RATIONAL CURVES ON HYPERSURFACES OF LOW DEGREE

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Abstract. We prove that for $n > 2$ and $d < \frac{n+1}{2}$, a *general* complex hypersurface $X \subset \mathbb{P}^n$ of degree d has the property that for each integer e the scheme $R_e(X)$ parametrizing degree e, smooth rational curves on X is an integral, local complete intersection scheme of "expected" dimension $(n+1-d)e + (n-4)$.

The techniques used in the proof include:

(1) Classical results about lines on hypersurfaces including a new result about flatness of the projection map from the space of *pointed lines*.
(2) The Kontsevich moduli space of stable maps, $\overline{\mathcal{M}}_{0,r}(X,e)$. In particular we use the deformation theory of stable maps, properness of the stack $\overline{\mathcal{M}}_{0,r}(X,e)$, and the decomposition of $\overline{\mathcal{M}}_{0,r}(X,e)$ described in [2].
(3) A version of Mori's bend-and-break lemma.

1. Summary

1.1. **Brief Summary.** All schemes we consider will be $\mathbb{C}$-schemes and all morphisms will be morphisms of $\mathbb{C}$-schemes. All (absolute) products will be over $\mathbb{C}$.

For a projective scheme X over $\mathbb{C}$ along with an ample line bundle L we define $R_e(X)$ to be the open subscheme of the Hilbert scheme $\mathrm{Hilb}^{et+1}(X/k)$ which parametrizes smooth rational curves of degree e lying in X.

Theorem 1.1. *Let $n > 2$ be an integer and let d be a positive integer such that $d < \frac{n+1}{2}$. For a general hypersurface $X \subset \mathbb{P}^n$ of degree d and for every integer $e \geq 1$, the scheme $R_e(X)$ is an integral, local complete intersection scheme of dimension $(n+1-d)e + (n-4)$.*

The idea of the proof is as follows. There is an embedding of $R_e(X)$ into the smooth scheme $R_e(\mathbb{P}^n)$. Denote by $\pi : U_e(\mathbb{P}^n) \to R_e(\mathbb{P}^n)$ the universal family of rational curves in $\mathbb{P}^n$ and by $\rho : U_e(\mathbb{P}^n) \to \mathbb{P}^n$ the *evaluation* morphism. Then $R_e(X)$ is the scheme of zeroes of a section of the locally free sheaf $\pi_*\rho^*\mathcal{O}_{\mathbb{P}^n}(d)$. Thus to prove that $R_e(X)$ is a local complete intersection scheme, it suffices to prove that the codimension of $R_e(X)$ in $R_e(\mathbb{P}^n)$ equals the rank of $\pi_*\rho^*\mathcal{O}_{\mathbb{P}^n}(d)$.

The remainder of the proof is a "deformation and specialization" argument: we embed the non-proper scheme $R_e(X)$ as an open subscheme of a proper scheme which is still modular, i.e. we choose a "modular compactification". Then we show that every generic point in $R_e(X)$ specializes to a point in the "boundary" of the compactification. We use deformation theory to study the irreducible components of the boundary of the compactification. In particular we show that a general point of each irreducible component of the boundary is a unibranch point of the

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compactification whose local ring is reduced and has the expected dimension. This reduces the proof to a combinatorial argument.

1.2. **Detailed Summary.**

In the next few paragraphs we will give a detailed summary of the proof. Our compactification consists of the embedding of $R_e(X)$ as an open subscheme in the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X,e)$ parametrizing stable maps to X. We recall the partition of $\overline{\mathcal{M}}_{0,0}(X,e)$ into locally closed subsets defined in [2]; we call this partition the *Behrend-Manin decomposition* (our partition differs slightly from that in [2]). In particular, the image of $R_e(X)$ is a dense open subset of a component of this partition. We identify certain *basic components* as those components of the partition parametrizing stable maps such that each irreducible component of the domain curve is mapped to a line in X.

We prove a new result about lines on X. We define the incidence correspondence of *pointed lines* in X:

$$F_{0,1}(X) = \{(p,l) | p \text{ a point}, l \text{ a line}, p \in l \subset X\}. \tag{1}$$

We prove that for a general hypersurface $X \subset \mathbb{P}^n$ of degree $d \leq n-1$, the projection morphism $F_{0,1}(X) \to X$ is flat of relative dimension $n-d-1$. From this theorem it easily follows that each basic component B is an integral scheme whose general point is a unibranch point of $\overline{\mathcal{M}}_{0,0}(X,e)$ at which $\overline{\mathcal{M}}_{0,0}(X,e)$ is reduced of dimension $(n+1-d)e+(n-4)$. Thus for each basic component B there is a unique irreducible component $M(B)$ of $\overline{\mathcal{M}}_{0,0}(X,e)$ which contains B, and $M(B)$ is reduced and has dimension $(n+1-d)e+(n-4)$.

Using a version of the bend-and-break lemma of Mori, we prove that every irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ is of the form $M(B)$ for some basic component B. Using this fact and results about flatness, we bootstrap to prove that each evaluation map $\overline{\mathcal{M}}_{0,r}(X,e) \to X$ is flat of the expected dimension and is *generically unobstructed*. This implies that each $\overline{\mathcal{M}}_{0,r}(X,e)$ (including $r=0$) has the expected dimension and is generically smooth. Thus $\overline{\mathcal{M}}_{0,r}(X,e)$ is a reduced, local complete intersection stack of the expected dimension and it only remains to prove that $\overline{\mathcal{M}}_{0,r}(X,e)$ is irreducible, i.e. it remains to prove that all of the irreducible components $M(B)$ are actually equal.

