the composition

$$\delta : \rho^*\mathcal{G} \xrightarrow{\rho^*\phi} \rho^*\mathcal{F} \xrightarrow{\beta} \mathcal{Q}'. \tag{32}$$

Both $\rho^*\mathcal{G}$ and $\mathcal{Q}'$ have rank g, so δ induces a well-defined morphism of invertible sheaves $\det(\delta) : \rho^*\det(\mathcal{G}) \to \mathcal{O}_{C'}(1)$. Denote by $\mathcal{D}_\phi \subset C'$ the zero scheme of this morphism.

Lemma 4.4. *The Cartier divisor $\mathcal{D}_\phi$ is irreducible and generically reduced, and the projection morphism $\pi : \mathcal{D}_\phi \to B$ admits a dualizing complex of the form*

$$\omega_{\mathcal{D}_\phi/B} = \rho^* \left(det(\mathcal{F})^{\otimes(r-1)} \otimes_{\mathcal{O}_B} det(\mathcal{E}) \right) \otimes_{\mathcal{O}_{C'}} \mathcal{O}_{C'}(-(f-1))|_{\mathcal{D}_\phi}[r(f-r)-1]. \tag{33}$$

If B is Cohen-Macaulay (resp. Gorenstein) then also $\mathcal{D}_\phi$ is Cohen-Macaulay (resp. Gorenstein). If B is Cohen-Macaulay and $codim_B(B_{g-1}) \geq 3$, then $\mathcal{D}_\phi$ is normal.

Proof. The proof is the same sort of argument as in the proofs of Lemma 2.3 and Lemma 2.6. The details are left to the reader. There is one extra detail in the proof of the last claim: over $B - B_{g-1}$, the divisor $\mathcal{D}_\phi$ is not necessarily smooth. However the singular locus of $\mathcal{D}_\phi$ is the locus where δ has rank at most $g-2$, and this has codimension at least 4 in $\rho^{-1}(B - B_{g-1})$. So the singular locus of $\mathcal{D}_\phi \cap \rho^{-1}(B - B_{g-1})$ has codimension at least 3 in $\mathcal{D}_\phi$. If $\text{codim}_B(B_{g-1}) \geq 3$, then the codimension of $\mathcal{D}_\phi \cap \rho^{-1}(B_{g-1})$ in C', i.e. the codimension of $\rho^{-1}(B_{g-1})$ in C', equals $\text{codim}_B(B_{g-1}) \geq 3$. Therefore the codimension of $\rho^{-1}(B_{g-1})$ in $\mathcal{D}_\phi$ is $\text{codim}_B(B_{g-1}) - 1 \geq 2$. So the singular locus of $\mathcal{D}_\phi$ has codimension at least 2 in $\mathcal{D}_\phi$, and, by Serre's criterion, $\mathcal{D}_\phi$ is normal. ☐

Remark 4.5. The last condition for $\mathcal{D}_\phi$ to be normal is not a necessary condition. It seems certain that $\mathcal{D}_\phi$ is normal provided that for every codimension 2 point η of B contained in B_{g-1}, B is regular at η and $\mathcal{I}_{g-1} \cdot \mathcal{O}_{B,\eta}$ equals the maximal ideal $\mathfrak{p}_\eta$. In fact it seems likely that $\mathcal{D}_\phi$ is normal provided that for every codimension 2 point η of B contained in B_{g-1}, B is regular at η and $\mathcal{I}_{g-1} \cdot \mathcal{O}_{B,\eta}/\mathfrak{p}_\eta^2$ inside of $\mathfrak{p}_\eta/\mathfrak{p}_\eta^2$ has dimension at least 1 as a $\kappa(\eta)$-vector space. For example, let $B = \mathbb{A}^2$ and $\phi : \mathcal{O}_{\mathbb{A}^2} \to \mathcal{O}_{\mathbb{A}^2}^{\oplus 2}$ be the map with matrix $(x, y^m)^\dagger$ for $m \geq 1$. Then $\mathcal{D}_\phi$ is normal.

Let $u : \widetilde{B} \to B$ be a resolution of $\mathcal{E}$, and let $\widetilde{\mathcal{E}}$ and $\widetilde{\phi} : \mathcal{G} \to u^*\mathcal{F}$ be as in Notation 2.10. Inside of the fiber product $\widetilde{B} \times_B C'$, there is the Cartier divisor $\mathcal{D}_{\widetilde{\phi}}$. The projection morphism $\text{pr}_2 : \widetilde{B} \times_B C' \to C'$ maps $\mathcal{D}_{\widetilde{\phi}}$ onto $\mathcal{D}_\phi$. Denote by $w : \mathcal{D}_{\widetilde{\phi}} \to \mathcal{D}_\phi$ the induced morphism.

Lemma 4.6. *The support of the inverse image Cartier divisor $pr_2^*\mathcal{D}_\phi$ is contained in the divisor $\mathcal{D}_{\widetilde{\phi}} \cup pr_1^{-1}(E_1 \cup \ldots E_k)$. Moreover,*

$$K_{\widetilde{B} \times_B C'} + \mathcal{D}_{\widetilde{\phi}} = pr_2^* (K_{C'} + \mathcal{D}_\phi) + \sum_{i=1}^{k} (a(E_i; B, B_{g-1}) - 1)\, pr_1^* E_i. \tag{34}$$

Proof. Over $\widetilde{B} - (E_1 \cup \cdots \cup E_k)$, it is clear that pr_2 is an isomorphism. Therefore,

$$K_{\widetilde{B} \times_B C'} + \mathcal{D}_{\widetilde{\phi}} = \text{pr}_2^* (K_{C'} + \mathcal{D}_\phi) + \sum_{i=1}^{k} (a_i - 1)\, \text{pr}_1^* E_i \tag{35}$$

for some sequence of rational numbers $a_1, \ldots, a_k$. To compute the integers a_i, first restrict to $\mathcal{D}_{\widetilde{\phi}}$. By adjunction, the restriction of $K_{\widetilde{B} \times_B C'} + \mathcal{D}_{\widetilde{\phi}}$ to $\mathcal{D}_{\widetilde{\phi}}$ is just $K_{\mathcal{D}_{\widetilde{\phi}}}$. And the restriction of

$\mathrm{pr}_2^*(K_{C'}+\mathcal{D}_\phi)$ equals $w^*K_{\mathcal{D}_\phi}$ (which is *defined* to be the sum of K_B and $C_1(\omega_{\mathcal{D}_\phi/B})$ if $\mathcal{D}_\phi$ is not normal). Applying Lemma 4.4 to both $\mathcal{D}_\phi$ and $\mathcal{D}_{\widetilde{\phi}}$,

$$
\begin{aligned}
&K_{\mathcal{D}_{\widetilde{\phi}}} - w^*K_{\mathcal{D}_\phi} = \\
&\widetilde{\pi}^*\left[K_{\widetilde{B}} - u^*K_B + C_1(\det(\widetilde{\mathcal{E}})) - C_1(\det(\mathcal{E}))\right].
\end{aligned}
\tag{36}
$$

As proved in Lemma 2.14, the divisor on the right is just $\sum_{i=1}^k (a(E_i;B,B_{g-1})-1)\,\widetilde{\pi}^*E_i$. So the restriction to $\mathcal{D}_{\widetilde{\phi}}$ of $\sum_{i=1}^k (a_i-1)\,\mathrm{pr}_1^*E_i$ equals the restriction of $\sum_{i=1}^k (a(E_i;B,B_{g-1})-1)\,\mathrm{pr}_1^*E_i$.

To finish the argument, it must be proved that the pullback map $\mathrm{Pic}(\widetilde{B})\to\mathrm{Pic}(\mathcal{D}_{\widetilde{\phi}})$ is injective. There are two possible cases: $f-g\le g$ or $f-g>g$.

Suppose first that $f-g\le g$. Form the Grassmannian bundle $(\sigma: C''\to\widetilde{B}, \epsilon:\sigma^*\widetilde{\mathcal{G}}\to\mathcal{Q}'')$ parametrizing rank $g-(f-g)$ locally free quotients of $\widetilde{\mathcal{G}}$. The coproduct of $\epsilon:\sigma^*\widetilde{\mathcal{G}}\to\mathcal{Q}''$ and $\sigma^*\widetilde{\phi}:\sigma^*\widetilde{\mathcal{G}}\to\sigma^*u^*\mathcal{F}$ gives a surjective morphism of sheaves $\sigma^*u^*\mathcal{F}\to\mathcal{R}$ where $\mathcal{R}$ is locally free of rank $\mathrm{rank}(\mathcal{F})-\mathrm{rank}(\widetilde{\mathcal{G}})+\mathrm{rank}(\mathcal{Q}'')=g$. There is an induced morphism from C'' to $\widetilde{B}\times_B C$ compatible with the projection to $\widetilde{B}$. And the image is contained in $\mathcal{D}_{\widetilde{\phi}}$. Since $\sigma:C''\to\widetilde{B}$ is a Grassmannian bundle, the pullback map on Picard groups is injective. And this map factors through the pullback map on Picard groups from $\widetilde{B}$ to $\mathcal{D}_{\widetilde{\phi}}$. Therefore the pullback map $\mathrm{Pic}(\widetilde{B})\to\mathrm{Pic}(\mathcal{D}_{\widetilde{\phi}})$ is injective.

Finally, suppose that $f-g>g$. In this case let $(\sigma:C''\to\widetilde{B},\epsilon:\sigma^*(\widetilde{\mathcal{E}})\to\mathcal{Q}'')$ be the Grassmannian bundle parametrizing rank g locally free quotients of $\widetilde{\mathcal{E}}$. There is an obvious closed immersion of C'' into $\widetilde{B}\times_B C'$. And the image clearly lies in $\mathcal{D}_{\widetilde{\phi}}$. As in the last paragraph, this implies that the pullback map $\mathrm{Pic}(\widetilde{B})\to\mathrm{Pic}(\mathcal{D}_{\widetilde{\phi}})$ is injective. □

As the scheme $\mathcal{D}_{\widetilde{\phi}}\to\widetilde{B}$ is typically not smooth, this is not a log resolution of $(C',\mathcal{D}_\phi)$. The construction of a log resolution of $(\widetilde{B}\times_B C',\mathcal{D}_{\widetilde{\phi}})$ is essentially equivalent to the construction in Section 3.

Notation 4.7. Denote by $C^{(0)}$ the fiber product $C^{(0)}=\widetilde{B}\times_B C'$ and denote by $M^{(0)}=M^{(0)}(\widetilde{B},\widetilde{\mathcal{G}},\mathcal{O}_{\widetilde{B}}^{\oplus g})$ the scheme constructed in Section 3. Denote by $p^{(0)}:T^{(0)}\to C^{(0)}$ the GL_g-torsor parametrizing sheaf isomorphisms $\mathrm{pr}_2^*\mathcal{Q}'\to\mathcal{O}_{C^{(0)}}^{\oplus g}$ (with the obvious left GL_g-action) and denote by $\lambda:(p^{(0)})^*\mathrm{pr}_2^*\mathcal{Q}'\to\mathcal{O}_{T^{(0)}}^{\oplus g}$ the universal isomorphism. Denote by ϵ the composition of $(p^{(0)})^*\delta:(p^{(0)})^*\mathrm{pr}_1^*\widetilde{\mathcal{G}}\to(p^{(0)})^*\mathrm{pr}_2^*\mathcal{Q}'$ with λ. Denote by $q^{(0)}:T^{(0)}\to M^{(0)}$ the morphism of $\widetilde{B}$-schemes induced by ϵ. Observe that this morphism is equivariant for the obvious GL_g-action on $M^{(0)}$. Denote by g' the maximum of 0 and $2g-f$.

Lemma 4.8. (i) *The image of $q^{(0)}$ equals $M^{(0)}-M^{(0)}_{g'-1}$.*

(ii) *The morphism $q^{(0)}:T^{(0)}\to\left(M^{(0)}-M^{(0)}_{g'-1}\right)$ factors as an open immersion into a torsor over $M^{(0)}$ for the vector bundle over $\widetilde{B}$ associated to $Hom_{\mathcal{O}_{\widetilde{B}}}\left(\widetilde{\mathcal{E}},\mathcal{O}_{\widetilde{B}}^{\oplus g}\right)$. In particular, $q^{(0)}$ is smooth.*

(iii) *The inverse image scheme $(p^{(0)})^{-1}(\mathcal{D}_{\widetilde{\phi}})$ equals the inverse image scheme $(q^{(0)})^{-1}(M^{(0)}_{g-1})$.*

Proof. **(i)**: This follows by considering the intersection of the subbundle $\mathrm{pr}_1^*\widetilde{\mathcal{G}}\subset\mathrm{pr}_1^*u^*\mathcal{F}$ with the kernel of $\mathrm{pr}_1^*u^*\mathcal{F}\to\mathrm{pr}_2^*\mathcal{Q}'$. The first subbundle has rank g at every point, and the second has rank $f-g$. Hence the maximal possible intersection is $f-g$ if $f-g\le g$, and g otherwise. So the minimal possible rank of ϵ is $g-(f-g)=2g-f$ if $2g-f\ge 0$, and 0 otherwise, i.e. it is g'. On the other hand, up to composing with an isomorphism $\mathrm{pr}_2^*\mathcal{Q}'\to\mathcal{O}_{C^{(0)}}^{\oplus g}$, any morphism $\widetilde{\mathcal{G}}\to\mathcal{O}_{\widetilde{B}}^{\oplus g}$ can be obtained as a fiber of ϵ over a geometric point of $T^{(0)}$. Therefore $q^{(0)}:T^{(0)}\to\left(M^{(0)}-M^{(0)}_{g'-1}\right)$ is surjective.

(ii): The torsor over $M^{(0)}$ is simply $M^{(0)}(\widetilde{B}, u^*\mathcal{F}, \mathcal{O}_{\widetilde{B}}^{\oplus g})$. The open immersion from $T^{(0)}$ to this scheme is clear.

(iii) By construction, $\mathcal{D}_{\widetilde{\phi}}$ is the scheme determined by the determinant of δ. And the pullback $(p^{(0)})^*\delta$ equals $(q^{(0)})^*\phi$ by construction. Therefore the inverse image of $\mathcal{D}_{\widetilde{\phi}}$ under $p^{(0)}$ is precisely the inverse image of $M^{(0)}_{g-1}$ under $q^{(0)}$. □

Notation 4.9. Denote by

$$T^{(g)} \to T^{(g-1)} \to \cdots \to T^{(1)} \to T^{(0)} \tag{37}$$

the sequence of morphisms obtained via base-change by $q^{(0)}$ from the sequence of morphisms

$$M^{(g)} \to M^{(g-1)} \to \cdots \to M^{(1)} \to M^{(0)} \tag{38}$$

constructed in Lemma 3.4.

In particular, each scheme $T^{(k)}$ has a natural GL_g-action and the each morphism $T^{(k+1)} \to T^{(k)}$ is GL_g-equivariant. For each $i = 1, \ldots, g$ the composition $T^{(i)} \to T^{(0)}$ is equivalent to the blowing up of an ideal sheaf $\mathcal{J}_T^{(i)}$ on $T^{(0)}$. Moreover, this ideal sheaf is GL_g-equivariant. Therefore it is of the form $(p^{(0)})^{-1}\mathcal{J}_C^{(i)}$ for some ideal sheaf $\mathcal{J}(i)_C$ on $C^{(0)}$.

Notation 4.10. Denote by

$$C^{(g)} \to C^{(g-1)} \to \cdots \to C^{(1)} \to C^{(0)} \tag{39}$$

the sequence of morphisms obtained by blowing up each of the ideal sheaves $\mathcal{J}(i)_C$.

For each $0 \leq r < s \leq g$, there is a Cartesian diagram

$$\begin{array}{ccc} T^{(s)} & \longrightarrow & T^{(r)} \\ \big\downarrow & & \big\downarrow \\ C^{(s)} & \longrightarrow & C^{(r)} \end{array} \tag{40}$$

where the vertical arrows are GL_g-torsors, and there is a Cartesian diagram

$$\begin{array}{ccc} T^{(s)} & \longrightarrow & T^{(r)} \\ \big\downarrow & & \big\downarrow \\ M^{(s)} & \longrightarrow & M^{(r)} \end{array} \tag{41}$$

where the vertical arrows are open subsets of torsors for a smooth group scheme. By Proposition 3.12 $M^{(g)} \to M^{(0)}$ gives a log resolution of the pair $(M^{(0)}, M^{(0)}_{g-1})$. So $C^{(g)} \to C^{(0)}$ gives a log resolution of the pair $(C^{(0)}, \mathcal{D}_{\widetilde{\phi}})$. Moreover $C^{(0)} \to \widetilde{B}$ is smooth.

Notation 4.11. Denote by $F_0, \ldots, F_{g-1}$ the Cartier divisors on $C^{(g)}$ corresponding to the divisors $E_0^{(g)}, \ldots E_{g-1}^{(g)}$ on $M^{(g)}$. Of course $F_i = \emptyset$ for $i < g'$.

Proposition 4.12. *There exists a log resolution $t : C^{(g)} \to \widetilde{B} \times_B C'$ of the pair $\left(\widetilde{B} \times_B C', \mathcal{D}_{\widetilde{\phi}}\right)$ with exceptional locus $F_{g'} \cup \cdots \cup F_{g-2}$ satisfying the following properties.*

(i) *The morphism $pr_1 \circ t : C^{(g)} \to \widetilde{B}$ is smooth, and the intersection of every fiber with $F_{g'} \cup \cdots \cup F_{g-2} \cup F_{g-1}$ is a simple normal crossings divisor.*
(ii) *The morphism $pr_2 \circ t : C^{(g)} \to C'$ is a log resolution of the pair $(C', \mathcal{D}_\phi)$ with exceptional locus $F_{g'} \cup \cdots \cup F_{g-2} \cup (pr_2 \circ t)^{-1}(E_1 \cup \cdots \cup E_k)$ and such that the strict transform of $\mathcal{D}_\phi$ is the divisor F_{g-1}.*

(iii) *There is an equivalence of $\mathbb{Q}$-Cartier divisor classes on $C^{(g)}$*

$$\begin{array}{l} K_{C^{(g)}} - (pr_2 \circ t)^*(K_{C'} + \mathcal{D}_\phi) = \\ -F_{g-1} + \sum_{j=g'}^{g-2} \left((g-1-j)(g-j)-1\right) F_j + \\ \sum_{i=1}^{k} \left(a(E_i; B, B_{g-1}) - 1\right) (pr_1 \circ t)^* E_i \end{array} \tag{42}$$

(iv) *The log discrepancy of $(C', \mathcal{D}_\phi)$ equals the minimum of 1 and the log discrepancy of (B, B_{g-1}). In particular, every exceptional divisor for (B, B_{g-1}) gives rise to an exceptional divisor for $(C', \mathcal{D}_\phi)$.*

(v) *The pair $(C', \mathcal{D}_\phi)$ is log canonical (resp. purely log terminal, canonical) iff the pair (B, B_{g-1}) is log canonical (resp. purely log terminal, canonical).*

(vi) *Assume that $\mathcal{D}_\phi$ is normal. Then the total discrepancy of $(\mathcal{D}_\phi, \emptyset)$ equals the total discrepancy of (B, B_{g-1}).*

(vii) *Assume that $\mathcal{D}_\phi$ is normal. Then $(\mathcal{D}_\phi, \emptyset)$ is log canonical (resp. Kawamata log terminal, canonical) iff the pair (B, B_{g-1}) is log canonical (resp. purely log terminal, canonical).*

(viii) *Assume that $\mathcal{D}_\phi$ is normal. If (B, B_{g-1}) is terminal, then $\mathcal{D}_\phi$ is terminal. If for every exceptional divisor E_i, $a(E_i; B, B_{g-1}) \neq 1$, then the converse also holds.*

Proof. **(i)**: This can be checked after performing the smooth, surjective base-change by $T^{(g)} \to C^{(g)}$. And again by smooth surjective base-change, the result on $T^{(g)}$ is equivalent to the statement that $M^{(g)}$ is smooth over $\widetilde{B}$ and every fiber intersects $E_{g'}^{(g)} \cup \cdots \cup E_{g-1}^{(g)}$ in a simple normal crossings divisor. This follows from (i) and (ii) of Proposition 3.12.

(ii): As mentioned, t is a log resolution of $\mathcal{D}_{\widetilde{\phi}}$. Moreover the divisor $F_{g'} \cup \dots F_{g-2}$ is flat over $\widetilde{B}$ and intersects every fiber in a simple normal crossings divisor. Therefore $F_{g'} \cup \dots F_{g-2} \cup (\mathrm{pr}_2 \circ t)^{-1}(E_1 \cup \cdots \cup E_k)$ is a simple normal crossings divisor. It follows that $\mathrm{pr}_2 \circ t$ is a log resolution of $(C', \mathcal{D}_\phi)$.

(iii), (iv) and (v): Item (iii) follows from Corollary 3.14 and Lemma 4.6. Items (iv) and (v) follow immediately from (iii).

(vi), (vii) and (viii): Observe that $F_{g-1} \to \mathcal{D}_\phi$ is a resolution of singularities. Also,

$$\begin{array}{l} K_{F_{g-1}} - (w \circ t)^* K_{\mathcal{D}_\phi} = \\ \sum_{j=g'}^{g-2} \left((g-1-j)(g-j)-1\right) F_j|_{F_{g-1}} + \\ \sum_{i=1}^{k} \left(a(E_i; B, B_{g-1}) - 1\right) (\mathrm{pr}_1 \circ t)^* E_i. \end{array} \tag{43}$$

By the same sort of argument as in Lemma 4.6, all of the relevant divisor classes are linearly independent on F_{g-1}. Since the coefficients $(g-1-j)(g-j)$ are at least 2 for $j = g', \dots, g-2$, the total discrepancy of $\mathcal{D}_\phi$ equals the total discrepancy of (B, B_{g-1}). This proves (vi). As always, it is possible that some exceptional divisors of (B, B_{g-1}) do not give rise to exceptional divisors of $\mathcal{D}_\phi$. Items (vii) and (viii) follow immediately from (vi). $\square$

Corollary 4.13. *Let $S \subset B$ be an irreducible Cartier divisor. Denote by ϕ_S the restriction of ϕ to S. Assume that $\mathcal{D}_{\phi_S}$ is irreducible.*

(i) *Suppose that S is Kawamata log terminal. Then (S, S_{g-1}) is log canonical iff there exists an open subscheme $U \subset B$ containing S such that $(U, S + U_{g-1})$ is log canonical. In particular, if (S, S_{g-1}) is log canonical, then (U, U_{g-1}) is log canonical.*

(ii) *Suppose that $\mathcal{D}_{\phi_S}$ is irreducible and normal. Then (S, S_{g-1}) is Kawamata log terminal iff there exists an open subscheme $U \subset B$ containing S such that $(U, S + U_{g-1})$ is purely log terminal. In particular, if (S, S_{g-1}) is Kawamata log terminal, then (U, U_{g-1}) is Kawamata log terminal.*

(iii) *Suppose that B is Gorenstein and that $\mathcal{D}_{\phi_S}$ is irreducible and normal. Then (S, S_{g-1}) is canonical iff there exists an open subscheme $U \subset B$ containing S such that (U, U_{g-1}) is canonical. In particular, if (S, S_{g-1}) is canonical then (U, U_{g-1}) is canonical.*

Proof. **(i)**: First observe that $\rho^{-1}(S) \subset C'$ is also Kawamata log terminal since ρ is smooth. By Proposition 4.12, (S, S_{g-1}) is log canonical iff $(\rho^{-1}(S), \mathcal{D}_{\phi_S})$ is log canonical. By a similar argument, $(B, S+B_{g-1})$ is log canonical iff $(C', \rho^{-1}(S)+\mathcal{D}_\phi)$ is log canonical. By [21, Thm. 7.5.2], $(\rho^{-1}(S), \mathcal{D}_{\phi_S})$ is log canonical iff $(C', \rho^{-1}(S)+\mathcal{D}_\phi)$ is log canonical near $\rho^{-1}(S)$. Therefore (S, S_{g-1}) is log canonical iff $(B, S+B_{g-1})$ is log canonical near S.