To prove that all of the irreducible components $M(B)$ are equal, we observe that there is a combinatorially defined equivalence relation defined on the set of basic components B such that equivalent basic components, $B \cong B'$ satisfy $M(B) = M(B')$. Thus we are reduced to a combinatorial argument which proves that all basic components are equivalent.

Along the way we generalize the strategy of proof above so that it could apply to smooth projective schemes X other than hypersurfaces $X \subset \mathbb{P}^n$ of degree $d < \frac{n+1}{2}$ (this is made completely explicit for complete intersections in $\mathbb{P}^n$). One is reduced to proving

1. The evaluation morphism $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat, generically unobstructed and the general fiber is geometrically irreducible.
2. For each positive integer e at most the *threshold degree* of X, the evaluation morphism $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat of the expected dimension.
3. For each positive integer e at most the threshold degree of X, the stack $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible.

The most difficult condition to verify seems to be (2), but it is our hope that this can be verified for a larger class of Fano schemes X than the hypersurfaces above.

In [8], Kim and Pandharipande proved irreducibility and rationality of the stacks $\overline{\mathcal{M}}_{0,r}(X,\beta)$ when X is a homogeneous variety for a linear algebraic group (and β is a numerical equivalence class of curves on X). In particular, when X is a linear or quadric hypersurface in $\mathbb{P}^n$, with $n \geq 4$, it follows from [8, Corollary 1] that $\overline{\mathcal{M}}_{0,r}(X,e)$ is irreducible and from [8, Theorem 3] that $\overline{\mathcal{M}}_{0,r}(X,e)$ is rational. This paper can be seen as a generalization of the irreducibilty result [8, Corollary 1] to hypersurfaces $X \subset \mathbb{P}^n$ of degree roughly $d \leq \frac{n}{2}$. In a forthcoming paper, [6], we give a partial generalization of the rationality result of [8, Theorem 3] to hypersurfaces $X \subset \mathbb{P}^n$ of degree roughly $d \leq \sqrt{n}$.

1.3. **Notation.** Given a $\mathbb{C}$-vector space W, $\mathbb{P}W$ denotes the projective space $\mathrm{Proj}(\oplus_{d\geq 0} S^d(W^*))$ which parametrizes one-dimensional linear subspaces of W (not one-dimensional quotient spaces of W). Given any integers $k \leq n$, $G(k,n)$ denotes the Grassmannian which parametrizes k-dimensional linear subspaces of $\mathbb{C}^n$. For a triple of integers $k \leq l \leq n$, $F((k,l),n)$ denotes the partial flag variety which parametrizes partial flags $V_1 \subset V_2 \subset \mathbb{C}^n$ of linear subspaces with $\dim(V_1) = k$ and $\dim(V_2) = l$.

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2. Lines on Hypersurfaces

We denote $W = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ so that $\mathbb{P}W$ parametrizes degree d hypersurfaces $X \subset \mathbb{P}^n$. Let $\mathcal{X} \subset \mathbb{P}W \times \mathbb{P}^n$ denote the universal family of degree d hypersurfaces in $\mathbb{P}^n$. For each degree d hypersurface $X \subset \mathbb{P}^n$, denote by $F_1(X)$ the subscheme of $G(2, n+1)$ which parametrizes lines $L \subset X \subset \mathbb{P}^n$. Denote by $F((1,2), n+1)$ the partial flag variety parametrizing pairs (p, L) where $L \subset \mathbb{P}^n$ is a line and $p \in L$ is a point. Denote by $F_{0,1}(X)$ the subscheme of $F((1,2), n+1)$ which parametrizes pairs (p, L) with $p \in L \subset X \subset \mathbb{P}^n$.

Similarly, let $F_1(\mathcal{X}) \subset \mathbb{P}W \times G(2, n+1)$ denote the subscheme parametrizing pairs (X, L) with $L \in F_1(X)$, and let $F_{0,1}(\mathcal{X}) \subset \mathbb{P}W \times F((1,2), n+1)$ denote the subscheme parametrizing triples (X, p, L) with $(p, L) \in F_{0,1}(X)$. There are projection morphisms

$$\pi_0 : \mathbb{P}W \times F((1,2), n+1) \to \mathbb{P}W, \tag{2}$$

$$\pi_1 : \mathbb{P}W \times F((1,2), n+1) \to \mathbb{P}^n, \tag{3}$$

$$\pi_2 : \mathbb{P}W \times F((1,2), n+1) \to G(2, n+1) \tag{4}$$

By construction, the morphism $(\pi_0, \pi_1) : F_{0,1}(\mathcal{X}) \to \mathbb{P}W \times \mathbb{P}^n$ factors through $\mathcal{X} \subset \mathbb{P}W \times \mathbb{P}^n$. Denote by $\rho : F_{0,1}(\mathcal{X}) \to \mathcal{X}$ the induced morphism. For a particular hypersurface $X \in \mathbb{P}W$, denote by $\rho_X : F_{0,1}(X) \to X$ the fiber of ρ.

The main result of this section is the following theorem:

Theorem 2.1. *Let d be a positive integer with $d \leq n-1$. For a general hypersurface $X \in \mathbb{P}W$ the morphism $\rho_X : F_{0,1}(X) \to X$ is flat of relative dimension $n-d-1$.*

We give the proof in the remainder of this section. From now on we assume that d is given such that $d \leq n-1$.

Denote by $\mathcal{O}$ the structure sheaf of $\mathbb{P}W \times F((1,2),n+1)$ and denote by $\mathcal{O}_F$ the $\mathcal{O}$-module which is the pushforward of the structure sheaf of $F_{0,1}(\mathcal{X})$. On $G(2,n+1)$ we have a universal rank 2 subbundle of $\mathcal{O}_{G(2,n+1)}^{\oplus n+1}$. Denote by S the pullback under π_2 of this universal subbundle to $\mathbb{P}W \times F((1,2),n+1)$. And denote by U the pullback under π_0 of the universal rank 1 subbundle $\mathcal{O}_{\mathbb{P}W}(-1) \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}W}$. By restricting a section of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ to a line $L \subset \mathbb{P}^n$ parametrized by a point $[L] \in G(2, n+1)$, we get a map $U \to \mathrm{Sym}^d(S^\vee)$. By adjunction, this gives rise to a map $\mathrm{Sym}^d(S) \otimes_{\mathcal{O}} U \to \mathcal{O}$, whose image is exactly the ideal sheaf of $F_{0,1}(\mathcal{X})$. In other words, there is a partial resolution of coherent sheaves:

$$\mathrm{Sym}^d(S) \otimes_{\mathcal{O}} U \xrightarrow{\ \sigma\ } \mathcal{O} \longrightarrow \mathcal{O}_F \longrightarrow 0. \tag{5}$$

In other words, $F_{0,1}(\mathcal{X})$ is the zero scheme of the global section $\mathcal{O} \to \mathrm{Sym}^d(S^\vee) \otimes_{\mathcal{O}} U^\vee$ which is the transpose of σ.

Since $\mathrm{Sym}^d(S) \otimes_{\mathcal{O}} U$ is locally free of rank $d+1$, every irreducible component of $\mathcal{X}$ has codimension at most $d+1$ in $\mathbb{P}W \times F((1,2),n+1)$. Therefore, every (nonempty) fiber of ρ has dimension at least $n-d-1$. We define $\mathcal{U} \subset \mathcal{X}$ as a set to be

$$\mathcal{U} = \left\{(X,p) \in \mathcal{X} \,|\, \dim(\rho^{-1}(X,p)) \leq n-d-1\right\}. \tag{6}$$

It follows by upper semicontinuity of the fiber dimension that $\mathcal{U}$ is a Zariski open subset of $\mathcal{X}$, and we give it the corresponding structure of open subscheme of $\mathcal{X}$. A priori $\mathcal{U}$ might contain points $(X,p) \subset \mathcal{X}$ for which $\rho^{-1}(X,p)$ is empty. But by [9, exercise V.4.6], it follows that ρ is surjective (also this exercise rederives the statement above about dimensions of fibers).

Notice that the projection morphism $\pi_0 : \mathcal{X} \to \mathbb{P}^n$ is a projective bundle whose fiber over $p \in \mathbb{P}^n$ is identified, as a subscheme of $\mathbb{P}W$, with the hyperplane parametrizing $X \in \mathbb{P}W$ with $p \in X$. In particular, $\mathcal{X}$ is a smooth k-scheme. Given the map σ above, we can form the Koszul complex of locally free $\mathcal{O}$-modules in the usual way. By [10, Theorem 17.4 (iii)(4)], this complex is acyclic over $\mathcal{U}$. Therefore the fibers of ρ over $\mathcal{U}$, considered as subschemes of the appropriate fiber of $\pi_1 : F((1,2),n+1) \to \mathbb{P}^n$, all have equal Hilbert polynomial. Since $\mathcal{U}$ is smooth, it follows from [7, Theorem III.9.9] that ρ is flat over $\mathcal{U}$.

Let $\mathcal{Y} \subset F_{0,1}(\mathcal{X})$ denote the complement of $\mathcal{U}$ with the induced, reduced scheme structure. Theorem 2 is equivalent to the statement that $\pi_0|_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{P}W$ is not surjective. Denote by e the codimension of $\mathcal{Y}$ in $\mathcal{X}$. Since the fiber dimension of $\mathcal{X} \to \mathbb{P}W$ is $n-1$, to prove that $\mathcal{Y}$ fails to dominate $\mathbb{P}W$, it suffices to prove that $e > n-1$. In the remainder of this section we will prove that $e > n-1$.

On $\mathbb{P}^n$ let Q denote the locally free quotient sheaf of $\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}$ by $\mathcal{O}_{\mathbb{P}V}(-1)$. The dual injection $Q^\vee \hookrightarrow \left(\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}\right)^\vee$ can be considered as a filtration of $\left(\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}\right)^\vee$. The dth symmetric product of this filtration is a filtration of $W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n}$:

$$W \otimes_{\mathbb{C}} \mathcal{O} = F^{0,d} \supset F^{1,d} \supset \cdots \supset F^{d,d} \supset F^{d+1,d} = 0. \tag{7}$$

Here $F^{i,d}$ is the locally free subsheaf of $W \otimes_{\mathbb{C}} \mathcal{O}$ which is the image of the multiplication map

$$\mathrm{Sym}^{d-i}\left(\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}\right)^\vee \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathrm{Sym}^i Q^\vee \to W \otimes_{\mathbb{C}} \mathcal{O}. \tag{8}$$

The associated graded sheaves of this filtration $G^{i,d} = F^{i,d}/F^{i+1,d}$ are canonically isomorphic to the sheaves $\mathcal{O}_{\mathbb{P}^n}(d-i) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathrm{Sym}^i Q^\vee$.