(ii) and (iii): For (ii), combine (vii) of Proposition 4.12 with [21, Thm. 7.5.1]. For (iii), combine (vii) of Proposition 4.12 with [32] (see also [21, Thm. 7.9]). ☐

Remark 4.14. If one further assumes that B is a local complete intersection scheme, then one can also use the results of Ein and Mustaţă [6] to prove Corollary 4.13 and to relate the minimal log discrepancy of (S, S_{g-1}) to the minimal log discrepancy of (B, B_{g-1}).

5. Deformation to the normal cone

Corollary 4.13 concludes results about (B, B_{g-1}) by analyzing the restriction of ϕ to an irreducible Cartier divisor S. In applications it is also natural to restrict ϕ to an irreducible closed subvariety $Y \subset B$ that is not necessarily a Cartier divisor; in particular, if r is the smallest integer such that $B_r \neq \emptyset$, then one natural choice is to take Y to be an irreducible component of B_r. In this section, the general case is reduced to the case of a Cartier divisor by using deformation to the normal cone. We briefly review the discussion of deformation to the normal cone from [9, Chapter 5]. All of the unproved assertions regarding deformation to the normal cone used here are proved there.

The following setup is a little more general than needed. Let $\phi : \mathcal{G} \to \mathcal{F}$ be a morphism of locally free $\mathcal{O}_X$-modules of ranks g and f (but do not assume that $f > g$). Denote by $\mathcal{E}$ the cokernel of ϕ. Let $Y \subset B$ be a closed subscheme with ideal sheaf $\mathcal{J}$. Denote by $\mathcal{K}_Y$ and $\mathcal{E}_Y$ the kernel and cokernel respectively of the map of $\mathcal{O}_Y$-modules

$$\phi \otimes \mathrm{Id} : \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y \to \mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{O}_Y. \tag{44}$$

In particular $\mathcal{E}_Y$ is simply $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{O}_Y$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{J}/\mathcal{J}^2 & \longrightarrow & \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_B/\mathcal{J}^2 & \longrightarrow & \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_2} & & \Big\downarrow{\scriptstyle \phi_3} & & \\ 0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{J}/\mathcal{J}^2 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{O}_B/\mathcal{J}^2 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{O}_Y & \longrightarrow & 0 \end{array} \tag{45}$$

Each map ϕ_i is just $\phi \otimes \mathrm{Id}$. Since tensor product is right exact, the cokernel of ϕ_1 is just $\mathcal{E}_Y \otimes_{\mathcal{O}_Y} \mathcal{J}/\mathcal{J}^2$. And by definition the kernel of ϕ_3 is $\mathcal{K}_Y$. By the Snake Lemma, there is an induced connecting map from $\mathrm{Ker}(\phi_3)$ to $\mathrm{Coker}(\phi_1)$.

Definition 5.1. The *connecting map*, denoted $\theta = \theta_{\phi,Y} : \mathcal{K}_Y \to \mathcal{E}_Y \otimes_{\mathcal{O}_Y} \mathcal{J}/\mathcal{J}^2$, is the map of $\mathcal{O}_Y$-modules induced by the Snake Lemma as above. The *induced map* of the connecting map is the map $\theta'_{\phi,Y} : \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{K}_Y) \to \mathcal{J}/\mathcal{J}^2$ induced from $\theta_{\phi,Y}$ by adjointness of Hom and tensor product.

Hypothesis 5.2. From now on, $\mathcal{E}_Y$ is assumed to be a locally free $\mathcal{O}_Y$-module. This implies that also $\mathcal{K}_Y$ is a locally free $\mathcal{O}_Y$-module. In the later sections, $\mathcal{J}/\mathcal{J}^2$ will also be a locally free $\mathcal{O}_Y$-module (but this is *not* a hypothesis in this section).

Lemma 5.3. (i) *For the transpose $\phi^\dagger$, the kernel of $\phi^\dagger|_Y$ is $\mathcal{E}_Y^\vee$, the cokernel of $\phi^\dagger|_Y$ is $\mathcal{K}_Y^\vee$, and the connecting map $\theta'_{\phi^\dagger,Y}$ is identified with $\theta'_{\phi,Y}$ under the canonical isomorphism*

$$\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{K}_Y^\vee, \mathcal{E}_Y^\vee) \cong \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{K}_Y).$$

(ii) *Let $\phi' : \mathcal{G}' \to \mathcal{F}'$ be a second morphism of locally free sheaves on B such that the kernel and cokernel of $\phi'|_Y$ are locally free $\mathcal{O}_Y$-modules, $\mathcal{K}'_Y$ and $\mathcal{E}'_Y$. Consider $\phi \oplus \phi' : \mathcal{G} \oplus \mathcal{G}' \to \mathcal{F} \oplus \mathcal{F}'$. The kernel of $(\phi \oplus \phi')|_Y$ is $\mathcal{K}_Y \oplus \mathcal{K}'_Y$, the cokernel of $(\phi \oplus \phi')|_Y$ is $\mathcal{E}_Y \oplus \mathcal{E}'_Y$, and $\theta_{\phi\oplus\phi',Y}$ equals $\theta_{\phi,Y} \oplus \theta_{\phi',Y}$ via the canonical isomorphisms.*

(iii) *Consider* $\phi \otimes \phi' : \mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{G}' \to \mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{F}'$. *The kernel of* $\phi \otimes \phi'$ *is the surjective image of* $(\mathcal{K}_Y \otimes_{\mathcal{O}_Y} \mathcal{G}'|_Y) \oplus (\mathcal{G}|_Y \otimes_{\mathcal{O}_Y} \mathcal{K}'_Y)$, *the cokernel of* $\phi \otimes \phi'$ *is a subsheaf of* $(\mathcal{E}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}'|_Y) \oplus (\mathcal{F}|_Y \otimes_{\mathcal{O}_Y} \mathcal{E}'_Y)$, *and* $\theta_{\phi\otimes\phi',Y}$ *is the unique morphism compatible with* $(\theta_{\phi,Y} \otimes \phi') \oplus (\phi \otimes \theta_{\phi',Y})$.

(iv) *Let* $\psi_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}'$ *and* $\psi_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}'$ *be morphisms of* $\mathcal{O}_B$*-modules such that there is a commutative diagram:*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\phi} & \mathcal{F} \\ {\scriptstyle\psi_{\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle\psi_{\mathcal{F}}} \\ \mathcal{G}' & \xrightarrow{\phi'} & \mathcal{F}' \end{array} \tag{46}$$

There are unique morphisms $\psi_K : \mathcal{K}_Y \to \mathcal{K}'_Y$ *and* $\psi_E : \mathcal{E}_Y \to \mathcal{E}'_Y$ *such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_Y & \longrightarrow & \mathcal{G}|_Y & \xrightarrow{\phi|_Y} & \mathcal{F}|_Y & \longrightarrow & \mathcal{E}_Y & \longrightarrow & 0 \\ & & {\scriptstyle\psi_{\mathcal{K}}}\downarrow & & {\scriptstyle\psi_{\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle\psi_{\mathcal{F}}} & & \downarrow{\scriptstyle\psi_{\mathcal{E}}} & & \\ 0 & \longrightarrow & \mathcal{K}'_Y & \longrightarrow & \mathcal{G}'|_Y & \xrightarrow{\phi'|_Y} & \mathcal{F}'|_Y & \longrightarrow & \mathcal{E}'_Y & \longrightarrow & 0 \end{array} \tag{47}$$

Moreover, the diagram of connecting maps commutes:

$$\begin{array}{ccc} \mathcal{K}_Y & \xrightarrow{\theta_{\phi,Y}} & \mathcal{E}_Y \otimes_{\mathcal{O}_Y} \mathcal{J}/\mathcal{J}^2 \\ {\scriptstyle\psi_{\mathcal{K}}}\downarrow & & \downarrow{\scriptstyle\psi_{\mathcal{E}}\otimes Id} \\ \mathcal{K}'_Y & \xrightarrow{\theta_{\phi',Y}} & \mathcal{E}'_Y \otimes_{\mathcal{O}_Y} \mathcal{J}/\mathcal{J}^2. \end{array} \tag{48}$$

Proof. Each of these follows by simple diagram-chasing. The details are left to the reader. ☐

Deformation to the normal cone, which is described in detail in [9, Chapter 5], is as follows. Form the product $B \times \mathbb{P}^1$ and consider the closed subscheme $Y \times \{\infty\} \subset B \times \mathbb{P}^1$. The ideal sheaf of this subscheme is

$$\mathcal{J}' = \mathrm{pr}_1^{-1}(\mathcal{J}) + \mathrm{pr}_2^{-1}(\mathcal{O}_{\mathbb{P}^1}(-\underline{\infty})). \tag{49}$$

This decomposition of the ideal sheaf yields a decomposition of the Rees algebra

$$\begin{aligned} &\oplus_{n=0}^{\infty}(\mathcal{J}')^n/(\mathcal{J}')^{n+1} \cong \\ &\mathrm{pr}_1^*\left(\oplus_{n=0}^{\infty}(\mathcal{J})^n/\mathcal{J}^{n+1}\right) \otimes \mathrm{pr}_2^*\left(\oplus_{n=0}^{\infty}\mathcal{O}_{\mathbb{P}^1}(-n\underline{\infty})/\mathcal{O}_{\mathbb{P}^1}(-(n+1)\underline{\infty})\right). \end{aligned} \tag{50}$$

The relative Spec of the Rees algebra is the *normal cone.* If the normal cone is the symmetric algebra of a locally free sheaf, the normal cone is called the *normal bundle.* The decomposition above gives an isomorphism of the normal cone $C_{Y\times\{\infty\}}(B \times \mathbb{P}^1)$ with the fiber product $\mathrm{pr}_1^*C_Y B \times_{Y\times\{\infty\}} \mathrm{pr}_2^* N_{\{\infty\}}\mathbb{P}^1$.

Notation 5.4. Denote $C = \mathrm{pr}_1^* C_Y B$ and $C' = C_{Y\times\{\infty\}}(B \times \mathbb{P}^1)$. The normal bundle $N_{\{\infty\}}\mathbb{P}^1$ is just the trivial rank 1 vector bundle, denoted by $\mathbf{1}$ in [9].

The isomorphism of algebras respects the $\mathbb{G}_m$-actions induced by the grading of the algebras. Therefore, in the notation of [9], there is an equivalence of cones $C' \cong C \oplus \mathbf{1}$.

Let $u : M \to B \times \mathbb{P}^1$ be the blowing up of $B \times \mathbb{P}^1$ along $Y \times \{\infty\}$. Denote by $\varrho : M \to \mathbb{P}^1$ the composition $\mathrm{pr}_2 \circ u$. This is a flat morphism. The preimage of $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ is isomorphic to $B \times \mathbb{A}^1$ (compatible with the projections to B and to $\mathbb{A}^1$). And the Cartier divisor $M_\infty = \varrho^{-1}(\infty)$ is the sum of two effective divisors B_Y and $\mathbb{P}(C') = \mathbb{P}(C \oplus \mathbf{1})$. Here B_Y is the blowing up of B along Y. And, as usual, for a cone K, the symbol $\mathbb{P}(K)$ means the relative Proj of the graded algebra associated to K. Denote by $\pi : \mathbb{P}(C \oplus \mathbf{1}) \to Y$ the obvious projection morphism.

The intersection of B_Y and $\mathbb{P}(C \oplus \mathbf{1})$ is the exceptional divisor on B_Y and is the "hyperplane section at infinity" $\mathbb{P}(C)$ in $\mathbb{P}(C \oplus \mathbf{1})$. The complement of the hyperplane section at infinity is identified with the cone C over Y. Finally, there is a closed immersion $\iota : Y \times \mathbb{P}^1 \to M$ such that

$u \circ \iota : Y \times \mathbb{P}^1 \to B \times \mathbb{P}^1$ is the obvious closed immersion. The fiber of $\iota(Y \times \mathbb{P}^1)$ over ∞ is identifed with the zero section of $C \subset \mathbb{P}(C \oplus \mathbf{1})$.

Definition 5.5. For a closed subscheme $Y \subset B$, the *deformation to the normal cone* is the datum $(\varrho : M \to \mathbb{P}^1, \iota : Y \times \mathbb{P}^1 \hookrightarrow M, B_Y \hookrightarrow M, \mathbb{P}(C \oplus \mathbf{1}) \hookrightarrow M)$. Denote by $\phi_M : \mathcal{G}_M \to \mathcal{F}_M$ the morphism of locally free sheaves $u^*\mathrm{pr}_1^*\phi$.

On $\mathbb{P}(C \oplus \mathbf{1})$ there is a rank 1 locally free quotient $\beta : \pi^* \left(\mathrm{pr}_1^*\mathcal{J}/\mathcal{J}^2 \oplus \mathcal{O}_Y\right) \to \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(1)$ (satisfying a well-known universal property). Denote by $\beta_1 : \pi^* \left(\mathrm{pr}_1^*\mathcal{J}/\mathcal{J}^2\right) \to \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(1)$ and $\beta_2 : \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})} \to \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(1)$ the two components of β. Of course, the zero scheme of the section β_2 is precisely the hyperplane section at infinity $\mathbb{P}(C) \subset \mathbb{P}(C \oplus \mathbf{1})$. The invertible sheaf $\mathcal{O}_M(-\mathbb{P}(C \oplus \mathbf{1}))|_{\mathbb{P}(C\oplus\mathbf{1})}$ is canonically isomorphic to $\mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(1)$; the isomorphism is induced by the isomorphism of ideal sheaves $\mathrm{u}^{-1}\mathcal{J}' \cong \mathcal{O}_M(-\mathbb{P}(C \oplus \mathbf{1}))$.

The pullback ϕ_M factors through an *elementary-transform-up* of $\mathcal{G}_M$. To describe this elementary-transform-up, first dualize all sheaves and form the adjoints of all sheaf morphisms. Consider the adjoint morphism $\phi_M^\dagger : \mathcal{F}_M^\vee \to \mathcal{G}_M^\vee$. The restriction of ϕ_M to $\mathbb{P}(C\oplus\mathbf{1})$ is just $\pi^*(\phi|_Y)$. In particular, the image of $(\mathcal{F}_M^\vee)|_{\mathbb{P}(C\oplus\mathbf{1})}$ is contained in the kernel of $\pi^* \left(\mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y\right)^\vee \to \pi^* \left(\mathcal{K}_Y\right)^\vee$. Define the subsheaf $(\widetilde{\mathcal{G}})^\vee \subset (\mathcal{G}_M)^\vee$ to be the kernel of the surjection

$$(\mathcal{G}_M)^\vee \to (\mathcal{G}_M)^\vee|_{\mathbb{P}(C\oplus\mathbf{1})} \cong \pi^* \left(\mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y\right)^\vee \to \pi^* \left(\mathcal{K}_Y\right)^\vee. \tag{51}$$

Then $\phi_M^\dagger$ factors through the subsheaf $(\widetilde{\mathcal{G}})^\vee$. Define $(\widetilde{\phi})^\dagger : \mathcal{F}_M^\vee \to (\widetilde{\mathcal{G}})^\vee$ to be the induced map.

Lemma 5.6. *Denote by $\widetilde{\mathcal{G}}$ the dual of $(\widetilde{\mathcal{G}})^\vee$, and denote by $\widetilde{\phi}$ the adjoint of $(\widetilde{\phi})^\dagger$.*

(i) *The sheaf $(\widetilde{\mathcal{G}})^\vee$ is locally free of rank g. Therefore also $\widetilde{\mathcal{G}}$ is locally free of rank g.*
(ii) *The cokernel of the sheaf map $\mathcal{G}_M \to \widetilde{\mathcal{G}}$ is canonically isomorphic to the push-forward from $\mathbb{P}(C \oplus \mathbf{1})$ of the locally free sheaf $\pi^*\mathcal{K}_Y \otimes \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(-1)$.*
(iii) *The restriction of $\mathcal{G}_M \to \widetilde{\mathcal{G}}$ to $\mathbb{P}(C \oplus \mathbf{1})$ fits into an exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^*\mathcal{K}_Y & \longrightarrow & \pi^*(\mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y) & \longrightarrow & \\ & & \widetilde{\mathcal{G}}|_{\mathbb{P}(C\oplus\mathbf{1})} & \longrightarrow & \pi^*\mathcal{K}_Y \otimes \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(-1) & \longrightarrow & 0 \end{array} \tag{52}$$

Proof. Item (i) is very easy and is just the fact that an *elementary-transform-down* along a Cartier divisor gives rise to a locally free sheaf. For (ii) and (iii), observe that the restriction to $\mathbb{P}(C \oplus \mathbf{1})$ of $(\widetilde{\mathcal{G}})^\vee \to (\mathcal{G}_M)^\vee$ fits into an exact sequence,

$$\begin{array}{ccccccc} 0 & \longrightarrow & Tor_1^{\mathcal{O}_M}(\mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}, \pi^*\mathcal{K}_Y^\vee) & \longrightarrow & (\widetilde{\mathcal{G}})^\vee|_{\mathbb{P}(C\oplus\mathbf{1})} & & \\ & \longrightarrow & \pi^*(\mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y)^\vee & \longrightarrow & \pi^*\mathcal{K}_Y^\vee & \longrightarrow & 0. \end{array} \tag{53}$$

Of course $Tor_1^{\mathcal{O}_M}(\mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}, \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})})$ is just $\mathcal{O}_M(-\mathbb{P}(C \oplus \mathbf{1}))|_{\mathbb{P}(C\oplus\mathbf{1})}$, i.e. $\mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(1)$. Therefore the left-most term in the exact sequence is just $\mathcal{K}_Y^\vee \otimes \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(1)$. Dualizing this sequence gives (ii) and (iii). ☐

Notation 5.7. The restriction of $\widetilde{\phi}$ to $\mathbb{P}(C \oplus \mathbf{1})$ induces a morphism of locally free sheaves

$$\widetilde{\mathcal{G}}|_{\mathbb{P}(C\oplus\mathbf{1})}/\pi^*(\mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y) \to \pi^*(\mathcal{F} \otimes_{\mathcal{O}_B} \mathcal{O}_Y)/\pi^*(\mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y). \tag{54}$$

Up to canonical isomorphism, this is the same as a morphism

$$\gamma : \pi^*\mathcal{K}_Y \otimes \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(-1) \to \pi^*\mathcal{E}_Y. \tag{55}$$

And the cokernel of $\widetilde{\phi}$ on $\mathbb{P}(C \oplus \mathbf{1})$ equals the cokernel of γ.

The next goal is to show that the map γ essentially is the same as the map $\theta_{\phi,Y}$.

The canonical inclusion $\widetilde{\mathcal{G}} \otimes \mathcal{O}_M(-\mathbb{P}(C \oplus \mathbf{1})) \to \widetilde{\mathcal{G}}$ factors through the kernel of $\widetilde{\mathcal{G}} \to \pi^*\mathcal{K}_Y \otimes \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(-1)$. So there is an induced inclusion $\widetilde{\mathcal{G}} \otimes \mathcal{O}_M(-\mathbb{P}(C \oplus \mathbf{1})) \to \mathcal{G}_M$ whose cokernel is $\pi^*\left(\mathcal{G} \otimes_{\mathcal{O}_B} \mathcal{O}_Y/\mathcal{K}_Y\right)$. In particular, there is a commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{G}} \otimes \mathcal{O}_M(-\mathbb{P}(C \oplus \mathbf{1})) & \longrightarrow & \mathcal{G}_M \\ {\scriptstyle \widetilde{\phi}\otimes\mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_M} \\ \mathcal{F}_M \otimes \mathcal{O}_M(-\mathbb{P}(C \oplus \mathbf{1})) & \longrightarrow & \mathcal{F}_M \end{array} \tag{56}$$

Lemma 5.8. *There is a commutative diagram of coherent sheaves:*

$$\begin{array}{ccc} \pi^*\mathcal{K}_Y & \xrightarrow{\pi^*\theta_{\phi,Y}} & \pi^*\mathcal{E}_Y \otimes \pi^*(\mathcal{J}/\mathcal{J}^2) \\ {\scriptstyle Id}\big\downarrow & & \big\downarrow{\scriptstyle Id\otimes\beta_1} \\ \pi^*\mathcal{K}_Y & \xrightarrow{\gamma\otimes Id} & \pi^*\mathcal{E}_Y \otimes \mathcal{O}_{\mathbb{P}(C\oplus\mathbf{1})}(1) \end{array} \tag{57}$$

Proof. To ease notation in this proof, denote $\mathbb{P} = \mathbb{P}(C \oplus \mathbf{1})$. Consider the commutative diagram with exact rows analogous to Equation(45) whose rows are

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{G}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M(-\mathbb{P})/\mathcal{O}_M(-2\mathbb{P}) & \longrightarrow & \mathcal{G}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M/\mathcal{O}_M(-2\mathbb{P}) \\ & \longrightarrow & \mathcal{G}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{\mathbb{P}} & \longrightarrow & 0 \end{array} \tag{58}$$

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M(-\mathbb{P})/\mathcal{O}_M(-2\mathbb{P}) & \longrightarrow & \mathcal{F}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M/\mathcal{O}_M(-2\mathbb{P}) \\ & \longrightarrow & \mathcal{F}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{\mathbb{P}} & \longrightarrow & 0 \end{array} \tag{59}$$

Associated to this commutative diagram, the snake lemma produces a connecting map $\theta_{\phi_M,\mathbb{P}} : \pi^*\mathcal{K}_Y \to \pi^*\mathcal{E}_Y \otimes \mathcal{O}_{\mathbb{P}}(1)$.

Observe that the ideal sheaf $u^{-1}\mathrm{pr}_1^{-1}(\mathcal{J})$ is contained in the ideal sheaf $\mathcal{O}_M(-\mathbb{P})$ (moreover, after dividing by the defining equation of $\mathbb{P}$, the residual ideal sheaf is the ideal sheaf of the closed immersion $\iota : Y \times \mathbb{P}^1 \to M$). Therefore there is a map from the pullback by $\mathrm{pr}_1 \circ u$ of the commutative diagram in Equation(45) to the commutative diagram above. In particular there is a commutative diagram of connecting maps

$$\begin{array}{ccc} \pi^*\mathcal{K}_Y & \xrightarrow{\pi^*\theta_{\phi,Y}} & \pi^*\mathcal{E}_Y \otimes \pi^*\left(\mathcal{J}/\mathcal{J}^2\right) \\ {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}\otimes\beta} \\ \pi^*\mathcal{K}_Y & \xrightarrow{\theta_{\phi_M,\mathbb{P}}} & \pi^*\mathcal{E}_Y \otimes \mathcal{O}_{\mathbb{P}}(1) \end{array} \tag{60}$$

Now consider the map $\widetilde{\mathcal{G}}(-\mathbb{P}) \to \mathcal{G}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M/\mathcal{O}_M(-2\mathbb{P})$ constructed above. The image of $\widetilde{\mathcal{G}}(-\mathbb{P})$ in the quotient $\mathcal{G}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{\mathbb{P}}$ is precisely $\pi^*\mathcal{K}_Y$. Moreover there is the map $\widetilde{\phi} \otimes \mathrm{Id} : \widetilde{\mathcal{G}}(-\mathbb{P}) \to \mathcal{F}_M(-\mathbb{P})$ and the diagram in Equation (56) commutes. Therefore $\widetilde{\mathcal{G}}(-\mathbb{P})$ can be used to compute the connecting map $\theta_{\phi_M,\mathbb{P}}$. But the construction of γ was by precisely the same construction, i.e. the connecting map $\theta_{\phi_M,\mathbb{P}}$ equals $\gamma \otimes \mathrm{Id}$. ☐

6. The stack of multiple covers of lines

Hypothesis 6.1. From this point on, the base field K is algebraically closed of characteristic 0.

Let V denote a K-vector space of dimension $n+1$ so that the projective space $\mathbb{P}(V)$ is isomorphic to $\mathbb{P}^n$. In this and the next sections, results from the previous sections are applied to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$, the Kontsevich moduli stack parametrizing stable maps from unmarked, genus 0 curves to $\mathbb{P}(V)$ of degree e. For more details about this stack, see [11]. The goal is to prove that for a positive integer d with $d + e \leq n$, for a general hypersurface $X \subset \mathbb{P}(V)$ of degree d, the closed substack $\overline{\mathcal{M}}_{0,0}(X, e) \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$ has at worst canonical singularities.

Remark 6.2. What does it mean to say that a pair of Deligne-Mumford stacks is canonical (resp. log canonical, etc.)? For a pair (B, Y), one can compute the log discrepancy $a(E; B, Y)$ étale locally on B, i.e. if $(f_i : B_i \to B)$ is an étale cover, then $a(E; B, Y) = \min(a(f_i^*E; B_i, f_i^{-1}Y)|\mathrm{center}(f_i^*E) \neq \emptyset)$. There is a standard way of extending any étale local notion for schemes to Deligne-Mumford stacks: the Deligne-Mumford stack has an étale local cover by schemes and the log discrepancies are defined using this cover by the formula above.

Consider the Kontsevich moduli stack $\overline{\mathcal{M}}_{0,r}(\mathbb{P}(V), e)$. Let $p : \mathcal{C} \to \overline{\mathcal{M}}_{0,r}(\mathbb{P}(V), e)$ denote the universal curve, and let $f : \mathcal{C} \to \mathbb{P}(V)$ denote the universal map. For each integer $d > 0$, on $\overline{\mathcal{M}}_{0,r}(\mathbb{P}(V), e)$ there is a locally free sheaf $\mathcal{P}_d$ of rank $ed+1$ defined by $\mathcal{P}_d = p_*f^*\mathcal{O}_{\mathbb{P}(V)}(d)$. Standard facts about stable maps together with cohomology and base change imply that the higher direct images of $f^*\mathcal{O}_{\mathbb{P}(V)}(d)$ vanish and that $\mathcal{P}_d$ is locally free of rank $ed + 1$. The case of interest is $r = 0$, but in fact this holds for arbitrary r. There is a canonical *evaluation morphism* of locally free sheaves on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$, $\phi_d^\dagger : H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) \otimes_\kappa \mathcal{O} \to \mathcal{P}_d$.

Definition 6.3. For each $e \geq 1$ and $d \geq 0$, define $\mathcal{G}_d$ to be the dual of $\mathcal{P}_d$, define $\mathcal{F}_d$ to be the trivial locally free sheaf $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))^\vee \otimes_\kappa \mathcal{O}$, and define the *co-evaluation morphism for degree d* to be the morphism,

$$\phi_d : \mathcal{G} \to \mathcal{F}, \tag{61}$$

adjoint to the evaluation morphism. Define $\mathcal{E}_d$ to be the cokernel of ϕ_d. Define $(\pi_d : C_d \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e), \alpha_d : \pi^*\mathcal{E}_d \to \mathcal{Q}_d)$ to be the projective Abelian cone parametrizing rank 1 locally free quotients of $\mathcal{E}_d$. When there is no risk of confusion, the subscripts will be dropped.

The goal is to analyze the singularities of C. Denote by $(\rho : C' \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e), \beta : H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))^\vee \otimes_\kappa \mathcal{O} \to \mathcal{Q}')$ the projective bundle parametrizing rank 1 locally free quotients of the trivial locally free sheaf $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) \otimes_\kappa \mathcal{O}$. Of course C' is the same as the product $\mathbb{P}H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) \times \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$. As in Section 2, define $h : C \to C'$ to be the tautological closed immersion. Our interest in C comes from the following easy result. The proof is left to the reader (but also see [15, Lemma 4.5]).

Lemma 6.4. *The Deligne-Mumford stack C parametrizes pairs $([X], [f : C \to X])$ consisting of $[X] \in \mathbb{P}H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$, a hypersurface of degree d in $\mathbb{P}(V)$, and $f : C \to X$, a Kontsevich stable map of genus 0 and degree e to X. In particular, for each $[X] \in \mathbb{P}H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$, the fiber of C over $[X]$ is canonically identified with the Kontsevich moduli stack $\overline{\mathcal{M}}_{0,0}(X, e)$.*

Notation 6.5. Denote by $\mathbb{G} = \mathbb{G}(2, V)$ the Grassmannian variety over κ parametrizing rank 2 locally free quotients of $V^\vee$, i.e. parametrizing 2-dimensional linear subspaces of V. Let $V^\vee \otimes_\kappa \mathcal{O}_\mathbb{G} \to S^\vee$ denote the universal quotient, so that the adjoint $S \hookrightarrow V \otimes_\kappa \mathcal{O}_\mathbb{G}$ is the universal rank 2 subbundle. Denote the quotient of the universal subbundle by $V \otimes_\kappa \mathcal{O}_\mathbb{G} \to T$. Observe that $T^\vee \hookrightarrow V^\vee \otimes_\kappa \mathcal{O}_\mathbb{G}$ is simply the annihilator of S. For each $d \geq 0$, there is an induced filtration on $\mathrm{Sym}^d(V^\vee) \otimes_\kappa \mathcal{O}_\mathbb{G}$

$$\begin{aligned} &\mathrm{Sym}^d(V^\vee) \otimes_\kappa \mathcal{O}_\mathbb{G} = F^0 \supset F^1 \supset \cdots \supset F^d \\ &F^i = \mathrm{Sym}^i(T^\vee) \cdot \mathrm{Sym}^{d-i}(V^\vee), \quad F^i/F^{i+1} \cong \mathrm{Sym}^i(T^\vee) \otimes_{\mathcal{O}_\mathbb{G}} \mathrm{Sym}^{d-i}(S^\vee) \end{aligned} \tag{62}$$

This filtration is the same as the filtration by order of vanishing along S.

Definition 6.6. There is an induced morphism $\mathbb{P}S \to \mathbb{P}(V)$ identifying $\mathbb{G}$ with the Hilbert scheme of lines in $\mathbb{P}(V)$. The *stack of multiple covers of lines* Y is defined to be the closed substack of $\mathbb{G} \times \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$ parametrizing pairs $([L], [f : C \to L])$ where $[L] \in \mathbb{G}$ is a line in $\mathbb{P}(V)$ and $f : C \to L$ is a Kontsevich stable map of genus 0 and degree e.

There are several equivalent definitions. Of course the projection $\mathrm{pr}_\mathbb{G} : Y \to \mathbb{G}$ is Zariski locally isomorphic to the product $\mathbb{G} \times \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$. An easy observation is that the projection $Y \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$ is a closed immersion.

Lemma 6.7. *The* 1*-morphism $Y \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$ *is representable by closed immersions and the image is the rank* 2 *locus* $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e))_2$ *for* $\phi_1 : \mathcal{G}_1 \to \mathcal{F}_1$*. Moreover, for each* ϕ_d *the reduced substack of the rank* $(d+1)$ *locus* $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e))_{d+1}$ *equals the image of* Y*.*

Proof. It is clear that $Y \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$ is injective on geometric points; after all for a pair $([L], [f : C \to L])$, we have that $L = f(C)$, so the line $[L]$ is uniquely determined by $f : C \to \mathbb{P}(V)$. Moreover for a stable map $f : C \to X$, the rank of ϕ_1 restricted to the residue field of $[f]$ is at least as big as $H^0(f(C), \mathcal{O}_{\mathbb{P}(V)}(1)|_{f(C)})$. For a pure 1-dimensional subscheme of $\mathbb{P}(V)$, this dimension is always at least 2, and equals 2 only if $f(C)$ is a line. Therefore every geometric point of $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e))_2$ is a geometric point of Y. The same sort of argument shows that for every d, the rank $(d+1)$ locus of ϕ_d equals Y (as sets of geometric points).

Moreover, since $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e))_1$ is empty, on $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e))_2$ the quotient $\mathcal{E}$ of ϕ_1 is a locally free sheaf of rank 2. Then $\mathcal{E}$ is a quotient of $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))^\vee$ which induces a morphism from $(\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e))_2$ to $\mathbf{G}$, from which it is easy to construct an inverse to $Y \to (\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e))_2$. □

Consider the restriction of ϕ_d to Y. It is simpler to phrase the results for the adjoint $\phi_d^\dagger$, but by (i) of Lemma 5.3, they are both equivalent.

Lemma 6.8. *The kernel of* $\phi_1^\dagger|_Y$ *equals* $pr_{\mathbb{G}}^* T^\vee \subset V^\vee \otimes_\kappa \mathcal{O}_Y$*. The cokernel of* $\phi_1^\dagger|_Y$ *is a locally free sheaf* $\mathcal{R}$ *of rank* $e-1$*. And the induced connecting map* $\theta'_{\phi_1^\dagger, Y} : Hom_{\mathcal{O}_Y}(\mathcal{R}, pr_{\mathbb{G}}^* T^\vee) \to \mathcal{J}/\mathcal{J}^2$ *is an isomorphism of* $\mathcal{O}_Y$*-modules.*

Proof. The first two assertions follow from the proof of Lemma 6.7; the details are left to the reader. The third assertion can probably be proved directly, but it also follows from the deformation theory of Kontsevich stable maps developed in [2] and [3] (see also [16, Sec. 3]). Since Y is smooth and since $\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$ is smooth, the conormal sheaf $\mathcal{J}/\mathcal{J}^2$ is a locally free $\mathcal{O}_Y$-module. The first step is to describe this sheaf.

As above, let $\pi : \mathcal{C} \to Y$ be the universal curve. Let $g : \mathcal{C} \to \mathbb{P}(S)$ be the universal map (compatible with projection to $\mathbb{G}$), and let $\mathrm{pr}_{\mathbb{P}(V)} : \mathbb{P}(S) \to \mathbb{P}(V)$ be the obvious projection so that $f = \mathrm{pr}_{\mathbb{P}(V)} \circ g$ is the universal map from $\mathcal{C}$ to $\mathbb{P}(V)$. There is a perfect complex of amplitude $[-1, 0]$ on $\mathcal{C}$, denoted L_f, such that the object $(\mathbb{R}\pi_* L_f^\vee)[1]$ in the derived category of Y is quasi-isomorphic to the restriction of the tangent bundle of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$. Similarly, there is a perfect complex of amplitude $[-1, 0]$, denoted L_g, such that the object $(\mathbb{R}\pi_* L_g^\vee)[1]$ in the derived category of Y is quasi-isomorphic to the vertical tangent bundle of the morphism $\mathrm{pr}_{\mathbb{G}} : Y \to \mathbb{G}$. These complexes are as follows:

$$\begin{array}{rcl} & -1 & & 0 \\ L_f : & f^*\Omega^1_{\mathbb{P}(V)} & \xrightarrow{(df)^\dagger} & \Omega^1_{\mathcal{C}/Y} \\ L_g : & g^*\Omega^1_{\mathbb{P}(S)/\mathbb{G}} & \xrightarrow{(dg)^\dagger} & \Omega^1_{\mathcal{C}/Y} \end{array} \tag{63}$$

Of course the derivative of the morphism $\mathrm{pr}_{\mathbb{P}(V)} : \mathbb{P}(S) \to \mathbb{P}(V)$ induces a surjective sheaf map from $f^*\Omega^1_{\mathbb{P}(V)}$ to $g^*\Omega^1_{\mathbb{P}(S)/\mathbb{G}}$ whose kernel is just $g^*\mathcal{O}_{\mathbb{P}(S)}(-1) \otimes \mathrm{pr}_{\mathbb{G}}^* T^\vee$. So there is a distinguished triangle

$$g^*\mathcal{O}_{\mathbb{P}(S)}(-1) \otimes \mathrm{pr}_{\mathbb{G}}^* T^\vee[1] \to L_f \to L_g \tag{64}$$

Applying the derived functor $\mathbb{R}Hom_{\mathcal{O}_\mathcal{C}}(*, \mathcal{O}_\mathcal{C})$ to the distinguished triangle above produces a distinguished triangle,

$$L_g^\vee \to L_f^\vee \to Hom_{\mathcal{O}_\mathcal{C}}\left(\mathrm{pr}_{\mathbb{G}}^* T^\vee, g^*\mathcal{O}_{\mathbb{P}(S)}(1)\right)[-1]. \tag{65}$$

Finally apply $\mathbb{R}\pi_*$ to this distinguished triangle to get a distinguished triangle,

$$(\mathbb{R}\pi_* L_g^\vee)[1] \to (\mathbb{R}\pi_* L_f^\vee)[1] \to Hom_{\mathcal{O}_Y}\left(\mathrm{pr}_{\mathbb{G}}^* T^\vee, \mathcal{P}_1\right)[0], \tag{66}$$

(obviously some minor details have been left out). The derivative map from the vertical tangent bundle of $\mathrm{pr}_{\mathbb{G}} : Y \to \mathbb{G}$ to the restriction of the tangent bundle of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$ has cokernel isomorphic to $\mathit{Hom}_{\mathcal{O}_Y}(\mathrm{pr}_{\mathbb{G}}^* T^\vee, \mathcal{P}_1)$.

The map $\phi_1^\dagger|_Y$ has image $\mathrm{pr}_{\mathbb{G}}^* S^\vee$. Therefore inside of this cokernel there is a subsheaf $\mathit{Hom}_{\mathcal{O}_Y}(\mathrm{pr}_{\mathbb{G}}^* T^\vee, \mathrm{pr}_{\mathbb{G}}^* S^\vee)$. This is just the pullback of the tangent bundle of $\mathbb{G}$. The cokernel of this subsheaf is the normal bundle of $Y \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$. And, since $\mathcal{R}$ is $\mathcal{P}_1/\mathrm{pr}_{\mathbb{G}}^* S^\vee$ by definition, this cokernel is canonically isomorphic to $\mathit{Hom}_{\mathcal{O}_Y}(\mathrm{pr}_{\mathbb{G}}^* T^\vee, \mathcal{R})$. Therefore the dual sheaf, $\mathcal{J}/\mathcal{J}^2$, is canonically isomorphic to $\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}, \mathrm{pr}_{\mathbb{G}}^* T^\vee)$. Obviously this is very close to what is needed.

Using the canonical isomorphism above, the induced connecting map $\theta'_{\phi_1^\dagger, Y}$ is an endomorphism of the locally free sheaf $\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}, \mathrm{pr}_{\mathbb{G}}^* T^\vee)$. An endomorphism of a locally free sheaf is an isomorphism iff the determinant of the endomorphism is invertible. Since Y is a proper, smooth, connected Deligne-Mumford stack, the global sections of $\mathcal{O}_Y$ are just the constants. So to prove that the determinant is invertible, it suffices to prove that it is nonzero at a single point.

The proof is reduced to a simple (slightly tedious) computation in local coordinates. Choose homogeneous coordinates $Y_0, \dots, Y_n$ on $\mathbb{P}(V)$, i.e. $Y_0, \dots, Y_n$ is an ordered basis for $V^\vee = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$. Choose homogeneous coordinates X_0, X_1 on $\mathbb{P}^1$. Let $\mathbb{A}$ be the affine space associated to the *dual vector space* W of linear transformations

$$W^\vee := \mathrm{Hom}_\kappa\left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e-2)), \mathrm{span}\{Y_2, \dots, Y_n\}\right). \tag{67}$$

A basis for the vector space $W^\vee$, i.e. for the vector space of linear forms on $\mathbb{A}$, is given by the tensors $(X_0^i X_1^{e-2-i})^\vee \otimes Y_j$ for $j = 2, \dots, n$ and $i = 0, \dots, e-2$. Define $F : \mathbb{P}^1 \times \mathbb{A} \to \mathbb{P}(V)$ to be the morphism with $F^*\mathcal{O}_{\mathbb{P}(V)}(1) = \mathrm{pr}_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(e)$ and where the pullback of homogeneous coordinates is defined by

$$\begin{array}{rcl} F^*Y_0 & = & \mathrm{pr}_{\mathbb{P}^1}^* X_0^e, \\ F^*Y_1 & = & \mathrm{pr}_{\mathbb{P}^1}^* X_1^e, \\ F^*Y_j & = & \sum_{i=0}^{e-2} \mathrm{pr}_{\mathbb{P}^1}^*(X_0^{i+1} X_1^{e-1-i}) \cdot \mathrm{pr}_{\mathbb{A}}^*((X_0^i X_1^{e-2-i})^\vee \otimes Y_j), \ j = 2, \dots, n \end{array} \tag{68}$$

The morphism F is a family of stable maps of degree e over $\mathbb{A}$ and defines a 1-morphism $\zeta : \mathbb{A} \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$. The pullback by ζ of $\mathcal{P}_1$ is simply the trivial vector bundle $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e))$ on $\mathbb{A}$, and the pullback of $\phi_1^\dagger$ is simply the map:

$$\begin{array}{rcl} \phi_1^\dagger(1 \otimes Y_0) & = & 1 \otimes X_0^e, \\ \phi_1^\dagger(1 \otimes Y_1) & = & 1 \otimes X_1^e, \\ \phi_1^\dagger(1 \otimes Y_j) & = & \sum_{i=0}^{e-2} \left((X_0^i X_1^{e-2-i})^\vee \otimes Y_j\right) \otimes X_0^{i+1} X_1^{e-1-i}, \ j = 2, \dots, n \end{array} \tag{69}$$

It follows that the rank 2 locus is the origin $0 \in \mathbb{A}$. The inverse image ideal sheaf $\zeta^{-1}\mathcal{J}$ is just the ideal of the origin, i.e. the ideal with generators $(X_0^i X_1^{e-2-i})^\vee \otimes Y_j$ for $i = 0, \dots, e$ and $j = 2, \dots, n$. The kernel of $\phi_1^\dagger|_0$ is $\mathrm{span}\{Y_2, \dots, Y_n\}$. The image of $\phi_1^\dagger|_0$ is $\mathrm{span}\{X_0^e, X_1^e\}$, and the cokernel is $X_0 X_1 \cdot H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e-2))$. So the pullback of $\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{K}_Y)$ is the vector space,

$$\mathrm{Hom}_\kappa\left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e-2)), \mathrm{span}\{Y_2, \dots, Y_n\}\right).$$

Chasing through the snake diagram associated to F, the pullback of the map $\theta'_{\phi_1^\dagger, Y} : \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{K}_Y) \to \mathcal{J}/\mathcal{J}^2$ is the map

$$(X_0^i X_1^{e-2-i})^\vee \otimes Y_j \mapsto (X_0^i X_1^{e-2-i})^\vee \otimes Y_j \tag{70}$$

i.e. it is the identity map. This proves that $\theta_{\phi_1^\dagger, Y}$ is an isomorphism when restricted to the image of $0 \in \mathbb{A}$. As mentioned above, this suffices to prove that $\theta_{\phi_1^\dagger, Y}$ is everywhere an isomorphism. $\square$

Remark 6.9. In fact, via the canonical isomorphism of $\mathcal{J}/\mathcal{J}^2$ with $\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{K}_Y)$ induced by the deformation theory computation, the endomorphism $\theta_{\phi_1^\dagger, Y}$ is the identity map. Since this is not used here, it is not proved.

Notation 6.10. For every nonnegative integer d, denote by $\mathcal{A}_d$ the locally free $\mathcal{O}_Y$-module, $\mathrm{pr}_{\mathbb{G}}^*(\mathrm{Sym}^{d-1}(S^\vee))^\vee$, denote by $\mathcal{R}_d$ the locally free $\mathcal{O}_Y$-module, $\mathcal{R} \otimes_{\mathcal{O}_Y} \mathrm{pr}_{\mathbb{G}}^*\mathrm{Sym}^d(S^\vee)$, and denote by $T_d^\vee$ the locally free $\mathcal{O}_{\mathbb{G}}$-module, $T^\vee \otimes_{\mathcal{O}_{\mathbb{G}}} \mathrm{Sym}^d(S^\vee)$.