In particular, notice that $F^{1,d}$ is simply the kernel of the evaluation map $W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d)$, i.e. the vector bundle parametrizes pairs (Φ, p) where $\Phi \in W$ is such that $\Phi(p) = 0$. We identify a nonzero section $\Phi \in W$, up to nonzero scaling, with the hypersurface it defines $X = V(\Phi)$. Then the associated projective bundle $\mathbb{P}F^{1,d}$ inside $\mathbb{P}W \times \mathbb{P}^n$ is the closed subscheme parametrizing pairs (X, p) with $p \in X$, i.e. $\mathbb{P}F^{1,d} = \mathcal{X}$. To prove the inequality $e > n-1$ from above, it suffices to prove that for each $p \in \mathbb{P}^n$ (equivalently for any $p \in \mathbb{P}^n$ by homogeneity) the intersection $\mathcal{Y} \cap \pi_0^{-1}(p) \subset \mathcal{X}$ has codimension greater than $n-1$ in $\pi_0^{-1}(p)$. In the remainder of this section we will prove this.

Observe the filtration above is not split on $\mathbb{P}^n$. But we can find a covering of $\mathbb{P}^n$ by open affine subschemes $A_\alpha \subset \mathbb{P}^n$ over which we do have a splitting (for example, the standard covering by complements of coordinate hyperplanes). Here by *splitting* we mean an isomorphism of bundles over A_α

$$s : W \otimes_{\mathbb{C}} \mathcal{O}_{A_\alpha} \longrightarrow \oplus_{j=0}^{d} \mathcal{O}_{\mathbb{P}^n}(d-j) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathrm{Sym}^j Q^\vee|_{A_\alpha} \tag{9}$$

which maps $F^{i,d}|_{A_\alpha}$ to the subbundle $\oplus_{j=i}^{d} \mathcal{O}_{\mathbb{P}^n}(d-j) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathrm{Sym}^j Q^\vee|_{A_\alpha}$ and such that the induced isomorphism $G^{i,d}|_{A_\alpha} \to \mathcal{O}_{\mathbb{P}^n}(d-i) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathrm{Sym}^i Q^\vee|_{A_\alpha}$ is the isomorphism from above.

Given an open affine $A_\alpha \subset \mathbb{P}^n$ we can form the projective bundle $\mathbb{P}_{A_\alpha}(F^{1,d}|_{A_\alpha})$ over A_α. Given a splitting s on A_α, denote by

$$\Delta_j(s) \subset \mathbb{P}_{A_\alpha}(F^{1,d}|_{A_\alpha}) \tag{10}$$

the closed subscheme which parametrizes pairs (Φ, x), $x \in A_\alpha, \Phi \in F^{1,d}|x$ such that the jth component of $s(\Phi)$ is zero. Thus $\Delta_0(s)$ is all of $\mathbb{P}_{A_\alpha}(F^{1,d}|_{A_\alpha})$. And, considering $\mathbb{P}_{A_\alpha}(F^{1,d}|_{A_\alpha})$ as an open subscheme of $\mathcal{X}$, $\Delta_1(s)$ is the intersection of $\mathbb{P}_{A_\alpha}(F^{1,d}|_{A_\alpha})$ with the singular locus of the projection morphism $\pi_0 : \mathcal{X} \to \mathbb{P}W$. Although $\Delta_0(s)$ and $\Delta_1(s)$ are independent of the choice of s, the same is not true for $\Delta_i(s)$ with $i > 1$. The next result follows immediately from the definition of the $\Delta_j(s)$.

Lemma 2.2. *For $j > 0$ the codimension of $\Delta_j(s)$ in $\mathbb{P}_{A_\alpha}(F^{1,d}|_{A_\alpha})$ equals*

$$\mathit{rank}\ \left(\mathcal{O}_{\mathbb{P}^n}(d-j) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathit{Sym}^j Q^\vee\right) = \begin{pmatrix} n-1+j \\ n-1 \end{pmatrix}. \tag{11}$$

In particular, for $j > 0$ the codimension of $\Delta_j(s)$ in $\mathbb{P}_{A_\alpha}(F^{1,d}|_{A_\alpha})$ is at least $n-1+j > n-1$. For each open subscheme $A_\alpha \subset \mathbb{P}V$, each splitting s, and each point $p \in A_\alpha$, define the locally closed subscheme

$$\mathcal{Y}_{p,s} := \left(\mathcal{Y} \cap \pi_0^{-1}(p)\right) - \cup_{j=1}^{d} \left(\Delta_j(s) \cap \pi_0^{-1}(p)\right). \tag{12}$$

To establish that $e > n-1$, it suffices to prove that the codimension of $\mathcal{Y}_{p,s}$ as a subscheme of $\pi_0^{-1}(p)$ has codimension greater than $n-1$. In the remainder of this section we prove this inequality.

On the complement of the closed subset $\Delta(s) := \cup_{j=1}^{d} (\Delta_j(s))$ there is a morphism

$$\mathbb{P}_{A_\alpha}\left(F^{1,d}|_{A_\alpha}\right) - \Delta(s) \xrightarrow{\beta} \prod_{j=1}^{d} \mathbb{P}_{A_\alpha}\left(\mathcal{O}_{\mathbb{P}^n}(d-j) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathrm{Sym}^j Q^\vee\right)|_{A_\alpha}. \tag{13}$$

We identify the space

$$\mathbb{P}_{A_\alpha}\left(\mathcal{O}_{\mathbb{P}^n}(d-j) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathrm{Sym}^j Q^\vee\right)|_{A_\alpha} \tag{14}$$

with the scheme parametrizing degree j hypersurfaces in fibers of the projection morphism $\mathbb{P}_{A_\alpha} Q|_{A_\alpha} \to A_\alpha$. Thus β assigns to each suitable pair $([\Phi], p)$ a sequence of hypersurfaces in $\mathbb{P}Q|_p$. Denote this sequence by $(X_1, \ldots, X_j, \ldots, X_d)$. Also, denote by $X = X_1 \cap \cdots \cap X_d$ the intersection of these hypersurfaces.