Proposition 6.11. *Using the isomorphisms from Lemma 6.8, the following hold.*

(i) *For each $d \geq 1$, the kernel of $\phi_d^\dagger|_Y$ is $pr_{\mathbb{G}}^* F^1 \subset Sym^d(V^\vee) \otimes_\kappa \mathcal{O}_Y$.*

(ii) *The cokernel of $\phi_d^\dagger|_Y$ is canonically isomorphic to $\mathcal{R}_{d-1}$.*

(iii) *The induced connecting map*

$$\theta'_{\phi_d^\dagger, Y} : Hom_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, pr_{\mathbb{G}}^* F^1) \to \mathcal{J}/\mathcal{J}^2 \tag{71}$$

is the zero map on the subsheaf $Hom_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, pr_{\mathbb{G}}^* F^2)$.

(iv) *Identify F^1/F^2 with $T^\vee \otimes_{\mathcal{O}_{\mathbb{G}}} Sym^{d-1}(S^\vee)$, i.e. $T_{d-1}^\vee$, and identify $\mathcal{J}/\mathcal{J}^2$ with $Hom_{\mathcal{O}_Y}(pr_{\mathbb{G}}^* T^\vee, \mathcal{R})$ using $\theta'_{\phi_1^\dagger, Y}$. The following map,*

$$\theta''_{\phi_d^\dagger, Y} : Hom_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, pr_{\mathbb{G}}^* T_{d-1}^\vee) \to Hom_{\mathcal{O}_Y}(\mathcal{R}, pr_{\mathbb{G}}^* T^\vee), \tag{72}$$

induced by $\theta'_{\phi_d^\dagger, Y}$, equals the map obtained by contracting the $Sym^{d-1}(S^\vee)$ factors.

Proof. **(i)**: The sheaf $\mathcal{P}_d$ is defined to be $\pi_* f^* \mathcal{O}_{\mathbb{P}(V)}(d)$. Consider the fiber product $\mathbb{P}(S) \times_{\mathbb{G}} Y$. There is an induced finite, flat, surjective morphism $(g, \pi) : \mathcal{C} \to \mathbb{P}(S) \times_{\mathbb{G}} Y$. And $f^*\mathcal{O}_{\mathbb{P}(V)}(d)$ equals $(g,\pi)^* \mathrm{pr}_{\mathbb{P}(S)}^* \mathcal{O}_{\mathbb{P}(S)}(d)$. Since $\pi = \mathrm{pr}_Y \circ (g, \pi)$,

$$\mathcal{P}_d = (\mathrm{pr}_Y)_*(g,\pi)_*(g,\pi)^* \mathrm{pr}_{\mathbb{P}(S)}^* \mathcal{O}_{\mathbb{P}(S)}(d). \tag{73}$$

There is a canonical sheaf map $\mathrm{pr}_{\mathbb{P}(S)}^* \mathcal{O}_{\mathbb{P}(S)}(d) \to (g,\pi)_*(g,\pi)^* \mathrm{pr}_{\mathbb{P}(S)}^* \mathcal{O}_{\mathbb{P}(S)}(d)$. Because (g, π) is faithfully flat, this sheaf map is injective. Since pushforward is left exact, there is an injective sheaf map from $(\mathrm{pr}_Y)_* \mathrm{pr}_{\mathbb{P}(S)}^* \mathcal{O}_{\mathbb{P}(S)}(d)$ to $\mathcal{P}_d$. But of course the first sheaf is just $\mathrm{pr}_{\mathbb{G}}^* \mathrm{Sym}^d(S^\vee)$. And the injective sheaf map $\mathrm{Sym}^d(S^\vee) \to \mathcal{P}_d$ is the image of $\phi_d^\dagger|_Y$. Therefore the kernel of $\phi_d^\dagger|_Y$ is the pullback of the kernel of $\mathrm{Sym}^d(V^\vee) \otimes_\kappa \mathcal{O}_{\mathbb{G}} \to \mathrm{Sym}^d(S^\vee)$, i.e. the first filtered subsheaf F^1 of $\mathrm{Sym}^d(V^\vee) \otimes_\kappa \mathcal{O}_{\mathbb{G}}$. This proves that the kernel of $\phi_d^\dagger|_Y$ equals $\mathrm{pr}_{\mathbb{G}}^* F^1$.

(ii): On $\mathbb{P}(S)$ there is a short exact sequence,

$$0 \to \mathrm{pr}_{\mathbb{G}}^* \mathbb{S}_{(d-1,1)}(S^\vee) \otimes \mathcal{O}_{\mathbb{P}(S)}(-1) \to \mathrm{pr}_{\mathbb{G}}^* \mathrm{Sym}^{d-1}(S^\vee) \to \mathcal{O}_{\mathbb{P}(S)}(d-1) \to 0, \tag{74}$$

where $\mathbb{S}_{(d-1,1)}$ is the *Schur functor* as defined in [10, Sec. 6.1]. Twist this sequence by $\mathcal{O}_{\mathbb{P}(S)}(1)$ and pullback by (g, π) to get a short exact sequence of $\mathcal{O}_{\mathcal{C}}$-modules. Pushing forward by π yields a long exact sequence of higher direct image sheaves. Since $\pi : \mathcal{C} \to Y$ is a flat family of at-worst-nodal curves of genus 0, $R^1\pi_*\mathcal{O}_{\mathcal{C}}$ is zero. Therefore, the long exact sequence reduces to the following short exact sequence,

$$0 \to \mathrm{pr}_{\mathbb{G}}^* \mathbb{S}_{(d-1,1)}(S^\vee) \to \mathrm{pr}_{\mathbb{G}}^* \mathrm{Sym}^{d-1}(S^\vee) \otimes \mathcal{P}_1 \to \mathcal{P}_d \to 0. \tag{75}$$

Of course, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{pr}_{\mathbb{G}}^* \mathbb{S}_{(d-1,1)}(S^\vee) & \to & \mathrm{pr}_{\mathbb{G}}^* \mathrm{Sym}^{d-1}(S^\vee) \otimes \mathrm{pr}_{\mathbb{G}}^* S^\vee & \to & \mathrm{pr}_{\mathbb{G}}^* \mathrm{Sym}^d(S^\vee) & \to & 0 \\ & & \mathrm{Id} \downarrow & & \downarrow \mathrm{Id} \otimes \phi_1^\dagger & & \downarrow \phi_d^\dagger & & \\ 0 & \to & \mathrm{pr}_{\mathbb{G}}^* \mathbb{S}_{(d-1,1)}(S^\vee) & \to & \mathrm{pr}_{\mathbb{G}}^* \mathrm{Sym}^{d-1}(S^\vee) \otimes \mathcal{P}_1 & \to & \mathcal{P}_d & \to & 0 \end{array} \tag{76}$$

Applying the snake lemma to this diagram yields an isomorphism of the cokernel of $\phi_d^\dagger$ with $\mathcal{R} \otimes \mathrm{pr}_{\mathbb{G}}^* \mathrm{Sym}^{d-1}(S^\vee)$, i.e. $\mathcal{R}_{d-1}$.

(iii) In the special case $d = 1$, (iii) follows from Lemma 6.8. Thus suppose that $d > 1$. The proof of (iii) uses the fact that there is a commutative diagram of sheaves on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$:

$$\begin{array}{ccc} V^\vee \otimes_K \mathrm{Sym}^{d-1}(V^\vee) \otimes_K \mathcal{O} & \xrightarrow{\phi_1^\dagger \otimes \mathrm{Id}} & \mathcal{P}_1 \otimes_K \mathrm{Sym}^{d-1}(V^\vee) \\ \psi_{\mathcal{G}} \downarrow & & \downarrow \psi_{\mathcal{F}} \\ \mathrm{Sym}^d(V^\vee) \otimes_K \mathcal{O} & \xrightarrow{\phi_d^\dagger} & \mathcal{P}_d \end{array} \tag{77}$$

Associated to $\phi_1^\dagger \otimes \mathrm{Id}$ is the induced connecting map,

$$\theta'_{\phi_1^\dagger \otimes \mathrm{Id}, Y} : \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R} \otimes_K \mathrm{Sym}^{d-1}(V^\vee), \mathrm{pr}_{\mathbb{G}}^* T^\vee \otimes_K \mathrm{Sym}^{d-1}(V^\vee)) \to \mathcal{J}/\mathcal{J}^2. \tag{78}$$

Of course this is obtained from $\theta'_{\phi_1^\dagger, Y}$ by contracting the $\mathrm{Sym}^{d-1}(V^\vee)$ factors. Define $\theta''_{\phi_1^\dagger \otimes \mathrm{Id}, Y}$ to be the restriction of $\theta'_{\phi_1^\dagger \otimes \mathrm{Id}, Y}$ to the subsheaf $\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, \mathrm{pr}_{\mathbb{G}}^* T^\vee \otimes_K \mathrm{Sym}^{d-1}(V^\vee))$. By (iv) of Lemma 5.3, there is a commutative diagram of induced connecting maps,

$$\begin{array}{ccc} \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, \mathrm{pr}_{\mathbb{G}}^* T^\vee \otimes_K \mathrm{Sym}^{d-1}(V^\vee)) & \xrightarrow{\theta'_{\phi_1^\dagger \otimes \mathrm{Id}, Y}} & \mathcal{J}/\mathcal{J}^2 \\ \psi \downarrow & & \downarrow \mathrm{Id} \\ \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}_{d-1} \mathrm{pr}_{\mathbb{G}}^* F^1) & \xrightarrow{\theta'_{\phi_d^\dagger, Y}} & \mathcal{J}/\mathcal{J}^2 \end{array} \tag{79}$$

Since $\theta'_{\psi_1^\dagger \otimes \mathrm{Id}, Y}$ is obtained by contracting the $\mathrm{Sym}^{d-1}(V^\vee)$ factors, in particular the kernel of $\theta_{\phi_1^\dagger \otimes \mathrm{Id}, Y}$ contains the subsheaf $\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, \mathrm{pr}_{\mathbb{G}}^* T^\vee \otimes_{\mathcal{O}_Y} F^1)$. Therefore the kernel of $\theta'_{\phi_d^\dagger, Y}$ contains the image under ψ of this subsheaf. But the image under ψ is just $\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, \mathrm{pr}_{\mathbb{G}}^* F^2)$. Forming the quotient by this sheaf gives another commutative diagram,

$$\begin{array}{ccc} \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, \mathrm{pr}_{\mathbb{G}}^* T^\vee \otimes_K \mathrm{Sym}^{d-1}(V^\vee)) & \xrightarrow{\theta'_{\phi_1^\dagger \otimes \mathrm{Id}, Y}} & \mathcal{J}/\mathcal{J}^2 \\ \psi'' \downarrow & & \downarrow \mathrm{Id} \\ \mathit{Hom}_{\mathcal{O}_Y}(\mathcal{R}_{d-1}, \mathrm{pr}_{\mathbb{G}}^* T_{d-1}) & \xrightarrow{\theta''_{\phi_d^\dagger, Y}} & \mathcal{J}/\mathcal{J}^2 \end{array} \tag{80}$$

Since ψ'' is surjective, $\theta''_{\phi_d^\dagger, Y}$ is the unique morphism making the above diagram commute. But using the fact that $\theta'_{\phi_1^\dagger \otimes \mathrm{Id}, Y}$ is obtained from contracting the $\mathrm{Sym}^{d-1}(V^\vee)$ factors, it is clear that the diagram also commutes when $\theta''_{\phi_d^\dagger, Y}$ is replaced by the map obtained from $\theta'_{\phi_1^\dagger, Y}$ by contracting the $\mathrm{Sym}^{d-1}(S^\vee)$ factors. Therefore $\theta''_{\phi_d^\dagger, Y}$ equals the map obtained from $\theta'_{\phi_1^\dagger, Y}$ by contracting the $\mathrm{Sym}^{d-1}(S^\vee)$ factors. $\square$

Corollary 6.12. *Regarding the restriction $\phi_d|_Y : \mathcal{G}_d|_Y \to \mathcal{F}_d|_Y$, the following hold.*

- (i) *The cokernel $\mathcal{E}_Y$ is canonically isomorphic to $pr_{\mathbb{G}}^*(F^1)^\vee$.*
- (ii) *The kernel $\mathcal{K}_Y$ is canonically isomorphic to $(\mathcal{R}_{d-1})^\vee$.*
- (iii) *The induced connecting map $\theta'_{\phi_d, Y}$ is canonically isomorphic to the induced connecting map $\theta'_{\phi_d^\dagger, Y}$.*

Proof. This follows from (i) of Lemma 5.3 and Proposition 6.11. $\square$

7. Proof of the main theorem

Fix an integer $e \geq 1$.

Notation 7.1. To simplify notation, in this section denote $B = \overline{\mathcal{M}}_{0,0}(\mathbb{P}(V), e)$. For each integer d, denote by $\pi : C_d \to B$ the projective Abelian cone of the coherent $\mathcal{O}_B$-module, $\mathcal{E}_d = \mathrm{Coker}(\phi_d)$. When there is no risk of confusion, C_d is denoted by C.

By Lemma 6.4, C_d is the Deligne-Mumford stack parametrizing pairs $([X], [f : C \to X])$ of a hypersurface of degree d, $X \subset \mathbb{P}(V)$, together with a stable map, $[f : C \to X]$, in $\overline{\mathcal{M}}_{0,0}(X, e)$. In this section the singularities of the cone C_d are described. The reader is reminded that K is an algebraically closed field of characteristic 0.

The simplest case is $e = 1$. The next two results are already known, in fact in arbitrary characteristic [20, Thm. V.4.3]. Only that part used here is proved (or rather reproved).

Proposition 7.2. *If $e = 1$, then B is a smooth projective scheme and for all $d \geq 1$ the morphism $\pi_d : C_d \to B$ is a projective bundle of the expected dimension. In particular C_d is a geometrically irreducible, smooth scheme of the expected dimension.*

Proof. Since $e = 1$, Y equals B, which is simply the Grassmannian $\mathbb{G}$. Thus $\mathcal{R}$ is the zero sheaf. And by Corollary 6.12, the cokernel $\mathcal{E}_d$ is locally free of the expected dimension. Therefore $\pi : C \to B$ is a projective bundle. $\square$

Theorem 7.3. [20, V.4.3] *If $e = 1$ and if $d > 2n - 3$, then the projection morphism $h_d : C_d \to \mathbb{P}H^0\left(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)\right)$ is not surjective, i.e. the general fiber is empty. If $d \leq 2n - 3$, then the projection morphism h_d is surjective, and the general fiber is a smooth scheme of the expected dimension $2n-d-3$. Moreover if $d < 2n-3$ and $(d, n) \neq (2, 3)$, then the general fiber is geometrically connected.*

Proof. For the full proof, the reader should consult [20]. Only that part of the theorem used later is proved here, namely that the general fiber is nonempty and smooth for $d \leq n - 1$ and that the general fiber is geometrically connected for $d \leq n - 2$.

Suppose that $d \leq n - 1$. Since C_d is irreducible of the expected dimension, h_d is surjective iff the general fiber has the expected dimension. To prove the general fiber has the expected dimension, it suffices to find one pair $([X], [L])$ consisting of a hypersurface $X \subset \mathbb{P}(V)$ and a line $L \subset X$ such that X is smooth along L and such that $H^1(L, N_{L/X})$ is zero. Then the Zariski tangent space of the fiber, i.e. $H^0(L, N_{L/X})$, has the expected dimension proving that on a nonempty (hence dense) open subset of C, h_d has the expected fiber dimension. Choose homogeneous coordinates $Y_0, Y_1, Y_2, \ldots, Y_n$ on $\mathbb{P}(V)$. Define L to be the vanishing locus of $Y_2, \ldots, Y_n$. Define $X \subset \mathbb{P}(V)$ to be the hypersurface with defining equation

$$F := \sum_{j=2}^{d+1} Y_0^{d+1-j} Y_1^{j-2} Y_j \tag{81}$$

At every point of L, either the partial derivative F_2 is nonzero or the partial derivative F_{d+1} is nonzero. Therefore X is smooth along L. Moreover, by the usual exact sequence,

$$0 \to N_{L/X} \to \mathcal{O}_L(1)^{n-1} \xrightarrow{F_2, \ldots, F_{d+1}, 0, \ldots, 0} \mathcal{O}_L(d) \to 0, \tag{82}$$

$N_{L/X}$ is isomorphic to $\mathcal{O}_L^{d-1} \oplus \mathcal{O}_L(1)^{n-d-1}$. Therefore $H^1(L, N_{L/X})$ is zero proving that h_d is surjective and the generic fiber has the expected dimension. Since C_d is smooth and since $\mathbb{P}H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$ is smooth, it follows by generic smoothness that the generic fiber of h_d is everywhere smooth.

Next it is proved that the generic fiber of h_d is connected if $d \leq n-2$. In this case, by the same sort of dimension computation as above, for a general hypersurface $X \subset \mathbb{P}(V)$, every irreducible component of $\overline{\mathcal{M}}_{0,1}(X, 1)$ surjects to X under the evaluation morphism $\text{ev} : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$. Therefore, to prove that $\overline{\mathcal{M}}_{0,1}(X, 1)$ is connected, and thus that $\overline{\mathcal{M}}_{0,0}(X, 1)$ is connected, it suffices to prove that every fiber of ev is connected. Now for a given $p \in X$, the set of lines in $\mathbb{P}(V)$ containing p is canonically isomorphic to the projective space $\mathbb{P}(V/L_p)$, where $L_p \subset V$ is the one-dimensional vector subspace corresponding to p. Choose homogeneous coordinates so that $p = [1 : 0 : \cdots : 0]$. Then the Taylor expansion of the defining equation F of X about the point p has the form,

$$F = Y_0^{d-1} F_1(Y_1, \ldots, Y_n) + Y_0^{d-2} F_2(Y_1, \ldots, Y_n) + \cdots + Y_0 F_{d-1}(Y_1, \ldots, Y_n) + F_d(Y_1, \ldots, Y_n), \tag{83}$$

where each F_i is a homogeneous polynomial of degree i in $Y_1, \ldots, Y_n$. A line passing through p with parametric equation $[(1-t) : tY_1 : tY_2 : \cdots : tY_n]$ is contained in X iff the equations $F_1(Y_1, \ldots, Y_n), \ldots, F_d(Y_1, \ldots, Y_n)$ are all zero. It should be observed that some of the homogeneous equations F_i may be identically zero. Nonetheless, the common zero locus of $F_1, \ldots, F_d$ in $\mathbb{P}(V/L_p)$

is an intersection of at most d hypersurfaces in a projective space of dimension $n-1$. In a projective space of dimension $n-1$, an intersection of at most $n-2$ hypersurfaces is always connected. Since $d \leq n-2$, the common zero locus of $F_1, \dots, F_d$ is connected. Therefore every fiber of ev is connected, proving that the fiber $\overline{\mathcal{M}}_{0,0}(X,1) = h_d^{-1}([X])$ is connected. $\square$

Now suppose that $e > 1$. Consider the closed immersion $Y \hookrightarrow B$.

Notation 7.4. Because the conormal sheaf of Y is locally free, the normal cone C is denoted by the letter N (also to avoid confusion with the cone C_d which we are studying).

Associated to the closed immersion $Y \to B$, there is the deformation to the normal cone ($\varrho : M \to \mathbb{P}^1, \iota : Y \times \mathbb{P}^1 \to M, B_Y \to M, \mathbb{P}(N \oplus \mathbf{1}) \to M$) as in Definition 5.5. Define $M^o = M - B_Y$. In particular, the intersection of M^o with $\mathbb{P}(N \oplus \mathbf{1})$ is just N.

Let $\widetilde{\phi}_d : \widetilde{\mathcal{G}} \to \mathcal{F}_M$ be the elementary-transform-up of the pullback of ϕ_d as described in Lemma 5.6. Let $\widetilde{\mathcal{E}}_d$ be the cokernel of $\widetilde{\phi}_d|_{M^o}$ and let $\widetilde{\pi}_d : \widetilde{C}_d \to M^o$ be the projective Abelian cone parametrizing rank 1 locally free quotients of $\widetilde{\mathcal{E}}_d$. Observe that over the open subset $\varrho^{-1}(\mathbb{A}^1) = B \times \mathbb{A}^1$, $\widetilde{\mathcal{E}}_d$ is simply $\mathrm{pr}_B^*\mathcal{E}_d$ and $\widetilde{C}_d$ is simply $C_d \times \mathbb{A}^1$.

Lemma 7.5. *Let $e \geq 2$. If $d+e \leq n$, then the fiber product $\widetilde{C}_d \times_{M^o} N$ is an integral, normal, local complete intersection scheme of the expected dimension having at worst canonical singularities.*

Proof. By Lemma 6.8, the normal bundle $N \to Y$ is the vector bundle associated to the locally free sheaf $Hom_{\mathcal{O}_Y}(\mathcal{R}^\vee, \mathrm{pr}_{\mathbb{G}}^*T)$, i.e. $N = M^{(0)}(Y, \mathcal{R}^\vee, \mathrm{pr}_{\mathbb{G}}^*T)$ in the notation of Section 3. Denote by $\mathcal{A}_{d-1}$ the locally free sheaf $\mathrm{pr}_{\mathbb{G}}^*\left(\mathrm{Sym}^{d-1}(S^\vee)\right)^\vee$, as in Notation 6.10. This is a locally free sheaf of rank d on Y.

By Corollary 6.12, the kernel of $\phi_d|_Y$ is the locally free sheaf

$$\mathcal{K}_Y = \mathcal{A}_{d-1} \otimes_{\mathcal{O}_Y} (\mathcal{R}^\vee) \tag{84}$$

and the cokernel $\mathcal{E}_Y$ of $\phi_d|_Y$ fits into a short exact sequence of locally free sheaves

$$0 \to \mathcal{A}_{d-1} \otimes_{\mathcal{O}_Y} (\mathrm{pr}_{\mathbb{G}}^*T) \to \mathcal{E}_Y \to \mathrm{pr}_{\mathbb{G}}^*(F^2)^\vee \to 0. \tag{85}$$

Denote the first sheaf in this sequence by $\mathcal{E}'_Y$ and denote the third sheaf in this sequence by $\mathcal{A}'$. By Notation 5.7, the cokernel of $\widetilde{\phi}_d$ on the Cartier divisor $N \subset M^o$ equals the cokernel of the sheaf map $\gamma : \pi_Y^*\mathcal{K}_Y \to \pi_Y^*\mathcal{E}_Y$. This uses the fact that $\mathcal{O}_{\mathbb{P}(N\oplus\mathbf{1})}(1)$ is canonically trivialized on N, so that all "twists" by this sheaf are canonically just "twists" by the structure sheaf. By Lemma 5.8, the map γ is the unique map induced by $\theta_{\phi_d,Y}$. By (i) of Lemma 5.3, this means that γ is the transpose of the unique map induced by $\theta_{\phi_d^\dagger,Y}$. By (iii) of Lemma 6.11, the map $\theta_{\phi_d^\dagger,Y}$ is the map induced by the universal homomorphism from $\mathrm{pr}_{\mathbb{G}}^*(T^\vee)$ to $\mathcal{R}$.

Putting all the pieces together, there are two conclusions: First, the image of γ is actually the image of a sheaf map,

$$\gamma' : \mathcal{K}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_N \to \mathcal{E}'_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_N, \tag{86}$$

so that there is a short exact sequence (which is split Zariski locally over Y),

$$0 \to \mathrm{Coker}(\gamma') \to \mathrm{Coker}(\gamma) \to \mathrm{pr}_Y^*\mathcal{A}' \to 0. \tag{87}$$

And second, denoting by $\psi : \mathcal{R}^\vee \otimes_{\mathcal{O}_Y} \mathcal{O}_N \to \mathrm{pr}_{\mathbb{G}}^*T \otimes_{\mathcal{O}_Y} \mathcal{O}_N$ the universal sheaf homomorphism on N, the sheaf map γ' is just $\mathrm{Id} \otimes \psi$,

$$\gamma' = \mathrm{Id} \otimes \psi : \mathcal{A}_{d-1} \otimes_{\mathcal{O}_Y} (\mathcal{R}^\vee) \otimes_{\mathcal{O}_Y} \mathcal{O}_N \to \mathcal{A}_{d-1} \otimes_{\mathcal{O}_Y} (\mathrm{pr}_{\mathbb{G}}^*T) \otimes_{\mathcal{O}_Y} \mathcal{O}_N. \tag{88}$$

Now the rank of $\mathcal{R}^\vee$ is $e-1$, the rank of $\mathrm{pr}_{\mathbb{G}}^*T$ is $n-1$ and the rank of $\mathcal{A}_{d-1}$ is d. By assumption, $d \leq (n-1)-(e-1)$. This means that, Zariski locally over Y, the hypotheses of Proposition 3.15 hold. So $\widetilde{C}_d \times_{M^o} N$ is a normal, integral, local complete intersection scheme of the expected dimension having at worst canonical singularities. $\square$

Notation 7.6. Denote by $W \subset B$ the maximal open substack over which C_d is an integral, normal scheme of the expected dimension having at worst canonical singularities. Observe that $W \times \mathbb{A}^1 \subset B \times \mathbb{A}^1$ is the maximal open substack over which $\widetilde{C}_d \times_{M^o} (B \times \mathbb{A}^1)$ is an integral, normal scheme of the expected dimension having at worst canonical singularities. Denote by $W' \subset M^o$ the maximal open substack over which $\widetilde{C}_d$ is an integral, normal scheme of the expected dimension having at worst canonical singularities.

Lemma 7.7. *If $e \geq 2$ and $d + e \leq n$, then the open substack $W \subset B$ contains the closed substack $Y \subset B$.*

Proof. By Lemma 7.5, by Corollary 2.7, by Proposition 2.15, and by (iii) of Corollary 4.13 applied to $N \subset M^o$, the open substack W' contains N.

There is one slight finesse in checking the hypothesis of (iii) of Corollary 4.13: In case $n - e \geq 2$, $\mathrm{codim}_N(N_{g-1}) = (f-(g-1))(g-(g-1)) = f - g + 1$ equals $(n-1)-(e-1)+1 = n-e+1 \geq 3$. Therefore Lemma 4.4 proves that the hypothesis of (iii) of Corollary 4.13 is satisfied.

The one remaining case is $d = 1, e = n-1$. In this case, there is an *ad hoc* argument. The morphism $\widetilde{\phi}_1$ restricted to N is just the universal sheaf map ψ. In this special case $\mathcal{D}_{\phi_N}$ sits inside $N \times_Y \mathbb{P}(\mathrm{pr}_{\mathbb{G}}^* T)$. The projection $\mathcal{D}_{\phi_N} \to \mathbb{P}(\mathrm{pr}_{\mathbb{G}}^* T)$ is a Zariski locally trivial bundle. Given a closed point $p \in Y$ and a one-dimensional subspace $L \subset \mathrm{pr}_{\mathbb{G}}^* T|_p$, the fiber over this point, considered as a subvariety of $N = M^{(0)}(Y, \mathcal{R}^\vee, \mathrm{pr}_{\mathbb{G}}^* T)$, equals the cone whose vertex set is $Hom_{\kappa(p)}(\mathcal{R}^\vee|_p, L)$ and whose quotient by the vertex set is the set of non-invertible linear maps in $Hom_{\kappa(p)}(\mathcal{R}^\vee|_p, \mathrm{pr}_{\mathbb{G}}^* T|_p/L)$. Observe that this second vector space is essentially just the vector space of square $(e-1) \times (e-1)$ matrices. In the special case $e = 2$, the cone is just a linear space and so it is smooth. In case $e \geq 3$ the vertex set has codimension $(e-1)^2 - 1 \geq 3$ in the fiber of $\mathcal{D}_{\phi_N}$, and the singular locus of the quotient has codimension $4 - 1 = 3$: $Hom_{\kappa(p)}(\mathcal{R}^\vee|_p, \mathrm{pr}_{\mathbb{G}}^* T|_p/L)_{e-3}$ has codimension 4. Therefore the singular locus has codimension 3 in the fiber of $\mathcal{D}_{\phi_N}$. So the fiber is normal, which implies that $\mathcal{D}_{\phi_N}$ is normal. Therefore when $d = 1$ and $e = n-1$, the hypotheses of (iii) of Corollary 4.13 are again satisfied.

Of course $W' \cap \varrho^{-1}(\mathbb{A}^1)$ equals $W \times \mathbb{A}^1$. Let $p \in Y$ be any point and consider $\iota(\{p\} \times \mathbb{P}^1) \subset M$. By construction of the deformation to the normal cone from Definition 5.5, $\iota(p, \infty)$ is the point on the zero section of $N \to Y$ over $p \in Y$. In particular $\iota(\{p\} \times \mathbb{P}^1) \subset M^o$ and $\iota(p, \infty) \in N$. Therefore $\iota(\{p\} \times \mathbb{P}^1)$ intersects W'. So $\iota(\{p\} \times \mathbb{A}^1)$ intersects $W' \cap \varrho^{-1}(\mathbb{A}^1)$, i.e $\{p\} \times \mathbb{A}^1$ intersects $W \times \mathbb{A}^1$. Therefore $p \in W$. Therefore Y is contained in W. $\square$

Theorem 7.8. *If $e \geq 2$ and if $d + e \leq n$, then C_d is an integral, normal, local complete intersection stack of the expected dimension having at worst canonical singularities.*

Proof. By Lemma 7.7, the open substack W contains Y. Now the automorphism group $\mathrm{GL}(V)$ acts on $\mathbb{P}(V)$ and thus on B. Moreover the sheaves $\mathcal{G}_d$ and $\mathcal{F}_d$ have natural $\mathrm{GL}(V)$-linearizations and the morphism ϕ_d is $\mathrm{GL}(V)$-equivariant. Therefore W is a $\mathrm{GL}(V)$-invariant open substack of B. So to prove that $W = B$, it suffices to prove that the closure of every $\mathrm{GL}(V)$-orbit intersects Y.

Let $f : D \to \mathbb{P}(V)$ be any stable map of genus 0 and degree e. Choose a direct sum decomposition $V = V_2 \oplus V_{n-1}$ so that $\mathbb{P}(V_{n-1}) \subset \mathbb{P}(V)$ is disjoint from $f(D)$. Consider the $\mathbb{G}_m$-action $m : \mathbb{G}_m \times \mathbb{P}(V) \to \mathbb{P}(V)$ given by $t \cdot (v, v') = (v, tv')$, where $v \in V_2, v' \in V_{n-1}$. This defines an action of a subgroup scheme of $\mathrm{GL}(V)$. Acting on $[f : D \to \mathbb{P}(V)]$ yields a 1-morphism $\zeta : \mathbb{G}_m \to B$. The limit as $t \to 0$ of this action is simply the stable map $g \circ f : D \to \mathbb{P}(V_2) \subset \mathbb{P}(V)$ where $g : \mathbb{P}(V) - \mathbb{P}(V_{n-1}) \to \mathbb{P}(V_2)$ is the projection map. In particular, $g \circ f$ is a multiple cover of the line $\mathbb{P}(V_2)$. Therefore the closure of the $\mathrm{GL}(V)$-orbit of $[f : D \to \mathbb{P}(V)]$ intersects Y in the point $[g \circ f : D \to \mathbb{P}(V)]$. It follows that W is all of B, i.e. C_d is an integral, normal, local complete intersection stack of the expected dimension having at worst canonical singularities. $\square$

Corollary 7.9. *If $e \geq 2$ and if $d + e \leq n$, then for a general hypersurface $X \subset \mathbb{P}(V)$ of degree d, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X, e)$ is an integral, normal, local complete intersection stack of the expected dimension $(n + 1 - d)e + (n - 3)$ having at worst canonical singularities.*

Proof. By Theorem 7.8, C_d is integral, normal, Gorenstein and canonical. Consider the projection $h_d : C_d \to \mathbb{P}H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$. The pullback of the hyperplane linear system gives a base-point-free linear system on C_d. By repeated application of [30, Thm. 1.13] (see also [21, Prop. 7.7]), the general fiber of h_d is a reduced, normal, local complete intersection stack having at worst canonical singularities. The one issue that remains is connectedness, i.e. it is a priori possible that $h_d : C_d \to \mathbb{P}H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$ has a nontrivial Stein factorization. But observe that by Corollary 7.3 (in fact by the part reproved there), the restriction of h_d to Y is surjective and has a trivial Stein factorization. So $h_d|_Y$ yields a section of the Stein factorization of h_d, which is irreducible and finite over $\mathbb{P}H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$. It follows that also the Stein factorization of h_d is trivial, i.e. the fibers of h_d are connected. So the general fiber of h_d is an integral, normal, local complete intersection stack of the expected dimension having at worst canonical singularities. □

8. Singularities of $\overline{M}_{0,0}(\mathbb{P}^n, e)$

In this section the Reid–Shepherd-Barron–Tai criterion is used to prove that (with a very few exceptions) the coarse moduli spaces $\overline{M}_{0,0}(\mathbb{P}^n, e)$ are terminal. The same computations are used to prove that if one carries out the deformation to the normal cone construction as in Section 7 over the coarse moduli space (instead of the stack, as in the last section), one obtains a family whose general fiber is the coarse moduli space of C_d and whose special fiber is a normal, $\mathbb{Q}$-Gorenstein, canonical variety. The "inversion of adjunction" conjecture then implies that C_d is itself canonical, and therefore $\overline{M}_{0,0}(X, e)$ is canonical for X general.

Let Γ be a finite cyclic group of order r and let $\zeta \in \mathrm{Hom}_{\mathrm{group}}(\Gamma, \mathbb{G}_m)$ be a generator for the character group of Γ. Let M be a finite dimensional Γ-representation over k (the field K is still of characteristic 0, but the following definition makes sense so long as the characteristic is prime to $\#\Gamma$). There is a direct sum decomposition,

$$M = \oplus_{i=0}^{r-1} L_{\zeta^i}^{\oplus a_i}, \tag{89}$$

where each L_{ζ^i} is the one-dimensional representation corresponding to the character ζ^i.

Definition 8.1. The *invariant of M with respect to ζ* (after Reid–Shepherd-Barron–Tai) is

$$\alpha_\zeta(M) = \frac{1}{r}\sum_{i=0}^{r-1} i a_i. \tag{90}$$

The *invariant of* M is $\alpha(M) = \min \alpha_\zeta(M)$ as ζ varies over all generators of the character group.

The importance of the invariant is the following theorem.

Theorem 8.2 (Reid–Shepherd-Barron–Tai criterion, [31])**.** *Let Y be a smooth k-variety, let G be a finite subgroup of the group of k-automorphisms of Y, and suppose that G acts without quasi-reflections. Then the quotient variety $X = Y//G$ is terminal (resp. canonical) iff for every cyclic subgroup $\Gamma \subset G$ and every closed point $x \in Y^\Gamma$, the invariant of the Zariski tangent space to Y at x satisfies*

$$\alpha(T_xY) > 1, \quad \textit{resp.} \quad \alpha(T_xY) \geq 1. \tag{91}$$

Corollary 8.3. *Let $\mathcal{X}$ be a smooth, connected Deligne-Mumford stack over K. Denote by $p : \mathcal{X} \to X$ the coarse moduli space, and suppose that p is an isomorphism over the complement of a closed subset of codimension ≥ 2. Then X is terminal (resp. canonical) iff for every geometric point x of $\mathcal{X}$, and for every cyclic subgroup Γ of the stabilizer group of x, the invariant of the Zariski tangent space to $\mathcal{X}$ at x satisfies*

$$\alpha(T_x\mathcal{X}) > 1, \quad \textit{resp.} \quad \alpha(T_xY) \geq 1. \tag{92}$$

Proof. This is just a rewording of Theorem 8.2 into the language of Deligne-Mumford stacks. ☐

Corollary 8.4. *Let $\mathcal{X}$ be a smooth, connected Deligne-Mumford stack over K. Let $f : \mathcal{Y} \to \mathcal{X}$ be a smooth, representable 1-morphism of Deligne-Mumford stacks. Suppose that the morphism $p : \mathcal{X} \to X$ is an isomorphism away from codimension 2 and that X is terminal (resp. canonical). Then the map to the coarse moduli space of $\mathcal{Y}$, say $q : \mathcal{Y} \to Y$, is an isomorphism away from codimension 2 and Y is terminal (resp. canonical).*

Proof. Let $U \subset \mathcal{X}$ denote the maximal open substack over which p is an isomorphism. Then $\mathcal{X} - U$ has codimension at least 2 in $\mathcal{X}$. Since f is smooth, in particular it is flat. Therefore $f^{-1}(\mathcal{X} - U)$ has codimension at least 2 in $\mathcal{Y}$. And $f^{-1}(U)$ is a scheme because f is representable. Therefore q is an isomorphism when restricted to $f^{-1}(U)$, which shows that q is an isomorphism away from codimension 2.

Next, apply Corollary 8.3. Let y be a geometric point of $\mathcal{Y}$ and let $x = f(y)$. Because f is representable, the homomorphism from the stabilizer group of y to the stabilizer group of x is injective. So a cyclic subgroup Γ of the stabilizer group of y is also a cyclic subgroup of the stabilizer group of x. By Corollary 8.3, the invariant of $T_x\mathcal{X}$ as a Γ-representation is greater than 1 (resp. at least 1). Since f is smooth, the differential $df : T_y\mathcal{Y} \to T_x\mathcal{X}$ is surjective. Therefore the invariant of $T_y\mathcal{Y}$ is greater than or equal to the invariant of $T_x\mathcal{X}$. Applying Corollary 8.3 one more time, Y is terminal (resp. canonical). ☐

Remark 8.5. Unfortunately, for *nice* representable 1-morphisms $f : \mathcal{Y} \to \mathcal{X}$ of smooth Deligne-Mumford stacks that are not smooth, Corollary 8.4 often fails. For instance, if $\mathcal{Z} \subset \mathcal{X}$ is a Zariski closed substack that is smooth and $f : \mathcal{Y} \to \mathcal{X}$ is the blowing up of $\mathcal{X}$ along $\mathcal{Z}$, it can happen that $p : \mathcal{X} \to X$ is an isomorphism away from codimension 2, that $q : \mathcal{Y} \to Y$ is an isomorphism away from codimension 2, that X is terminal (resp. canonical), but Y is not terminal (resp. canonical). For instance, consider the action of the group of third roots of unity μ_3 on affine 4-space $\mathbb{A}^4$ by $\omega \cdot (X_1, X_2, X_3, X_4) = (\omega X_1, \omega X_2, \omega X_3, \omega X_4)$. Let $Z \subset \mathbb{A}^4$ be the variety associated to the invariant ideal $\langle X_1, X_2, X_3 \rangle$. Let $Y \to \mathbb{A}^4$ denote the blowing up along Z. Then $f : [Y/\mu_3] \to [X/\mu_3]$ is a 1-morphism of smooth Deligne-Mumford stacks satisfying the hypotheses above and $X//\mu_3$ is terminal. But $Y//\mu_3$ is not even canonical.

Let Γ be a finite cyclic group of order r and let $\Delta \subset \Gamma$ be a subgroup of index s. Let $\gamma : \Gamma \to \mathbb{G}_m$ be a generator for the character group of Γ. The restriction of γ to Δ is a generator for the character group of Δ. The following lemma is a rewording of the argument on pp. 33–34 of [14].

Lemma 8.6. [14, pp. 33–34] *Let V be a finite-dimensional representation of Δ and let $V \otimes_{K[\Delta]} K[\Gamma]$ be the induced Γ-representation. The relation between the invariant of $V \otimes_{K[\Delta]} K[\Gamma]$ as a Γ-representation and the invariant of V as a Δ-representation is,*

$$\alpha_{\Gamma,\gamma}\left(V \otimes_{K[\Delta]} K[\Gamma]\right) = \alpha_{\Delta,\gamma|_\Delta}(V) + \frac{s-1}{2} dim_K(V). \tag{93}$$

Proof. Each side of the equation is additive in V, therefore it suffices to consider the case that V is an irreducible representation, i.e. a character $V = L_{\gamma|_\Delta^l}$ for some integer $l = 0, \dots \frac{r}{s} - 1$. Let ϕ be a generator for Γ, so that ϕ^s is a generator for Δ. Let $\overline{\epsilon}$ be a nonzero element of V. For each integer $j = 0, \dots, s-1$, denote $m = -l - \frac{j \cdot r}{s}$ and define the element $\epsilon_j \in V \otimes_{K[\Delta]} K[\Gamma]$ to be,

$$\epsilon_j = \sum_{i=0}^{s-1} \gamma^m(\phi^i)\overline{\epsilon} \otimes \phi^i. \tag{94}$$

It is trivial to compute that $\epsilon_j \cdot \phi = \epsilon_j \cdot \gamma^{-m}(\phi)$. So ϵ_j spans an irreducible subrepresentation of $V \otimes K[\Delta]K[\Gamma]$ that is isomorphic to $L_{\gamma^{-m}}$.

This gives s different irreducible subrepresentations of $V \otimes_{K[\Delta]} K[\Gamma]$, which is also the dimension as a K-vector space. So there is an irreducible decomposition,

$$V \otimes_{K[\Delta]} K[\Gamma] \cong \bigoplus_{j=0}^{s-1} L_{\gamma^{l+j\cdot\frac{r}{s}}}. \tag{95}$$

It follows that the invariant of $V \otimes_{K[\Delta]} K[\Gamma]$ as a Γ representation is

$$\begin{aligned} \tfrac{1}{r}\left[l + (l+\tfrac{r}{s}) + \cdots + (l+(s-1)\tfrac{r}{s})\right] = l/\left(\tfrac{r}{s}\right) + \tfrac{s-1}{2} = \\ \alpha_{\Delta,\gamma|_\Delta}(V) + \tfrac{s-1}{2}\dim_K(V). \end{aligned} \tag{96}$$

□

Recall that $\mathcal{M}_{0,0}(\mathbb{P}^n, e) \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is the open substack parametrizing stable maps with irreducible domain, and $M_{0,0}(\mathbb{P}^n, e)$ is the coarse moduli space of $\mathcal{M}_{0,0}(\mathbb{P}^n, e)$.

Proposition 8.7. *Let x be a geometric point of $\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ and let Γ be a subgroup of the stabilizer group of x. Denote $r = \#\Gamma$. The invariant of the Zariski tangent space to $\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ at x equals*

$$\alpha(T_x\mathcal{M}_{0,0}(X,e)) = \frac{e(n+1)}{2}\left(1 - \frac{1}{r}\right) - 1. \tag{97}$$

Except in the cases $(e,n) = (2,1)$ and $(e,n) = (2,2)$, the map $p : \mathcal{M}_{0,0}(\mathbb{P}^n, e) \to M_{0,0}(\mathbb{P}^n, e)$ is an isomorphism away from codimension 2. Except for the cases $(e,n) = (2,1), (2,2), (3,1)$, and $(2,3)$, $M_{0,0}(\mathbb{P}^n, e)$ is terminal. In the the cases $(e,n) = (3,1), (2,3)$, $M_{0,0}(\mathbb{P}^n, e)$ is canonical.

Proof. By the same GL_{n+1}-invariance argument as in the proof of Theorem 7.8, it suffices to prove the result when x is a geometric point of $Y \cap \mathcal{M}_{0,0}(\mathbb{P}^n, e)$. At such a point the Zariski tangent space decomposes as a direct sum of the Zariski tangent space to Y and the normal bundle to Y. The Zariski tangent space to Y further decomposes as the direct sum of the Zariski tangent space to $\mathbb{G}$ and the vertical tangent bundle of $\mathrm{pr}_{\mathbb{G}} : Y \to \mathbb{G}$. And by Lemma 6.8, the normal bundle to Y is a direct sum of $n-1$ copies of $\mathcal{R}^\vee$. What is needed is to compute the invariants (with respect to some ζ) of the vertical tangent bundle of $\mathrm{pr}_{\mathbb{G}}$ and of $\mathcal{R}^\vee$.

The vertical tangent bundle of $\mathrm{pr}_{\mathbb{G}}$ is the same as the tangent bundle of $\mathcal{M}_{0,0}(\mathbb{P}^1, e)$, so suppose now that $n = 1$. Let the geometric point x parametrize a stable map $f : C \to \mathbb{P}^1$. Choose a generator for Γ, which will be an automorphism $\phi : C \to C$ such that $f \circ \phi = f$ and such that $\phi^s = \mathrm{Id}$ iff r divides s. It is easy to show that, up to a choice of homogeneous coordinates, $\phi : C \to C$ is just the isomorphism $[X_0 : X_1] \mapsto [X_0 : \xi X_1]$ for some primitive r^{th} root of unity.