Lemma 2.3. *If we denote by X the hypersurface in $\mathbb{P}V$ corresponding to Φ, then $X_1 \cap \cdots \cap X_d$ is the fiber of $p \in X$ under $\rho_X : F_{0,1}(X) \to X$.*

Proof. This is most easily seen by passing to local coordinates. Let $(x_0, \ldots, x_n)$ be a system of homogeneous coordinates on $\mathbb{P}V$ (i.e. a basis for $V^\vee$) and let p be the point with homogeneous coordinates $[0, \ldots, 0, 1]$. We define a splitting s as follows: for each degree d homogeneous polynomial Φ in $(x_0, \ldots, x_n)$ we have a unique decomposition

$$\Phi = \Phi_d + \Phi_{d-1}x_n + \cdots + \Phi_{d-i}x_n^i + \cdots + \Phi_0 x_n^d \tag{15}$$

where each Φ_i is a homogeneous polynomial of degree i in $(x_0, \ldots, x_{n-1})$. Then the fiber of $F^{1,d}$ at p consists of those polynomials such that $\Phi_0 = 0$ and $\beta(\Phi) = (\Phi_d, \ldots, \Phi_1)$. For any line L passing through p there is a unique point of the form $y = (a_0, \ldots, a_{n-1}, 0)$ contained in L. Let $\mathbb{P}^1 \to \mathbb{P}^n$ be the morphism given by

$$(t_0, t_1) \mapsto (t_1 a_0, t_1 a_1, \ldots, t_1 a_{n-1}, t_0 + t_1 a_n). \tag{16}$$

The image of this morphism is just L. Substituting into Φ yields the polynomial on $\mathbb{P}^1$ given by

$$\begin{aligned} t_0^d \Phi_d(a_0, \ldots, a_{n-1}) + \cdots + t_0^{d-i} t_1^i \Phi_{d-i}(a_0, \ldots, a_{n-1}) + \ldots \\ + t_0 t_1^{d-1} \Phi_1(a_0, \ldots, a_{n-1}). \end{aligned}$$

The line L is contained in X iff this polynomial is identically zero iff each of the terms $\Phi_i(a_0, \ldots, a_n)$ is zero. One can show that the homogeneous ideal generated by the terms Φ_i is independent of our particular splitting. $\square$

In particular, we conclude that every fiber of β which intersects $\mathcal{Y}$ is contained in $\mathcal{Y}$. Therefore the codimension of $\mathcal{Y}_{p,s}$ in $\pi_0^{-1}(p)$ equals the codimension of the subvariety

$$\beta(\mathcal{Y}) \subset \prod_{j=1}^{d} \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^n}(d-j) \otimes_{\mathcal{O}} \mathrm{Sym}^j Q^\vee\right)|_p. \tag{17}$$

By construction, $\beta(\mathcal{Y})$ is the locus parametrizing sequences of hypersurfaces in $\mathbb{P}Q|_p$, $(X_1, \ldots, X_d)$, of degrees $1, \ldots, d$ respectively such that the intersection

$$X_{(1,\ldots,d)} := X_1 \cap \cdots \cap X_d \tag{18}$$

has dimension greater than $n - d - 1$. So we have reduced Theorem 2.1 to the following theorem:

Theorem 2.4. *Let Q be a vector space over k of dimension n and let d be an integer such that $1 \leq d \leq n-1$. Let $\mathbb{P}_d$ denote the scheme $\prod_{j=1}^d \mathbb{P}Sym^j Q^\vee$, which parametrizes d-tuples $(X_1, \ldots, X_d)$ of hypersurfaces $X_i \in \mathbb{P}Q$ of degree i. Denote by D_d the closed subscheme of $\mathbb{P}_d$ which parametrizes sequences $(X_1, \ldots, X_d)$ such that*

$$dim\left(X_{(1,\ldots,d)}\right) > n - d - 1. \tag{19}$$

The codimension of D_d in $\mathbb{P}_d$ is greater than $n - 1$.

Proof. We will prove this by induction on d. Consider first the case $d = 1$. Since $D_1 = \emptyset$ and the dimension of $\mathbb{P}_d = \mathbb{P}Q^\vee$ is $n-1$, the result is true for $d = 1$.

Let U_d denote the open subscheme of $\mathbb{P}_d$ which is the complement of D_d. Then for $1 \leq d \leq n-2$, U_{d+1} is contained in $U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$. To see this, note that if $X_{(1,\ldots,d)}$ has dimension larger than $n-d-1$, then $X_{1,\ldots,d+1}$ is nonempty and has dimension greater than $n-d-2$: it is nonempty since X_{d+1} is ample, it has dimension larger than $n-d-1$ by the Hauptidealsatz. So we see that the codimension of D_{d+1} in $\mathbb{P}_{d+1}$ is the minimum of the codimension of D_d in $\mathbb{P}_d$ and the codimension of $D_{d+1} \cap (U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee)$ in $U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$. So by induction we are reduced to showing that the codimension of $D_{d+1} \cap (U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee)$ in $U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ is larger than $n-1$.