Denote by $g : C \to C_0$ the quotient of C by ϕ and let $h : C_0 \to \mathbb{P}^1$ be the unique morphism such that $f = h \circ g$. The Zariski tangent space to $\mathcal{M}_{0,0}(\mathbb{P}^1, e)$ is just the vector space of global sections of the torsion sheaf $f^*T_{\mathbb{P}^1}/T_C$. And this fits into an exact sequence,

$$0 \longrightarrow g^*T_{C_0}/T_C \longrightarrow g^*h^*T_{\mathbb{P}^1}/T_C \longrightarrow g^*\left(h^*T_{\mathbb{P}^1}/T_{C_0}\right) \longrightarrow 0. \tag{98}$$

Now, as a representation of Γ, $g^*\left(h^*T_{\mathbb{P}^1}/T_{C_0}\right)$ is isomorphic to the tensor product $(h^*T_{\mathbb{P}^1}/T_{C_0}) \otimes_K K[\Gamma]$, where the first factor is a trivial representation. In particular, the invariant with respect to any generator ζ is just,

$$\alpha_\zeta(g^*(h^*T_{\mathbb{P}^1}/T_{C_0})) = 2(\frac{e}{r} - 1) \cdot \left(\frac{0}{r} + \frac{1}{r} + \cdots + \frac{r-1}{r}\right) = (e-r)\left(1 - \frac{1}{r}\right). \tag{99}$$

By direct computation, as a representation of Γ, $g^*T_{C_0}/T_C$ is isomorphic to

$$g^*T_{C_0}/T_C \cong L_{\xi^0}^{\oplus 2} \oplus L_{\xi^1}^{\oplus 1} \oplus L_{\xi^{r-1}}^{\oplus 1} \oplus \bigoplus_{i=1}^{r-2} L_{\xi^i}^{\oplus 2}. \tag{100}$$

It follows that the invariant with respect to any generator ζ is just,

$$\alpha_\zeta(g^*T_{C_0}/T_C) = 2\left(\frac{0}{r} + \frac{1}{r} + \cdots + \frac{r-1}{r}\right) - \left(\frac{i}{r} + \frac{r-i}{r}\right) = r\left(1 - \frac{1}{r}\right) - 1. \tag{101}$$

Here i is the unique integer such that $\{\xi^1, \xi^{r-1}\} = \{\zeta^i, \zeta^{r-i}\}$. Summing up, with respect to any generator ζ,

$$\alpha_\zeta(T_x\mathcal{M}_{0,0}(\mathbb{P}^1, e)) = e\left(1 - \frac{1}{r}\right) - 1. \tag{102}$$

Next consider the invariant of $\mathcal{R}|_x$. For clarity, denote this vector space by $\mathcal{R}_f$. Each of $g : C \to C_0$ and $h : C_0 \to \mathbb{P}^1$ is also a stable map of a genus 0 curve to $\mathbb{P}^1$. So each of these also has a canonically associated vector space $\mathcal{R}_g$, respectively $\mathcal{R}_h$. As Γ representations, $\mathcal{R}_g$ is just $K[\Gamma]/K1$ (by direct computation), and $\mathcal{R}_h$ is a trivial representation of dimension $\frac{e}{r} - 1$. It is easy to see that the relationship between these spaces is that $\mathcal{R}_f$ is isomorphic as a Γ-representation to $\mathcal{R}_h \otimes_K \mathcal{R}_g \oplus \mathcal{R}_g \oplus \mathcal{R}_h$. Therefore the invariant of $\mathcal{R}_f$ is

$$\alpha_\zeta(\mathcal{R}_f) = \frac{e}{2}\left(1 - \frac{1}{r}\right). \tag{103}$$

This is also the invariant of $\mathcal{R}_f^\vee$.

As mentioned above, the normal bundle of Y at x is isomorphic as a Γ-representation to a direct sum of $n-1$ copies of $\mathcal{R}_f^\vee$. And the vertical tangent space to $\mathrm{pr}_{\mathcal{G}}$ is the same as the tangent space to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$. Therefore,

$$\alpha_\zeta(T_x\mathcal{M}_{0,0}(\mathbb{P}^n, e)) = \frac{(n-1)e}{2}\left(1 - \frac{1}{r}\right) + e\left(1 - \frac{1}{r}\right) - 1 = \frac{(n+1)e}{2}\left(1 - \frac{1}{r}\right) - 1. \tag{104}$$

For $(e, n) \neq (2,1), (2,2)$, the invariant is at least 1, proving that the stabilizer group of x acts without quasi-reflections and the coarse moduli space has canonical singularities. Moreover, except in the extra cases $(e, n) = (3,1), (2,3)$, the invariant is actually larger than 1, proving that the coarse moduli space has terminal singularities. □

Remark 8.8. In case $e = 2$, $n = 1$, every geometric point of $\mathcal{M}_{0,0}(\mathbb{P}^1, 2)$ has nontrivial stabilizer. In fact the coarse moduli space $M_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to the complement of a smooth plane conic in $\mathbb{P}^2$ (via the *branch morphism*, c.f. [8]), and $p : \mathcal{M}_{0,0}(\mathbb{P}^1, 2) \to M_{0,0}(\mathbb{P}^1, 2)$ is a $\mathbb{Z}/2\mathbb{Z}$-gerbe. In case $e = 2$, $n = 2$, the coarse moduli space $M_{0,0}(\mathbb{P}^2, 2)$ is smooth and is isomorphic to an open subset of the blowing up of $\mathbb{P}^5$ along a Veronese surface (the open subset is the complement of the proper transform of the discriminant hypersurface). In this case the morphism $p : \mathcal{M}_{0,0}(\mathbb{P}^2, 2) \to M_{0,0}(\mathbb{P}^2, 2)$ is an isomorphism on the complement of the exceptional divisor, and over the exceptional divisor it is a $\mathbb{Z}/2\mathbb{Z}$-gerbe.

Lemma 8.9. *Let $(e, n) \neq (2,1), (2,2)$, be a pair of positive integers. If at least one of e and n is odd, then $M_{0,0}(\mathbb{P}^n, e)$ is Gorenstein. If both e and n are even, then $M_{0,0}(\mathbb{P}^n, e)$ is not Gorenstein.*

Proof. Since $(e, n) \neq (2,1), (2,2)$, $p : \mathcal{M}_{0,0}(\mathbb{P}^n, e) \to M_{0,0}(\mathbb{P}^n, e)$ is an isomorphism away from codimension 2. It follows from [24, Prop. 5.75] that the dualizing sheaf of $M_{0,0}(\mathbb{P}^n, e)$ is the pushforward by p_* of the dualizing sheaf of $\mathcal{M}_{0,0}(\mathbb{P}^n, e)$. Given a geometric point x of $\mathcal{M}_{0,0}(\mathbb{P}^n, e)$, the dualizing sheaf of $M_{0,0}(\mathbb{P}^n, e)$ is invertible at $p(x)$ iff there exists a section of the dualizing sheaf near $p(x)$ whose pullback to $\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ is non-zero at x. Such a section corresponds to a nonzero element in the one-dimensional vector space $\det(T_x\mathcal{M}_{0,0}(\mathbb{P}^n, e))^\vee$ that is invariant under the action of the stabilizer group of x. Therefore $M_{0,0}(\mathbb{P}^n, e)$ is Gorenstein iff for every geometric point x of $\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ and for every cyclic subgroup Γ of the stabilizer group of x, the induced character $\det(T_x\mathcal{M}_{0,0}(\mathbb{P}^n, e))$ is trivial.

As in the proof of Proposition 8.7, it suffices to compute the character for geometric points x of $Y \cap \mathcal{M}_{0,0}(\mathbb{P}^n, e)$. In the proof of Proposition 8.7 the characters of all relevant Γ-representations were computed. The character of $g^*\left(h^*T_{\mathbb{P}^1}/T_{C_0}\right)$ is just $\det(K[\Gamma])^{\otimes 2(\frac{e}{r}-1)}$ where $2(\frac{e}{r} - 1)$ is the dimension of $h^*T_{\mathbb{P}^1}/T_{C_0}$. Similarly, the character of $T_x\mathcal{M}_{\mathbb{P}^1,e}()$ is $\det(K[\Gamma])^{\otimes 2}$ (the missing L_{ξ^1} and $L_{\xi^{r-1}}$ factors tensor to give a trivial character). Finally, the character of $\mathcal{R}_g$ is $\det(K[\Gamma])$ and the character of $\mathcal{R}_f \cong (\mathcal{R}_h \otimes_K \mathcal{R}_g) \oplus (\mathcal{R}_h) \oplus (\mathcal{R}_g)$ is $\det(K[\Gamma])^{\otimes \frac{e}{r}}$. Altogether, the character of $T_x\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ is $\det(K[\Gamma])^{\otimes (n+1)\frac{e}{r}}$.

If r is odd, then the character $\det(K[\Gamma])$ is trivial so that the character of $T_x\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ is trivial. But if r is even, the character $\det(K[\Gamma])$ equals $L_{\zeta^{\frac{r}{2}}}$ for any generator ζ of the character group of Γ. This is a nontrivial character whose square is trivial. For r even, the character of $T_x\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ is nontrivial iff $\frac{e}{r}$ is odd and $n+1$ is odd. Therefore if e is odd or n is odd, then the character of $T_X\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ is trivial for every geometric point, i.e. $M_{0,0}(\mathbb{P}^n, e)$ is Gorenstein.

On the other hand, suppose that n and e are both even. Then for any line $L \subset \mathbb{P}^n$ and any reduced degree 2 divisor on L, the cyclic cover $f : C \to L$ of degree e branched over that divisor gives a stable map of degree e whose stabilizer group is cyclic of order $r = e$. Therefore the character of $T_x\mathcal{M}_{0,0}(\mathbb{P}^n, e)$ is nontrivial, i.e. $M_{0,0}(\mathbb{P}^n, e)$ is not Gorenstein. $\square$

Proposition 8.10. *Let $(e, n) \neq (2, 1), (2, 2)$, be a pair of positive integers. If $e \geq 3$ and $n = 1$, the coarse moduli space $\overline{M}_{0,0}(\mathbb{P}^1, e)$ is canonical. If $(e, n) = (2, 3)$, the coarse moduli space $\overline{M}_{0,0}(\mathbb{P}^3, 2)$ is canonical. In all other cases, the coarse moduli space $\overline{M}_{0,0}(\mathbb{P}^n, e)$ is terminal.*

Proof. The proof follows closely the argument on pp. 33–34 of [14]. As in the proof of Proposition 8.7, to prove that $\overline{M}_{0,0}(\mathbb{P}^n, e)$ is canonical (resp. terminal), it suffices to check that for every geometric point x of Y and for every cyclic subgroup Γ of the stabilizer group of x, the invariant of $T_x\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is bigger than 1 (resp. at least 1). Choose some line $\mathbb{P}^1 \subset \mathbb{P}^n$; the geometric point x will belong to the closed substack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e) \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$. The proof that the invariant of $T_x\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is bigger than 1 (resp. at least 1) proceeds by induction on the number δ of nodes of C. The base case is when $\delta = 0$, i.e. x is in $M_{0,0}(\mathbb{P}^n, e)$, and follows from Proposition 8.7. Therefore suppose $\delta > 0$, and, by way of induction, suppose the result has been proved for all points with fewer than δ nodes.

Let ϕ be a generator for Γ and let $\{q, \phi q, \phi^2 q, \ldots, \phi^{s-1}q\}$ be an orbit of Γ on C such that each $\phi^i q$ is a node. Of course, s divides the order of Γ, which is denoted by r. The language of [4] for stable A-graphs is used. Denote by τ the stable A-graph of $f : C \to \mathbb{P}^1$, i.e. τ is the dual graph of C labelled by the degree of f, and denote by $\overline{\mathcal{M}}(\mathbb{P}^1, \tau)$ the corresponding Behrend-Manin moduli stacks (essentially this is the closed substack of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ parametrizing stable maps obtained as specializations of deformations of f that do not smooth the nodes of C). Let $E_0, E_1, \ldots, E_{s-1}$ denote the edges of τ corresponding to the nodes $q, \phi q, \ldots, \phi^{s-1}q$. Let $\psi : \tau \to \sigma$ be the maximal contraction of τ not contracting any of the edges $E_0, \ldots, E_{s-1}$, i.e. σ is the same as the dual graph of a curve obtained by smoothing all the nodes of C except $q, \ldots, \phi^{s-1}q$. Then $f : C \to \mathbb{P}^1$ gives a geometric point of the Behrend-Manin moduli stack $\overline{\mathcal{M}}(\mathbb{P}^1, \sigma)$, i.e. the moduli space of stable maps to $\mathbb{P}^1$ whose dual graph has a contraction to σ.

There is a canonical 1-morphism $\overline{\mathcal{M}}(\mathbb{P}^1, \sigma) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ that is unramified and whose normal sheaf is locally free. Therefore the tangent bundle of $\overline{\mathcal{M}}(\mathbb{P}^1, \sigma)$ at $[f : C \to X]$ is a vector subspace of the tangent bundle of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ at $[f : C \to X]$. Moreover the cokernel, i.e. the normal bundle, is precisely,

$$N_{[f]} = \bigoplus_{i=0}^{s-1} T'_{\phi^i q} \otimes_k T''_{\phi^i q}, \tag{105}$$

where $T'_{\phi^i q}$ and $T''_{\phi^i q}$ are the tangent spaces of the two branches of C at $\phi^i q$ (there isn't any canonical ordering of the two branches; the notation T' and T'' is just for convenience).

Now suppose there exists a nonzero Γ-invariant section $\epsilon \in N_{[f]}$. Then there exists a Γ-invariant section $\widetilde{\epsilon}$ of $T[f]\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ mapping to ϵ. Over an étale neighborhood of the image of $[f]$ in $\overline{M}_{0,0}(\mathbb{P}^1, e)$, the stack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ is a finite group quotient $[M/G]$, where G is the stabilizer group of f and M is a smooth scheme. In particular the invariant locus $M^\Gamma \subset M$ is a smooth closed subscheme whose Zariski tangent space at $[f]$ is the Γ-invariant subspace of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$. In particular there exists a smooth, connected curve $B \subset M^\Gamma$ such that B contains the point $[f]$ and the tangent space to B at $[f]$ equals $\mathrm{span}(\widetilde{\epsilon})$. Since $\widetilde{\epsilon}$ has nonzero image in $N_{[f]}$, the curve B is not contained in the image of $\overline{\mathcal{M}}(\mathbb{P}^1,\sigma)$. So a general point of B parametrizes a stable map with fewer nodes than $f : C \to \mathbb{P}^1$. On the other hand, the invariant of the Zariski tangent space is constant in connected families. So by the induction hypothesis, the invariant of f equals the invariant of a general point of B which is greater than 1 (resp. at least 1).

By the last paragraph, the proof reduces to the case when the Γ-invariant subspace of $\bigoplus_{i=0}^{s-1} T'_{\phi^i q} \otimes T''_{\phi^i q}$ is trivial for every node q of C. Let $\Delta \subset \Gamma$ denote the subgroup generated by ϕ^s. There is an action of Δ on $T'_q \otimes T''_q$, and the Γ-representation $N_{[f]}$ is simply the induced representation $\left(T'_q \otimes T''_q\right) \otimes_{K[\Delta]} K[\Gamma]$. By Lemma 8.6, the invariant of $N_{[f]}$ as a Γ-representation is simply,

$$\alpha_\gamma(N_{[f]}) = l \cdot \frac{s}{r} + \frac{s-1}{2}, \tag{106}$$

where the character $T'_q \otimes T''_q$ of Δ is $\gamma|_\Delta^l$ for $l = 0, \dots, \frac{r}{s} - 1$.

If $s \geq 3$, then already the invariant of $N_{[f]}$ is greater than 1. Thus assume that $s = 1$ or $s = 2$. If $s = 2$, then the invariant of $N_{[f]}$ is $\frac{1}{2} + l \cdot \frac{2}{r}$. If $s = 1$, then the invariant is $l \cdot \frac{1}{r}$. The only possibilities that don't give an invariant larger than 1 are,

(i) every node of C is fixed by ϕ, or
(ii) there is precisely one pair of nodes $q, \phi q$ not fixed by ϕ.

These possibilities are considered in turn.

(i): Suppose first that every node of C is fixed by ϕ. Then every irreducible component of C is stabilized by ϕ (as a set, not pointwise). Since ϕ is nontrivial, there exists an irreducible component C_i of C on which the action of ϕ is nontrivial. Let C_1 be an irreducible component so that the restriction $\phi|_{C_1}$ has maximal order r_1 (i.e. $\phi|_{C_1}^m = \mathrm{Id}$ iff r_1 divides m). The irreducible component C_1 contains at least one node of C and contains no more than two nodes of C since the only automorphism of $\mathbb{P}^1$ fixing three points is the identity. In particular, C_1 is not contracted by f.

Let τ be the stable A-graph of C and let $\psi : \tau \to \sigma$ be the maximal contraction not contracting the edges corresponding to nodes on C_1. There is a morphism $\overline{\mathcal{M}}(\mathbb{P}^1,\sigma) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ and the normal bundle, as mentioned above, is the direct sum over nodes q on C_1 of $T'_q \otimes T''_q$. Let $\xi : \sigma \hookrightarrow \tau'$ be the combinatorial morphism that is the inclusion of the maximal sub-A-graph of τ whose only vertex is v_1, the vertex of the irreducible component of C_1. More simply, τ' is the graph with the single vertex v_1 corresponding to C_1 and with one tail for each node of C contained in C_1 (i.e. either one or two tails depending on whether C_1 contains one or two nodes of C). Let $f_1 : (C_1, q) \to \mathbb{P}^1$ or $f_1 : (C_1, q, q') \to \mathbb{P}^1$ be the marked stable map that is the restriction of f to the irreducible component C_1 marked by the nodes of C contained on C_1.

There is a commutative diagram of 1-morphisms:

$$\begin{array}{ccc} \overline{\mathcal{M}}(\sigma,\mathbb{P}^1) & \longrightarrow & \overline{\mathcal{M}}(\sigma,\mathbb{P}^n) \\ {\scriptstyle \overline{\mathcal{M}}(\xi,\mathbb{P}^1)}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mathcal{M}}(\xi,\mathbb{P}^n)} \\ \overline{\mathcal{M}}(\tau',\mathbb{P}^1) & \longrightarrow & \overline{\mathcal{M}}(\tau',\mathbb{P}^n) \end{array} \tag{107}$$

The horizontal arrows are closed immersions and the vertical arrows are smooth. So the invariant of the tangent space $T_{[f]}\overline{\mathcal{M}}(\sigma,\mathbb{P}^n)$ is greater than or equal to the invariant of the tangent space $T_{[f_1]}\overline{\mathcal{M}}(\tau',\mathbb{P}^n)$. Let e_1 be the degree of $f_1 : C_1 \to \mathbb{P}^1$. As a Γ-representation, $T_{[f_1]}\overline{\mathcal{M}}(\tau',\mathbb{P}^n)$ is the direct sum of $T_{[f_1]}\mathcal{M}_{0,0}(\mathbb{P}^n, e_1)$ with the tangent space $T_q C_1$ (or $T_q C_1 \oplus T_{q'} C_1$ if C_1 contains two nodes). By Proposition 8.7, the invariant of $T_{[f_1]}\mathcal{M}_{0,0}(\mathbb{P}^n, e_1)$ is $\frac{e_1(n+1)}{2}\left(1 - \frac{1}{r_1}\right) - 1$. Except for the four cases $(e_1, n) = (2,1), (2,2), (2,3), (3,2)$, this invariant is already greater than 1, hence the invariant of $T[f]\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is also greater than 1. To finish the proof in case (i), each case $(e_1, n) = (2,1), (2,2), (2,3), (3,2)$ is considered.

(i); $(e_1, n) = (2,3), (3,2)$: If $(e_1, n) = (2,3)$ or $(3,2)$ the invariant $\frac{e_1(n+1)}{2}\left(1 - \frac{1}{r_1}\right) - 1$ equals 1. And then the invariant of T_qC is either $\frac{1}{2}$ for $(2,3)$ or $\frac{1}{3}$ or $\frac{2}{3}$ for $(3,2)$. Therefore the invariant of $\overline{\mathcal{M}}(\tau', \mathbb{P}^n)$ is greater than 1. So the invariant of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is also greater than 1.

(i); $(e_1, n) = (2,2)$: If $(e_1, n) = (2,2)$, the invariant of $T_{[f_1]}\mathcal{M}_{0,0}(\mathbb{P}^2, 2)$ is $\frac{1}{2}$. The invariant of T_qC_1 is also $\frac{1}{2}$. So if there are two nodes on C_1, the invariant is already greater than 1. If there is only one node, so far the invariant only equals 1. But also the invariant of $T'_q \otimes T''_q$, is positive. So the invariant of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is greater than 1.

(i); $(e_1, n) = (2,1)$: Finally, suppose $(e_1, n) = (2,1)$. The invariant of $T_{[f_1]}\mathcal{M}_{0,0}(\mathbb{P}^1, 2)$ is zero. The invariant of T_qC_1 is $\frac{1}{2}$. If there are two nodes on C_1, then the invariant is 1 and then the contributions of $T'_q \otimes T''_q$ and $T'_{q'} \otimes T''_{q'}$ will make the total invariant of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ greater than 1. Therefore assume there is only one node. Then the invariant of $T_{[f_1]}\overline{\mathcal{M}}(\mathbb{P}^1, \sigma)$ equals $\frac{1}{2}$. Consider the Γ-representation $T'_q \otimes T''_q$. Let C_2 denote the irreducible component of C that intersects C_1 at q. Because the order r_1 is the maximal among all orders of $\phi|_{C_i}$, either ϕ acts trivially on C_2 or the order of $\phi|_{C_2}$ is 2. In the second case, both T'_q and T''_q give characters of Γ that are $\gamma^{\frac{r}{2}}$, the unique character of order 2. So the tensor product is the trivial character. This violates the assumption that for every node there are no non-zero Γ-invariant sections of $\bigoplus_{i=0}^{s-1} T'_{\phi^i q} \otimes T''_{\phi^i q}$. Therefore ϕ acts trivially on C_2 and the invariant of $T'_q \otimes T''_q$ is $\frac{1}{2}$. So the invariant of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ is at least $\frac{1}{2} + \frac{1}{2} = 1$. Unfortunately, this is the best one can do – it is quite easy to write down a degree $e \geq 3$ cover of $\mathbb{P}^1$ with reducible domain where the invariant equals 1. So for $n = 1$ and $e \geq 3$, the conclusion is that the invariant is ≥ 1, i.e. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$ has canonical singularities (of course we still have to dispense with the case that there are two nodes $q, \phi q$ interchanged by ϕ!).

(ii): This finishes the analysis when ϕ fixes all nodes. Next suppose that there is exactly one pair of nodes $\{q, \phi q\}$ which are not fixed by ϕ. The node q disconnects C into a union of two connected subcurves, denoted D_q and C_2. Let C_2 denote the subcurve containing ϕq. The node ϕq disconnects C_2 into a union of two connected subcurves, denoted $D_{\phi q}$ and C_1. Let C_1 denote the subcurve containing q. So C_1 is the maximal connected subcurve of C containing q and ϕq on which both q and ϕq are nonsingular points. Observe that $\phi(D_q) = D_{\phi q}$, $\phi(D_{\phi q}) = D_q$ and both D_q and $D_{\phi q}$ are smooth. Let $C_q \subset C_1$ be the irreducible component containing q and let $C_{\phi q} \subset C_1$ be the irreducible component containing ϕq. Observe that $\phi(C_q) = C_{\phi q}$ and $\phi(C_{\phi q}) = C_q$. There are two possibilities depending on whether C_q (and thus $C_{\phi q}$) is contracted by f or not.

(ii); C_q is contracted: Suppose that C_q is contracted by f and suppose that $C_q \neq C_{\phi q}$. Then C_q contains at least three nodes, q and two other nodes. By assumption, each of the two other nodes is fixed by ϕ. Also $\phi(C_q) = \phi(C_{\phi q})$. Since C_q is not equal to $C_{\phi q}$, then $C_q \cap C_{\phi q}$ is at most one node. But then the second of the other nodes cannot be fixed by ϕ. This is a contradiction. Therefore if C_q is contracted by f, then $C_q = C_{\phi q}$ and C_q contains at least one other node q' of C_1 that is one of the two fixed points of $\phi|_{C_q}$. Also, since $\phi|_{C_q}$ contains the orbit $\{q, \phi q\}$ of order 2, $\phi|_{C_q}$ has order 2.

The node q' disconnects C_1 into C_q and a connected subcurve C_0. Suppose that $\phi|_{C_0}$ is the identity. Then the Γ-representation $T'_r \otimes T''_r$ has invariant $\frac{1}{2}$. Combined with the invariant $\frac{1}{2} + \frac{l \cdot r}{2}$ coming from the nodes $\{q, \phi q\}$, the total invariant is greater than 1 so that the invariant of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is greater than 1. Next suppose that $\phi|_{C_0}$ is not the identity. Then by the same analysis as in (i), the invariant coming from C_0 is at least $\frac{1}{2}$ (except when $(e_1, n) = (2,1)$, in which case the invariant is at least 1). Combined with the invariant $\frac{1}{2} + \frac{l \cdot r}{2}$ coming from the nodes $\{q, \phi q\}$, the total invariant is greater than 1 so that the invariant of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is greater than 1.

(ii); C_q is not contracted: The final case is when f does not contract C_q. By the same analysis as above, D_q and $D_{\phi q}$ are irreducible and are not contracted by f. Let τ be the stable A-graph of f and let $\psi : \tau \to \sigma$ be the maximal contraction not contracting the edges corresponding to the nodes q and ϕq. Then σ has three vertices: v_0 corresponding to the connected subcurve D_q, v_1 corresponding to the connected subcurve C_1, and v_2 corresponding to the connected subcurve

$D_{\phi q}$. The stable map $f : C \to \mathbb{P}^1$ determines a point of the Behrend-Manin stack, $\overline{\mathcal{M}}(\mathbb{P}^n, \sigma)$. The Zariski tangent space of $\overline{\mathcal{M}}(\mathbb{P}^n, \sigma)$ at $[f]$ is a Γ-sub-representation of the Zariski tangent space of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ and the cokernel is $(T'_q \otimes T''_q) \oplus (T'_{\phi q} \otimes T''_{\phi q})$.

Let $\xi : \sigma \hookrightarrow \tau'$ be the maximal *disconnected* subgraph of σ containing the vertices v_0 and v_2, i.e. the stable A-graph with vertices v_0 and v_2 and one flag attached to each vertex corresponding to the marked point q and ϕq respectively. There is an associated 1-morphism $\overline{\mathcal{M}}(\mathbb{P}^n, \sigma) \to \overline{\mathcal{M}}(\mathbb{P}^n, \tau')$. Because C_1 is not contracted by f, this 1-morphism is smooth. In particular, the invariant of $T_{[f]}\overline{\mathcal{M}}(\mathbb{P}^n, \sigma)$ is at least as large as the invariant of $T_{[f]}\overline{\mathcal{M}}(\mathbb{P}^n, \tau')$. Of course $\overline{\mathcal{M}}(\mathbb{P}^n, \tau')$ is simply a product of the two factors from v_0 and v_2, $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, e_1) \times \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, e_1)$, where e_1 is the degree of $f|_{D_q} : D_q \to \mathbb{P}^n$. The Zariski tangent space is then a direct sum of the two factors from v_0 and v_2. The automorphism ϕ permutes the two factors and ϕ^2 acts as an automorphism of each factor.

Let γ be a generator for the character group of $\Gamma = \langle \phi \rangle$. Then also γ is a generator for the character group of $\langle \phi^2 \rangle$. The rank of $T\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, e_1)$ equals $(n+1)e_1 + (n-3) + 1$. Consider the invariant α'_γ of this $\langle \phi^2 \rangle$-representation with respect to γ. One contribution comes from the marked point q; this contribution is positive. So the invariant is positive. By Lemma 8.6, the invariant of $T\overline{\mathcal{M}}(\mathbb{P}^n, \tau')$ as a Γ-representation with respect to γ is,

$$\alpha_\gamma(T\overline{\mathcal{M}}(\mathbb{P}^n, \tau')) \geq \alpha'_\gamma + ((n+1)e_1 + (n-3) + 1)\,\frac{1}{2}. \tag{108}$$

The right-hand-side of the equation is a minimum when $n = 1$ and $e_1 = 1$, in which case it is still larger than $\frac{1}{2}$ (since α'_γ is positive). So the invariant of $\overline{\mathcal{M}}(\mathbb{P}^n, \sigma)$ is larger than $\frac{1}{2}$. And the invariant of $(T'_q \otimes T''_q) \oplus \left(T'_{\phi q} \otimes T''_{\phi q}\right)$ is larger than $\frac{1}{2}$. Therefore the invariant of $T_{[f]}\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is larger than 1. $\square$

Next consider the coarse moduli space of C_d. As in Section 7, let $(\varrho : M \to \mathbb{P}^1, \iota : Y \times \mathbb{P}^1 \to M, B_y \to M, \mathbb{P}(N \oplus \mathbf{1}) \to M)$ denote the deformation to the normal cone associated to the inclusion $Y \hookrightarrow B$, where $B = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$. Let $\widetilde{\pi}_d : \widetilde{C}_d \to M^o$ denote the projective Abelian cone.

Lemma 8.11. *If $e \geq 3$ and if $d + e \leq n$, then the map to the coarse moduli space*

$$\widetilde{C}_d \times_{M^\circ} N \to (\widetilde{C}_d \times_{M^\circ} N)_{coarse} \tag{109}$$

is an isomorphism away from codimension 2, and the coarse moduli space $(\widetilde{C}_d \times_{M^\circ} N)_{coarse}$ is a normal, $\mathbb{Q}$-Gorenstein variety with only canonical singularities.

Proof. Of course $\widetilde{C}_d \times_{M^\circ} N$ is normal and Gorenstein, therefore the coarse moduli space is normal and $\mathbb{Q}$-Gorenstein. To see that the coarse moduli map is an isomorphism away from codimension 2 and that the coarse moduli space is canonical, recall the resolution of the Deligne-Mumford stack, $\widetilde{C}_d \times_{M^\circ} N$. The resolution is constructed as follows. First of all, the projection $N \to Y$ is $M^{(0)}(Y, \mathcal{R}^\vee, \mathrm{pr}_{\mathbb{G}}^* T)$. By Proposition 3.12, there is a 1-morphism of stacks $u^{e-1,0} : M^{(e-1)} \to N$ such that $M^{(e-1)} \to Y$ is representable and smooth. There is a projective bundle C'_d over $M^{(e-1)}$ and a morphism $C'_d \to \widetilde{C}_d \times_{M^\circ} N$ that is a resolution of singularities. Observe that also $C'_d \to Y$ is representable and smooth. By Corollary 3.14, the relative canonical divisor of $C'_d \to \widetilde{C}_d \times_{M^\circ} N$ is effective.

Now consider the coarse moduli spaces $Y \to Y_{\text{coarse}}$, $C'_d \to C'_{d,\text{coarse}}$ and $\widetilde{C}_d \times_{M^\circ} N \to (\widetilde{C}_d \times_{M^\circ} N)_{\text{coarse}}$. By Proposition 8.10, the morphism $Y \to Y_{\text{coarse}}$ is an isomorphism away from codimension 2 and Y_{coarse} is canonical (this corresponds to the case $(e, 1)$ where $e \geq 3$). By Corollary 8.4, also $C'_d \to C'_{d,\text{coarse}}$ is an isomorphism away from codimension 2 and $C'_{d,\text{coarse}}$ is canonical. There is an open substack $U \subset \widetilde{C}_d \times_{M^\circ} N$ such that $C'_d \to \widetilde{C}_d \times_{M^\circ} N$ is an isomorphism over U and such that the complement of U has codimension at least 2. And the morphism $U \to U_{\text{coarse}}$ is an isomorphism away from codimension 2. Therefore also the morphism

$$\widetilde{C}_d \times_{M^\circ} N \to (\widetilde{C}_d \times_{M^\circ} N)_{\text{coarse}} \tag{110}$$

is an isomorphism away from codimension 2.

Because $C'_d \to C'_{d,\text{coarse}}$ is an isomorphism away from codimension 2, the relative canonical divisor of $C'_{d,\text{coarse}} \to (\widetilde{C}_d \times_{M^\circ} N)_{\text{coarse}}$ equals the image of the canonical divisor of $C'_d \to \widetilde{C}_d \times_{M^\circ} N$. Therefore the relative canonical divisor is effective. But also $C'_{d,\text{coarse}}$ is canonical. It follows that also $(\widetilde{C}_d \times_{M^\circ} N)_{\text{coarse}}$ is canonical. This finishes the proof. $\square$

Remark 8.12. If $e = 2$ and $d + 3 \le n$, the second part of the lemma also holds by a slightly more *ad hoc* argument (note that this inequality is worse than the inequality $d+e \le n$). In this case Y is a $\mathbb{Z}/2\mathbb{Z}$-gerbe over its coarse moduli space. And $N = M^{(0)}$ is a vector bundle of $1 \times (n-1)$-matrices. So the only stratum to blow up to form $M^{(1)}$ is the zero section. Doing this, the $\mathbb{Z}/2\mathbb{Z}$-invariant locus of $M^{(1)}$ is the whole exceptional divisor E. A simple computation shows that the 1-morphism $C'_d \to M^{(1)}$ preserves all stabilizer groups of geometric points (i.e. the induced homomorphisms of stabilizer groups are isomorphisms). Therefore the morphism $C'_d \to C'_{d,\text{coarse}}$ is a morphism to a smooth variety ramified of ramification index 1 along the preimage of E, i.e. it does not satisfy the first part of the lemma. However, it is straightforward to compute that the relative canonical divisor of $C'_{d,\text{coarse}} \to (\widetilde{C}_d \times_{M^\circ} N)_{\text{coarse}}$ is $\frac{n-3-d}{2} E_{\text{coarse}}$. Therefore $(\widetilde{C}_d \times_{M^\circ} N)_{\text{coarse}}$ is canonical when $d + 3 \le n$.

9. Conjectures about $\overline{M}_{0,0}(X, e)$

Conjecture 9.1 (Inversion of Adjunction, Conj. 7.3 [21])**.** Let X be a normal variety, S a normal Cartier divisor and $B = \sum b_i B_i$ a $\mathbb{Q}$-divisor. Assume that $K_X + S + B$ is $\mathbb{Q}$-Cartier. Then

$$\text{totaldiscrep}(S, B|S) = \text{discrep}(\text{Center} \cap S \neq \emptyset, X, S + B), \tag{111}$$

where the notation on the right means that we compute the discrepancy using only those divisors whose center on X intersects S.

Conjecture 9.2. If $e \ge 3$ and $d+e \le n$, the coarse moduli space $C_{d,\text{coarse}}$ is a normal, $\mathbb{Q}$-Gorenstein variety with only canonical singularities. If $e = 2$ and $d + 3 \le n$, the coarse moduli space $C_{d,\text{coarse}}$ is a normal, $\mathbb{Q}$-Gorenstein variety with only canonical singularities.

Conjecture 9.3. If $e \ge 3$, $d + e \le n$, and if $X \subset \mathbb{P}^n$ is a general hypersurface of degree d, then $\overline{M}_{0,0}(X, e)$ is a normal, $\mathbb{Q}$-Gorenstein variety with only canonical singularities. If $e = 2$, $d + 3 \le n$ and if $X \subset \mathbb{P}^n$ is a general hypersurface of degree d, then $\overline{M}_{0,0}(X, 2)$ is a normal, $\mathbb{Q}$-Gorenstein variety with only canonical singularities.

Proposition 9.4. *Suppose that $d + e \le n$.*

(i) *If $e \ge 2$ then the coarse moduli space $(C_d)_{coarse}$ is normal, $\mathbb{Q}$-Gorenstein and Kawamata log terminal.*

(ii) *If $e \ge 3$ and $d + e \le n$, then the coarse moduli map*

$$C_d \to (C_d)_{coarse} \tag{112}$$

is an isomorphism away from codimension 2.

(iii) *Conjecture 9.1 implies Conjecture 9.2.*

Proof. **(i)**: First of all, the stack C_d is normal and Gorenstein with canonical singularities by Theorem 7.8. So the coarse moduli space $C_{d,\text{coarse}}$ is normal and $\mathbb{Q}$-Gorenstein. And by [21, Prop. 3.16], $C_{d,\text{coarse}}$ is Kawamata log terminal.

(ii): Denote by $Z \subset C_d$ the closed substack where the map $C_d \to (C_d)_{\text{coarse}}$ is not an isomorphism. Denote by $\widetilde{Z} \subset \widetilde{C}_d$ the closed substack where the map $\widetilde{C}_d \to (\widetilde{C}_d)_{\text{coarse}}$ is not an isomorphism.

Clearly $\widetilde{Z} \cap \rho^{-1}(\mathbb{A}^1) = Z \times \mathbb{A}^1$. Of course Z is invariant under the action of GL_{n+1}. By the same argument as in the proof of Theorem 7.8, every irreducible component of $\widetilde{Z}$ has non-empty intersection with the fiber over ∞, i.e. $\widetilde{C}_d \times_{M^o} N$. By Lemma 8.11, if $e \geq 3$, then every irreducible component of $\widetilde{Z} \cap (\widetilde{C}_d \times_{M^o} N)$ has codimension at least 2. By Krull's Hauptidealsatz, every irreducible component of $\widetilde{Z}$ has codimension at least 2. Therefore every irreducible component of $Z \subset C_d$ has codimension at least 2.

(iii): As in the proof of Theorem 7.8, let $W \subset \overline{M}_{0,0}(\mathbb{P}^n, e)$ be the largest open substack over which $C_{d,\text{coarse}}$ is canonical. This is a GL_{n+1}-invariant open set, so to prove that W is all of $\overline{M}_{0,0}(\mathbb{P}^n, e)$, it suffices to prove that W contains the image of Y.

Let $\widetilde{C}_d \to M^o$ be as in Section 7. Let $M^o \to M^o_{\text{coarse}}$ and $\widetilde{C}_d \to \widetilde{C}_{d,\text{coarse}}$ be the coarse moduli spaces. Let $W' \subset M^o_{\text{coarse}}$ be the largest open subset over which $\widetilde{C}_{d,\text{coarse}}$ is canonical. Of course $W' \cap \varrho^{-1}(\mathbb{A}^1) = W \times \mathbb{A}^1$. Now by Lemma 8.11, the Cartier divisor $(\widetilde{C}_d \times_{M^o} N)_{\text{coarse}}$ in $\widetilde{C}_{d,\text{coarse}}$ is normal and canonical. Assuming Conjecture 9.1 is true, there is an open subvariety of $\widetilde{C}_{d,\text{coarse}}$ containing $(\widetilde{C}_d \times_{M^o} N)_{\text{coarse}}$ that is canonical, i.e. W' contains $\varrho^{-1}(\infty)$. By the same argument as in the proof of Theorem 7.8, W contains the image of Y, i.e. W is all of $\overline{M}_{0,0}(\mathbb{P}^n, e)$. So Conjecture 9.2 is true. □

Proposition 9.5. *Let d, e and n be positive integers such that $d + e \leq n$. Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d.*

(i) *If $e \geq 2$, the coarse moduli space $\overline{M}_{0,0}(X, e)$ is normal, $\mathbb{Q}$-Gorenstein and Kawamata log terminal.*

(ii) *If $e \geq 3$, the coarse moduli map*
$$\overline{\mathcal{M}}_{0,0}(X, e) \to \overline{M}_{0,0}(X, e) \tag{113}$$
is an isomorphism away from codimension 2.

(iii) *Conjecture 9.2 implies Conjecture 9.3.*

Proof. This is the same argument as in the proof of Corollary 7.9. □

Remark 9.6. In (ii), if $e = 2$ then the coarse moduli map fails to be an isomorphism precisely on the locus $Y \cap \overline{\mathcal{M}}_{0,0}(X, 2)$. By direct computation, for X general this locus has dimension $2n - d - 1$. And $\overline{\mathcal{M}}_{0,0}(X, 2)$ has dimension $3n - d - 2$. Therefore, if $d + 3 \leq n$, then the coarse moduli map $\overline{\mathcal{M}}_{0,0}(X, 2) \to \overline{M}_{0,0}(X, 2)$ is an isomorphism away from codimension 2.

10. The canonical class on $\overline{\mathcal{M}}_{0,r}(X, e)$

Let $X \subset \mathbb{P}^n$ be a complete intersection of c hypersurfaces of degrees $\mathbf{d} = (d_1, \dots, d_c)$. Associated to the inclusion morphism, there is a 1-morphism of Kontsevich moduli spaces $\overline{\mathcal{M}}_{0,r}(X, e) \to \overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$. This 1-morphism is representable and is a closed immersion. The image is the zero locus of a section σ of a locally free sheaf $\mathcal{P}_{\mathbf{d}}$ in the small étale site of $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$.

If σ is a *regular* section, the dualizing sheaf ω' on $\overline{\mathcal{M}}_{0,r}(X, e)$ can be expressed as the pullback from $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ of the tensor product $\omega \otimes \det(\mathcal{P}_{\mathbf{d}})$, where ω is the dualizing sheaf on $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$. Pandharipande computed the $\mathbb{Q}$-Picard group of $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ in [29]. And he computed the $\mathbb{Q}$-divisor class of ω in [28]. The purpose of this section is to compute the class $\det(\mathcal{P}_{\mathbf{d}})$ in terms of the standard generators of the $\mathbb{Q}$-Picard group, and thereby compute the $\mathbb{Q}$-divisor class of ω' in the case that σ is a regular section.

Let $p \colon \mathcal{C} \to \overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ denote the universal curve. Let $f \colon \mathcal{C} \to \mathbb{P}^n$ denote the universal morphism. For each integer d, form the pullback sheaf $f^*\mathcal{O}_{\mathbb{P}^r}(d)$. Define $\mathcal{E}_d$ to be the pushforward $p_*(f^*\mathcal{O}_{\mathbb{P}^r}(d)$. More generally, given an ordered sequence $\mathbf{d} = (d_1, \dots, d_c)$ of integers, define $\mathcal{O}_{\mathbb{P}^n}(\mathbf{d})$ to be $\mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_c)$ and define $\mathcal{P}_{\mathbf{d}}$ to be $\mathcal{P}_{d_1} \oplus \cdots \oplus \mathcal{P}_{d_c}$.

There is a pullback map on global sections,

$$H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)\right) \to H^0\left(\mathcal{C}, f^*\mathcal{O}_{\mathbb{P}^n}(d)\right) \to H^0\left(\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e), \mathcal{P}_d\right). \tag{114}$$

Denote the composite map by f^*. More generally, given an ordered sequence $\mathbf{d}$, there is a pullback map on global sections,

$$f^*: H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathbf{d})\right) \to H^0\left(\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e), \mathcal{P}_{\mathbf{d}}\right). \tag{115}$$

Lemma 10.1. *If $d \geq 0$, then $\mathcal{P}_d$ is a locally free sheaf of rank $de+1$ in the small étale site of $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ and $R^i p_*\left(f^*\mathcal{O}_{\mathbb{P}^n}(d)\right)$ is zero for $i > 0$. More generally, if $\boldsymbol{d} = (d_1, \dots, d_c)$ and $d_1, \dots, d_c \geq 0$, then $\mathcal{P}_{\boldsymbol{d}}$ is a locally free sheaf of rank $|\boldsymbol{d}|e + c$, where $|\mathbf{d}| = d_1 + \cdots + d_c$ and $R^i p_*\left(f^*\mathcal{O}_{\mathbb{P}^n}(\boldsymbol{d})\right)$ is zero for $i > 0$.*

Proof. This has been proved in other places, in particular it is proved as part of the the proof of [15, Lemma 4.5]. ☐

Now let $d_1, \dots, d_c$ be a sequence of positive integers, and let $s = (s_1, \dots, s_c)$ be a global section of $H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathbf{d})\right)$. Let $X \subset \mathbb{P}^n$ be the zero locus of s. Let σ denote the pullback section $f^*s \in H^0\left(\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e), \mathcal{P}_{\mathbf{d}}\right)$.

Lemma 10.2. *The zero locus of σ as a closed substack of $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ is the image of the closed immersion $\overline{\mathcal{M}}_{0,r}(X, e) \to \overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$.*

Proof. This is also proved as part of the proof of [15, Lemma 4.5]. ☐

Since $\mathcal{P}_{\mathbf{d}}$ is a locally free sheaf of rank $e|\mathbf{d}| + c$, and since $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ is smooth, σ is a regular section iff the codimension of $\overline{\mathcal{M}}_{0,r}(X, e)$ in $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ equals $e|\mathbf{d}| + c$. In this case, the generalized version of the adjunction theorem proves the dualizing sheaf ω' on $\overline{\mathcal{M}}_{0,r}(X, e)$ is the pullback of the sheaf $\omega \otimes \det(\mathcal{P}_{\mathbf{d}})$, where ω is the dualizing sheaf on $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$.