Now suppose that $(X_1, \ldots, X_d, X_{d+1})$ is a point in $D_{d+1} \cap (U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee)$. By assumption every irreducible component of $X_{(1,\ldots,d)}$ has dimension $n-d-1$. Since also $X_{(1,\ldots,d+1)}$ has dimension $n-d-1$, we conclude that there is an irreducible component $C \subset X_{(1,\ldots,d)}$ such that $C \subset X_{d+1}$. If $X_{(1,\ldots,d)} = C_1 \cup \cdots \cup C_r$ is the irreducible decomposition, then the fiber of $D_{d+1} \cap (U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee)$ over $(X_1, \ldots, X_d)$ (which we consider as a subscheme of $\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$) is just the union of $i = 1, \ldots, r$ of the set $B_i \subset \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ parametrizing hypersurfaces X_{d+1} such that $C_i \subset X_{d+1}$. We are reduced to showing that the codimension of each B_i in $\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ is greater than $n-1$. We prove this in a lemma:

Lemma 2.5. *Let $Y \subset \mathbb{P}Q$ be an irreducible subscheme such that $\dim Y = n-d-1$. Let $B(Y) \subset \mathbb{P}Sym^{d+1}Q^\vee$ be the locus of hypersurfaces X_{d+1} such that $Y \subset X_{d+1}$. The codimension of $B(Y)$ is greater than $n-1$.*

Proof. Let $\Lambda \subset \mathbb{P}Q$ be a $(d-1)$-plane which is disjoint from Y. Choose coordinates on $\mathbb{P}Q$, $(x_0, \ldots, x_{n-1})$ with respect to which $\Lambda = Z(x_d, \ldots, x_{n-1})$. Let $\mathbb{G}_m$ denote the multiplicative group Spec $\mathbb{C}[t, t^{-1}]$. Let $\mu : \mathbb{G}_m \times \mathbb{P}Q \to \mathbb{P}Q$ be the group action given by

$$\mu(t, (x_0, \ldots, x_{d-1}, x_d, \ldots, x_{n-1})) = (t^{-1}x_0, \ldots, t^{-1}x_{d-1}, tx_d, \ldots, tx_{n-1}). \tag{20}$$

Since the Hilbert scheme of $\mathbb{P}Q$ is proper, the valuative criterion implies that the closed subscheme

$$\mu^{-1}(Y) \subset \mathbb{G}_m \times \mathbb{P}Q \tag{21}$$

which is flat over $\mathbb{G}_m$, extends over 0 to yield a closed subscheme

$$\mathcal{Y} \subset \mathbb{A}^1 \times \mathbb{P}Q \tag{22}$$

which is flat over $\mathbb{A}^1$. It is easy to see that the fiber of $\mathcal{Y}$ over 0 is a scheme whose reduced scheme is just

$$Z(x_0, \ldots, x_{d-1}) \subset \mathbb{P}Q. \tag{23}$$

Now we can form the family

$$\mathcal{B} \subset \mathbb{A}^1 \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee, \mathcal{B}_t = B(\mathcal{Y}_t). \tag{24}$$

Over $\mathbb{G}_m$ the fibers of $\mathcal{B}$ are isomorphic. It follows by upper semicontinuity that for $t \neq 0$ we have $\dim(\mathcal{B}_t) \leq \dim(\mathcal{B}_0)$. And of course we have

$$\mathcal{B}_0 = B(\mathcal{Y}_0) \subset B\left(Z(x_0, \ldots, x_{d-1})\right). \tag{25}$$

So we are reduced to proving the lemma for the special case $Y = Z(x_0, \ldots, x_{d-1})$. The set B of hypersurfaces $X_{d+1} \subset \mathbb{P}Q$ which contain $Z(x_0, \ldots, x_{d-1})$ is just the projectivization of the kernel of the surjective linear map

$$H^0(\mathbb{P}Q, \mathcal{O}_{\mathbb{P}Q}(d+1)) \to H^0(Y, \mathcal{O}_Y(d+1)).$$

So the codimension of B in $\mathbb{P}Q$ equals

$$\dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(d+1)) = \binom{n}{d+1}.$$

For $d+1 \leq n-1$ (which is one of our hypotheses) we see that $\binom{n}{d+1} \geq n > n-1$. We conclude that the codimension of $B(Y)$ in $\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ is greater than $n-1$. This proves the lemma. $\square$

By the above lemma, we conclude that the codimension of each B_i in $\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ is greater than $n-1$. So we have proved Theorem 2.4 $\square$

Since we had reduced Theorem 2.1 to Theorem 2.4, we have proved Theorem 2.1.

While we are discussing results about lines on hypersurfaces, let us mention two other results about lines on hypersurfaces.

Lemma 2.6. [9, exercise V.4.4.2] *For general X and a general line $L \subset X$, the normal bundle $N_{L/X}$ is of the form $\mathcal{O}_L^{\oplus d-1} \oplus \mathcal{O}_L(1)^{\oplus n-1-d}$.*

Theorem 2.7. [9, Theorem V.4.3.2] *For general X, the Fano scheme $F_1(X)$ is smooth. Therefore $F_{0,1}(X)$ is smooth. By generic smoothness, the general fiber of $F_{0,1}(X) \to X$ is smooth.*

3. Stable A-graphs and Stable Maps

We follow the notation from [2] regarding stable A-graphs. However, we shall only need to use genus 0 trees.

3.1. Graphs and Trees.

Definition 3.1. A *graph* τ is a 4-tuple $(F_\tau, W_\tau, j_\tau, \partial_\tau)$ defined as follows:

1. F_τ is a finite set called the set of *flags*
2. W_τ is a finite set called the set of *vertices*
3. $j_\tau : F_\tau \to F_\tau$ is an involution
4. $\partial_\tau : F_\tau \to W_\tau$ is a map called the *evaluation* map.