In [29], Pandharipande described the $\mathbb{Q}$-Picard group $\mathrm{Pic}\left(\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)\right) \otimes \mathbb{Q}$. For simplicity, assume that $n > 1$ and $e > 0$ and also that $(n, e) \neq (2, 2)$. The divisor class $\mathcal{H}$ is defined as the image of a positive generator $h^2 \in \mathrm{CH}^2(\mathbb{P}^n)$ under the composition

$$\mathrm{CH}^2(\mathbb{P}^n) \xrightarrow{f^*} \mathrm{CH}^2(\mathcal{C}) \otimes \mathbb{Q} \xrightarrow{p_*} \mathrm{CH}^1(\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)) \otimes \mathbb{Q}. \tag{116}$$

For each of the r sections $g_i : \overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e) \to \mathcal{C}$, the divisor class $\mathcal{L}_i$ is defined as the image of a positive generator $h \in \mathrm{CH}^1(\mathbb{P}^n)$ under pullback by $f \circ g_i$. Finally, for each *weighted partition* $P = (A \cup B, e_A, e_B)$ of $(\{1, \dots, r\}, e)$, there is the class Δ_P of the corresponding boundary stratum of $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$. A weighted partition is a datum consisting of a partition $A \cup B$ of $\{1, \dots, r\}$, and of a pair of nonnegative integers e_A, e_B such that $e_A + e_B = e$, satisfying the condition $|A| \geq 2$ (resp. $|B| \geq 2$) if $e_A = 0$ (resp. $e_B = 0$). The corresponding boundary stratum is the closure of the locally closed substack parametrizing stable maps whose dual graph is of type $(A \cup B, e_A, e_B)$. Pandharipande's result is that the $\mathbb{Q}$-Picard group is a $\mathbb{Q}$-vector space with basis,

$$\{\mathcal{H}\} \cup \{\mathcal{L}_i | i = 1, \dots, r\} \cup \{\Delta_P | P = (A \cup B, e_A, e_B)\}. \tag{117}$$

In the case that $r = 0$, for $i = 0, \dots, \left[\frac{e}{2}\right]$, denote by $\mathcal{D}_{i,0}$ the $\mathbb{Q}$-divisor class Δ_P where $P = (\emptyset \cup \emptyset, i, e-i)$. And for $r > 0$, for $i = 0, \dots, \left[\frac{e}{2}\right]$ and $j = 0, \dots, r$ denote by $W_{i,j}$ the set of weighted partitions $\{(A \cup B, e_A, e_B) | |A| = j, e_A = i\}$. Denote by $\mathcal{D}_{i,j}$ the $\mathbb{Q}$-divisor class

$$\mathcal{D}_{i,j} = \sum_{P \in W_{i,j}} \Delta_P. \tag{118}$$

In [28], Pandharipande computed the $\mathbb{Q}$-divisor class of the dualizing sheaf ω in terms of the basis above.

Proposition 10.3 (Pandharipande, Prop. 2 [28]). *The dualizing sheaf ω on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ has $\mathbb{Q}$-divisor class,*

$$\omega = \frac{1}{2e}\left[-(n+1)(e+1)\mathcal{H} + \sum_{i=1}^{\left[\frac{e}{2}\right]} \left((n+1)(e-i)i - 4e\right)\mathcal{D}_{i,0}\right]. \tag{119}$$

Proposition 10.4 (Pandharipande, Prop. 3 [28]). *The first Chern class of the dualizing sheaf ω on $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ has $\mathbb{Q}$-divisor class,*

$$C_1(\omega) = \frac{1}{2e^2}\left[-(n+1)(e+1)e + 2r\right]\mathcal{H} - \frac{1}{2e}\sum_{p=1}^{n} \mathcal{L}_p +$$

$$\frac{1}{2e^2}\sum_{i=0}^{\left[\frac{e}{2}\right]}\sum_{j=0}^{r}\left[(n+1)e(e-i)i + 2e^2 j - 4eij + 2ri^2 - 4e^2\right]\mathcal{D}_{i,j}.$$

It remains to compute the $\mathbb{Q}$-divisor class of the first Chern class $C_1(\mathcal{P}_{\mathbf{d}})$. First compute for each integer $d \geq 0$, the first Chern class $C_1(\mathcal{P}_d)$. This computation is an application of the Grothendieck-Riemann-Roch formula [9, Thm. 15.2]. Observe that p is a representable morphism between smooth Deligne-Mumford stacks with projective coarse moduli space. So one can deduce Grothendieck-Riemann-Roch for p from Grothendieck-Riemann-Roch for the coarse moduli spaces using [34]. Alternatively, one can use the Grothendieck-Riemann-Roch theorem of Toen [33].

By Lemma 10.1, the element in K-theory, $Rp_![f^*\mathcal{O}_{\mathbb{P}^n}(d)]$ equals $[\mathcal{P}_d]$. So, by the Grothendieck-Riemann-Roch formula,

$$\mathrm{ch}[\mathcal{P}_d] = p_*\left(f^*\mathrm{ch}[\mathcal{O}_{\mathbb{P}^n}(d)] \cap \mathrm{todd}(p)\right). \tag{120}$$

Denote by $h \in \mathrm{CH}^1(\mathbb{P}^n)$ the first Chern class of $\mathcal{O}_{\mathbb{P}^n}(1)$. Then, up to terms in $\mathrm{CH}^3(\mathbb{P}^n)$,

$$\mathrm{ch}[\mathcal{O}_{\mathbb{P}^n}(d)] = 1 + dh + \frac{d^2}{2}h^2 + \dots \tag{121}$$

By [13, Section 3.E], up to terms in $\mathrm{CH}^3(\mathcal{C}) \otimes \mathbb{Q}$,

$$\mathrm{todd}(p) = 1 - \frac{1}{2}C_1(\omega_p) + \frac{1}{12}\left(\eta + C_1(\omega_p)^2\right) + \dots, \tag{122}$$

where η is the $\mathbb{Q}$-divisor class of the ramification locus of p. By [29, Lemma 2.1.2], $p_*\left(\eta + C_1(\omega_p)^2\right)$ equals zero. Therefore, up to terms in $\mathrm{CH}^2(\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)) \otimes \mathbb{Q}$,

$$p_*\left(f^*\mathrm{ch}[\mathcal{O}_{\mathbb{P}^n}(d)] \cap \mathrm{todd}(p)\right) =$$

$$p_*\left(df^*(h) - \frac{1}{2}C_1(\omega_p)\right) + \frac{d^2}{2}p_*f^*(h^2) - \frac{d}{2}p_*\left(f^*(h) \cap C_1(\omega_p)\right).$$

Clearly $p_*(f^*(h))$ is just e and $p_*(C_1(\omega_p))$ is just -2. By definition, $p_*f^*(h^2)$ is the divisor class $\mathcal{H}$.

Lemma 10.5. *In the $\mathbb{Q}$-Picard group of $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$,*

$$p_*\left(f^*(h) \cap C_1(\omega_p)\right) = \frac{1}{d}\left[-\mathcal{H} + \sum_{i=1}^{\left[\frac{e}{2}\right]}\sum_{j=0}^{r}(e-i)i\mathcal{D}_{i,j}\right]. \tag{123}$$

Proof. Denote by α the difference of the right-hand-side of the equation from the left-hand-side. The proposition is that α equals 0 in the $\mathbb{Q}$-Picard group.

The method of proof is the same as in [29, Section 1.2]. Consider the class $\mathcal{S}$ of all pairs (B, ζ) where B is a smooth complete curve, $\zeta : B \to \overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ is a 1-morphism and such that (B, ζ) satisfies,

(i) for the pullback of the universal curve, $p_\zeta : \mathcal{C}_\zeta \to B$, $\mathcal{C}_\zeta$ is a smooth surface,
(ii) the general fiber of p_ζ is a smooth, rational curve,
(iii) every singular fiber of p_ζ has exactly two irreducible components, and
(iv) blowing down one irreducible component in each singular fiber yields a ruled surface over B.

In [29], it is proved that for any nonzero divisor class β in the $\mathbb{Q}$-Picard group of $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$, there is a pair (B, ζ) in $\mathcal{S}$ such that $\zeta^*(\beta)$ has nonzero degree on B. So to prove the proposition, it suffices to prove that for every pair (B, ζ) in $\mathcal{S}$, $\zeta^*(\alpha)$ has degree zero.

Suppose (B, ζ) is in $\mathcal{S}$. Let $(E_1 \cup E_1', \ldots, E_m \cup E_m')$ denote the irreducible components of the singular fibers of p_ζ. Let $s : B \to \mathcal{C}_\zeta$ denote a section of p_ζ not intersecting any of $E_1, \ldots, E_m$ (by (iv), such a section exists), and let S denote $s(B)$. Let F denote any smooth fiber of p_ζ. In the group of numerical equivalence classes, $N^1(\mathcal{C}_\zeta)$, the classes $[S], [F], [E_1], \ldots, [E_m]$ give a basis for $N^1(\mathcal{C}_\zeta)$ as a free $\mathbb{Z}$-module.

Denote $k = -\deg([S] \cap [S])$. By straightforward computation,

$$\begin{cases} \deg([F] \cap [F]) & = 0 \\ \deg([F] \cap [S]) & = 1 \\ \deg([F] \cap [E_i]) & = 0 \\ \deg([S] \cap [E_i]) & = 0 \\ \deg([E_i] \cap [E_i]) & = -1 \\ \deg([E_i] \cap [E_j]) & = 0, \quad i \neq j \end{cases} \tag{124}$$

By the adjunction formula, $\left(\zeta^*\omega_p \otimes \mathcal{O}_{\mathcal{C}_\zeta}(E_i)\right)|_{E_i} \cong \omega_{E_i}$. Therefore, $\deg\left(\zeta^* C_1(\omega_p) + [E_i]\right) \cap [E_i] = -2$, i.e. $\deg(\zeta^* C_1(\omega_p) \cap [E_i]) = -1$. Similarly, $\left(\zeta^*\omega_p \otimes \mathcal{O}_{\mathcal{C}_\zeta}(F)\right)|_F \cong \omega_F$. Therefore $\deg\left(\zeta^* C_1(\omega_p) + [F]\right) \cap [F] = -2$, i.e. $\deg(\zeta^* C_1(\omega_p) \cap [F]) = -2$. Finally, by adjunction $\left(\zeta^*\omega_p \otimes \mathcal{O}_{\mathcal{C}_\zeta}(S)\right)|_S$ is isomorphic to the relative dualizing sheaf of $p_\zeta|_S : S \to B$. But this is an isomorphism, so the relative dualizing sheaf is just $\mathcal{O}_S$. Therefore $\deg\left(\zeta^* C_1(\omega_p) + [S]\right) \cap [S] = 0$, i.e. $\deg(\zeta^* C_1(\omega_p) \cap [S]) = k$. Putting this all together, the numerical equivalence class of $\zeta^* C_1(\omega_p)$ equals

$$\zeta^* C_1(\omega_p) = -2[S] - k[F] + \sum_{i=1}^{m} [E_i]. \tag{125}$$

Denote $l = \deg(\zeta^* f^*(h)) \cap [S]$. For each $i = 1, \ldots, m$, define $e_i = \deg(\zeta^* f^*(h) \cap [E_i])$. By a similar computation as above, the numerical equivalence class of $\zeta^* f^*(h)$ equals

$$\zeta^* f^*(h) = e[S] + (l + ek)[F] - \sum_{i=1}^{m} e_i [E_i]. \tag{126}$$

Thus, $\deg(\zeta^* C_1(\omega_p) \cap \zeta^* f^*(h)) = -2l - ek + \sum_{i=1}^m e_i$. Similarly, $\deg(\zeta^* f^*(h) \cap \zeta^* f^*(h)) = 2el + e^2 k - \sum_{i=1}^m e e_i + \sum_{i=1}^m e_i(e - e_i)$, i.e. $-e\deg(\zeta^* C_1(\omega_p) \cap \zeta^* f^*(h)) + \sum_{i=1}^m e_i(e - e_i)$. Finally, observe that,

$$\deg\left(\zeta^* \sum_{i=1}^{[\frac{e}{2}]} \sum_{j=0}^{r} i(e-i)\mathcal{D}_{i,j}\right) = \sum_{i=0}^{m} e_i(e - e_i). \tag{127}$$

In conclusion,

$$\deg\zeta^* p_* \left(C_1(\omega_p) \cap f^*(h)\right) = -\frac{1}{e}\deg(\zeta^*\mathcal{H}) + \frac{1}{e}\sum_{i=1}^{[\frac{e}{2}]} \sum_{j=0}^{r} i(e-i)\deg(\zeta^*\mathcal{D}_{i,j}), \tag{128}$$

just as required. $\square$

Proposition 10.6. *On $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$, the $\mathbb{Q}$-divisor class of the first Chern class of $\mathcal{P}_d = p_*(f^*\mathcal{O}_{\mathbb{P}^n}(d))$ equals,*

$$C_1(\mathcal{P}_d) = \frac{d}{2e}\left[(ed+1)\mathcal{H} - \sum_{i=1}^{\left[\frac{e}{2}\right]}\sum_{j=0}^{r} i(e-i)\mathcal{D}_{i,j}\right]. \tag{129}$$

More generally, for $\mathbf{d} = (d_1, \dots, d_c)$, the $\mathbb{Q}$-divisor class of the first Chern class of $\mathcal{P}_{\mathbf{d}} = p_*(f^*\mathcal{O}_{\mathbb{P}^n}(\mathbf{d}))$ equals,

$$C_1(\mathcal{P}_{\mathbf{d}}) = \frac{1}{2e}\left(\prod_{k=1}^{c}(ed_k+1)\right)\left[\left(\sum_{k=1}^{c} d_k\right)\mathcal{H} + \left(\sum_{k=1}^{c}\frac{d_k}{ed_k+1}\right)\sum_{i=1}^{\left[\frac{e}{2}\right]}\sum_{j=0}^{c} i(e-i)\mathcal{D}_{i,j}\right]. \tag{130}$$

Proof. Substituting the result from Lemma 10.5 into the Grothendieck-Riemann-Roch formula yields, up to terms in $\mathrm{CH}^2(\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)) \otimes \mathbb{Q}$,

$$\mathrm{ch}[\mathcal{P}_d] = (ed+1) + \frac{d^2}{2}\mathcal{H} + \frac{d}{2e}\left(\mathcal{H} - \sum_{i=1}^{\left[\frac{e}{2}\right]}\sum_{j=0}^{r} i(e-i)\mathcal{D}_{i,j}\right) + \dots \tag{131}$$

Since $\mathrm{ch}[\mathcal{P}_d] = \mathrm{rank}(\mathcal{P}_d) + C_1(\mathcal{P}_d) + \dots$, the first part of the proposition follows.

Since $\mathcal{P}_{\mathbf{d}} \cong \oplus_{k=1}^{c}\mathcal{P}_{d_i}$, we have the formula

$$C_1(\mathcal{P}_{\mathbf{d}}) = \left(\prod_{k=1}^{c}\mathrm{rank}(\mathcal{P}_{d_k})\right)\sum_{k=1}^{c}\frac{C_1(\mathcal{P}_{d_k})}{\mathrm{rank}(\mathcal{P}_{d_k})}. \tag{132}$$

Substituting the first part of the proposition gives the second part of the proposition. □

The following corollaries follow immediately from Proposition 10.6. They are stated separately for notational convenience.

Let $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be a section with zero locus $X \subset \mathbb{P}^n$. Consider the locally free sheaf $\mathcal{P}_d$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$. Denote by $\sigma = f^*s$ the induced section of $\mathcal{P}_d$; the closed substack $\overline{\mathcal{M}}_{0,0}(X, e) \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is the zero locus of σ.

Corollary 10.7. *If the section $\sigma = f^*s$ of $\mathcal{P}_d$ is a regular section, i.e. if $\overline{\mathcal{M}}_{0,0}(X, e)$ has the expected codimension $ed + 1$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$, then the $\mathbb{Q}$-divisor class of the first Chern class of the dualizing sheaf on $\overline{\mathcal{M}}_{0,0}(X, e)$ equals,*

$$\frac{1}{2e}\left[\left((d^2-n-1)e - (n+1-d)\right)\mathcal{H} + \sum_{i=1}^{\left[\frac{e}{2}\right]}\left((n+1-d)i(e-i) - 4e\right)\mathcal{D}_{i,0}\right]. \tag{133}$$

Let $\mathbf{d} = (d_1, \dots, d_c)$ be a sequence of positive integers. Let $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathbf{d}))$ be a section with zero locus $X \subset \mathbb{P}^n$. Consider the locally free sheaf $\mathcal{P}_{\mathbf{d}}$ on $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$. Denote by $\sigma = f^*s$ the induced section; the closed substack $\overline{\mathcal{M}}_{0,0}(X, e) \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is the zero locus of σ.

Corollary 10.8. *If the section $\sigma = f^*s$ is a regular section, i.e. $\overline{\mathcal{M}}_{0,r}(X, e)$ has the expected codimension $e|\mathbf{d}| + c$ in $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$, then the $\mathbb{Q}$-divisor class of the first Chern class of the dualizing* *sheaf on $\overline{\mathcal{M}}_{0,r}(X,e)$ equals,*

$$\frac{1}{2e^2}\left[e\left(\prod_{k=1}^{c} ed_k+1\right)\left(\sum_{k=1}^{c} d_k\right)+2r-(n+1)(e+1)\right]\mathcal{H}-\frac{2}{e}\sum_{j=1}^{r}\mathcal{L}_j+$$
$$\sum_{j=0}^{r} j\mathcal{D}_{0,j}+\frac{1}{2e^2}\sum_{i=1}^{\left[\frac{e}{2}\right]}\sum_{j=0}^{r}\left[\left(\prod_{k=1}^{c} ed_k+1\right)\left(\sum_{k=1}^{c}\frac{ed_k}{ed_k+1}\right)i(e-i)+\right.$$
$$\left.(n+1)ei(e-i)+2e^2j-4eij+2ri^2-4e^2\right]\mathcal{D}_{i,j}$$

Lemma 10.9. *Let $X \subset \mathbb{P}^n$ be a projective scheme.*

(i) *If every geometric generic point of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizes a stable map mapping birationally to its image, then the pullback of the $\mathbb{Q}$-divisor class $\mathcal{H}$ in the $\mathbb{Q}$-Picard group of $\overline{\mathcal{M}}_{0,0}(X,e)$ is big. Moreover, the pullback of this divisor class to the seminormalization of $\overline{\mathcal{M}}_{0,0}(X,e)$ is Cartier and base-point-free.*

(ii) *If every geometric generic point of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizes a an a-normal smooth rational curve in X, then the pullback of the $\mathbb{Q}$-divisor class $C_1(\mathcal{P}_a)$ is an effective Cartier divisor.*

(iii) *If every geometric generic point of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizes a stable map with irreducible domain, then for $i=1,\dots,\left[\frac{e}{2}\right]$, the pullback of $\mathcal{D}_{i,0}$ is an effective $\mathbb{Q}$-Cartier divisor.*

Proof. **(i)**: To prove (i), replace $\overline{\mathcal{M}}_{0,0}(X,e)$ by its seminormalization $\overline{\mathcal{M}}_{0,0}(X,e)_{\text{sn}}$. Consider the universal curve $p:\mathcal{C}\to\overline{\mathcal{M}}_{0,0}(X,e)_{\text{sn}}$ and the universal morphism $f:\mathcal{C}\to X$. Form the closed image subscheme $C\subset\overline{\mathcal{M}}_{0,0}(X,e)_{\text{sn}}\times X$ of (p,f). Now C is a *well defined family of algebraic cycles* in the sense of [20, Defn. I.3.10]. By [20, Thm. I.3.21], there is a Chow variety $\text{Chow}_{1,e}(X)$ and an induced morphism $\overline{\mathcal{M}}_{0,0}(X,e)_{\text{sn}}\to\text{Chow}_{1,e}(X)$, the *Kontsevich-Chow morphism*. By the construction in [20, Section I.3.23], there is an ample invertible sheaf on $\text{Chow}_{1,e}(X)$ such that $\mathcal{H}$ is the pullback of this ample invertible sheaf. By the hypothesis that every geometric generic point of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizes a stable map that maps birationally to its image, the Kontsevich-Chow morphism is generically finite. Therefore $\mathcal{H}$ is base-point-free and big.

(ii): Let $W\subset H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(a))$ be a general vector subspace of dimension $ae+1$. There is an induced map $W\otimes_{\mathbb{C}}\mathcal{O}_{\overline{\mathcal{M}}_{0,0}(X,e)}\to\mathcal{E}_a$ and the first Chern class is simply the locus where this map fails to be an isomorphism. By the assumption that every geometric generic point of $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizes an a-normal stable map which, there is no irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ contained in this locus. Therefore this locus is an effective Cartier divisor.

(iii): Finally, by construction the boundary divisors are effective $\mathbb{Q}$-Cartier divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$. If no irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ is contained in the boundary, then the pullbacks of the boundary divisors are effective $\mathbb{Q}$-Cartier divisors on $\overline{\mathcal{M}}_{0,0}(X,e)$. □

Corollary 10.10. *Let $X\subset\mathbb{P}^n$ be a general hypersurface of degree d.*

(i) *If $d<min(n-3,\frac{n+1}{2})$ and $d^2\geq n+2$, then for $e>>0$ the canonical divisor of $\overline{\mathcal{M}}_{0,0}(X,e)$ is big.*

(ii) *If $d<min(n-6,\frac{n+1}{2})$ and $d^2+d\geq 2n+2$, then for every $e>0$ the canonical divisor of $\overline{\mathcal{M}}_{0,0}(X,e)$ is big.*

In particular, if also $e\geq 3$ and $d+e\leq n$ or $e=2$ and $d+3\leq n$, then Conjecture 9.3 implies that $\overline{M}_{0,0}(X,e)$ is of general type.

Proof. When $d<\frac{n+1}{2}$, then [15, Prop. 7.4] implies that $\overline{\mathcal{M}}_{0,0}(X,e)$ satisfies the hypotheses of (i) and (iii) of Lemma 10.9. Combining this with the formula from Corollary 10.7 gives (i) and (ii).

Finally, by Proposition 9.5 and Remark 9.6, if $e\geq 3$ and $d+e\leq n$ or if $e=2$ and $d+3\leq n$, then the coarse moduli map $\overline{\mathcal{M}}_{0,0}(X,e)\to\overline{M}_{0,0}(X,e)$ is an isomorphism away from codimension

2 so that the canonical bundle of $\overline{M}_{0,0}(X,e)$ equals the canonical bundle of $\overline{\mathcal{M}}_{0,0}(X,e)$. Therefore Conjecture 9.3 implies that $\overline{M}_{0,0}(X,e)$ is of general type. $\square$

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