In addition we have the auxiliary definitions

1. the set of *tails* $S_\tau \subset F_\tau$ is the set of fixed points of j_τ
2. the set of *edges* E_τ is the quotient of $F_\tau \setminus S_\tau$ by j_τ
3. for a vertex $v \in W_\tau$, the *valence* of v is defined to be $\mathrm{val}(v) = \#(\partial^{-1}(v))$.

We shall often write $\mathrm{Flag}(\tau)$ in place of F_τ, $\mathrm{Vertex}(\tau)$ in place of W_τ, $\mathrm{Tail}(\tau)$ in place of S_τ, $\mathrm{Edge}(\tau)$ in place of E_τ, and $\overline{f}$ in place of $j_\tau(f)$.

We can associate to a graph its *geometric realization* $|\tau|$ which is a CW-complex defined as follows. The set of 0-cells of $|\tau|$ is

$$|\tau|^0 = \mathrm{Vertex}(\tau) \sqcup \mathrm{Tail}(\tau). \tag{26}$$

The set of 1-cells of $|\tau|$ is

$$|\tau|^1 = \mathrm{Edge}(\tau) \sqcup \mathrm{Tail}(\tau). \tag{27}$$

If $[0,1]$ is a 1-cell associated to an edge $\{f, \overline{f}\}$, the point 0 is glued to the 0-cell ∂f, and the point 1 is glued to the 0-cell $\partial \overline{f}$. If $[0,1]$ is the 1-cell associated to a tail f, the point 0 is glued to the 0-cell ∂f, and the point 1 is glued to the 0-cell f.

Definition 3.2. A *tree* is a connected graph such that $H_1(|\tau|, \mathbb{Z}) = 0$, i.e. a graph which contains no closed loops.

One important tree is the *empty tree* $\lambda_\emptyset$, i.e. the tree such that $\mathrm{Vertex}(\lambda_\emptyset) = \emptyset$. For each nonnegative integer r define λ_r to be the tree with one vertex, $\mathrm{Vertex}(\lambda_r) = \{v\}$, and with r flags (all of which are tails), $\mathrm{Tail}(\lambda_r) = \{f_1, \ldots, f_r\}$. Also, for each pair of nonnegative integers (r_1, r_2), define λ_{r_1,r_2} to be the connected tree with two vertices v_1, v_2, with r_1 tails attached to v_1 and with r_2 tails attached to v_2.

Definition 3.3. An *A-graph* is a pair (τ, β_τ) where τ is a tree and

$$\beta : \mathrm{Vertex}(\tau) \to \mathbb{Z}_{\geq 0} \tag{28}$$

is a map called the *A-structure*. We shall often abbreviate (τ, β_τ) by just writing τ. We say that an A-graph τ is *stable* if for each vertex $v \in \mathrm{Vertex}(\tau)$ such that $\beta_\tau(v) = 0$, there are at least 3 distinct flags $f \in \mathrm{Flag}(\tau)$ such that $\partial f = v$ (i.e. the valence of v is at least 3).

One important A-graph is the empty graph, $\tau_\emptyset$. This is the unique A-graph whose underlying graph is $\lambda_\emptyset$. For each pair of nonnegative integers r and e, define $\tau_r(e)$ to be the unique A-graph whose underlying graph is λ_r and such that $\beta(v) = e$. Obviously $\tau_r(e)$ is stable iff either $r \geq 3$ or $e > 0$. For each pair of pairs (r_1, r_2) and (e_1, e_2) where r_1, r_2, e_1 and e_2 are nonnegative integers, define $\tau_{r_1,r_2}(e_1, e_2)$ to be the unique A-graph whose underlying graph is λ_{r_1,r_2}, such that $\beta(v_1) = e_1$ and such that $\beta(v_2) = e_2$.

There is a category whose objects are the stable A-graphs. The morphisms in this category are each a composition of two basic types of morphisms: *contractions* and *combinatorial morphisms*, c.f. [2] for the precise definitions. Essentially a contraction of A-graphs $\phi : \tau \to \sigma$ is a map from the vertices of τ onto the vertices of σ which maps adjacent vertices to adjacent vertices (here two vertices are adjacent if they are equal or if they are connected by an edge). And a combinatorial morphism $\tau \hookleftarrow \sigma$ is the inclusion of a subgraph σ into a graph τ. The functor which associates to a stable A-graph the corresponding *Behrend-Manin stack* is covariant for contractions. But it is contravariant for combinatorial morphisms. Therefore we think of a combinatorial morphism $\tau \hookleftarrow \sigma$ as a morphism from τ to σ (which explains our terminology $\tau \hookleftarrow \sigma$ for combinatorial morphisms).

Particularly important are morphisms of graphs which remove tails. For each stable A-graph τ we define $r_{>0}(\tau)$ to be the stable A-graph obtained by removing every tail $f \in \mathrm{Tail}(\tau)$ such that $\beta(\partial f) > 0$. We define $\tau \hookleftarrow r_{>0}(\tau)$ to be the canonical combinatorial morphism. For each stable A-graph τ we define $r_0(\tau)$ to be the stabilization of the A-graph obtained by removing all tails $f \in \mathrm{Tail}(\tau)$ such that $\beta(\partial f) = 0$. Technically the canonical morphism of graphs from τ to $r_0(\tau)$ consists of both a combinatorial morphism and a contraction. But we shall denote it by $\tau \hookleftarrow r_0(\tau)$ just as if it were a combinatorial morphism. Finally, we define $r(\tau) := r_{>0}(r_0(\tau)) = r_0(r_{>0}(\tau))$.

There are numerical invariants associated to an A-graph.

Definition 3.4. Given an A-graph τ, define

$$\beta(\tau) = \sum_{v \in \mathrm{Vertex}(\tau)} \beta(v). \tag{29}$$

If (X, L) is a polarized variety such that $K_X \overset{\text{num}}{=} mL$ for some integer m, define the *expected dimension*

$$\dim(X, \tau) = -m\beta(\tau) + \#\mathrm{Tail}(\tau) - \#\mathrm{Edge}(\tau) + (\dim(X) - 3). \tag{30}$$

3.2. **Prestable Curves and Dual Graphs.**

Definition 3.5. A *prestable curve* with r marked points $(C, (x_1, \ldots, x_r))$ is a pair where C is a complete, reduced, at worst nodal curve and $x_i \in C, i = 1, \ldots, r$ are distinct, nonsingular points of C.

Suppose that $(C, (x_1, \ldots, x_r))$ is a connected, prestable curve whose arithmetic genus is 0. One associates to $(C, (x_1, \ldots, x_n))$ a *dual graph*, Δ: a tree whose vertices $\{v_1, v_2, \ldots\}$ correspond to the irreducible components $\{C_1, C_2, \ldots\}$ of C, whose edges $\{\{f_1, \overline{f_1}\}, \{f_2, \overline{f_2}\}, \ldots\}$ correspond to the nodes $\{q_1, q_2, \ldots\}$ of C, and whose tails $\{g_1, \ldots, g_r\}$ correspond to the marked points $\{p_1, \ldots, p_r\}$ of C.

Definition 3.6. Let (X, L) be a polarized variety. A *prestable map* is a pair

$$((C, (x_1, \ldots, x_n)), C \xrightarrow{h} X) \tag{31}$$

where $(C, (x_1, \ldots, x_n))$ is a prestable curve, and where $C \xrightarrow{h} X$ is a morphism of $\mathbb{C}$-schemes.

Just as one associates to a connected prestable curve $(C, x_1, \ldots, x_n)$ of arithmetic genus 0 a tree $\Delta(C, x)$, one can associate an A-graph to a prestable map $\left((C, x_1, \ldots, x_n), C \xrightarrow{h} X\right)$ from a connected prestable curve of arithmetic genus 0. The underlying tree of $\Delta(C, x, h)$ is simply $\Delta(C, x)$. And, given a component C_i of C with corresponding vertex $v_i \in \mathrm{Vertex}(\Delta(C, x))$, one defines

$$\beta(v_i) = \int_{C_i} h_i^*(c_1(L)). \tag{32}$$

The A-graph $\Delta(C, x, h)$ is a *stable* A-graph iff (C, x, h) is a *stable* map.

3.3. **Behrend-Manin stacks.**

We refer the reader to [2] for the definition of the stacks $\overline{\mathcal{M}}(X, \tau)$. These are proper Deligne-Mumford stacks which parametrize stable maps along with some extra data. We shall sometimes deal with these stacks, but more often we shall deal with the open substack $\mathcal{M}(X, \tau) \subset \overline{\mathcal{M}}(X, \tau)$ of *strict maps* which we now define.

Definition 3.7. Let X be a variety, L a line bundle on X, and let τ be a stable A-graph. A *strict τ-map* is a datum

$$((C_v), (h_v : C_v \to X), (q_f)) \tag{33}$$

defined as follows:

(1) (C_v) is a set parametrized by $v \in \mathrm{Vertex}(\tau)$ of smooth rational curves, i.e. each $C_v \cong \mathbb{P}^1$
(2) $(h_v : C_v \to X)$ is a set parametrized by $v \in \mathrm{Vertex}(\tau)$ of morphisms of $\mathbb{C}$-schemes,
(3) (q_f) is a set parametrized by $f \in \mathrm{Flag}(\tau)$ of closed points $q_f \in C_{\partial f}$

and satisfying the following conditions

(1) for $v \in \mathrm{Vertex}(\tau)$, the degree of $h_v^*(L)$ as a line bundle on C_v is $\beta_\tau(v)$,
(2) for $f_1, f_2 \in \mathrm{Flag}(\tau)$ distinct flags with $\partial f_1 = \partial f_2$, $q_{f_1} \neq q_{f_2}$,
(3) for $f \in \mathrm{Flag}(\tau)$, we have $h_{\partial f}(q_f) = h_{\partial \overline{f}}(q_{\overline{f}})$.

Convention: For the empty graph, $\tau_\emptyset$, we define a strict $\tau_\emptyset$-map to simply be a point in X. Thus the set of strict $\tau_\emptyset$-maps is simply X.

Definition 3.8. If T is a $\mathbb{C}$-scheme, then a *family of strict τ-maps* over T is a datum

$$((\pi_v : \mathcal{C}_v \to T), (h_v : \mathcal{C}_v \to X), (q_f : T \to \mathcal{C}_{\partial f})) \tag{34}$$

defined as follows:

(1) $(\pi_v : \mathcal{C}_v \to T)$ is a set parametrized by $v \in \mathrm{Vertex}(\tau)$ of smooth, proper morphisms whose geometric fibers are rational curves

(2) $(h_v : \mathcal{C}_v \to X)$ is a set parametrized by $v \in \mathrm{Vertex}(\tau)$ of morphisms of $\mathbb{C}$-schemes

(3) $(q_f : T \to \mathcal{C}_{\partial f})$ is a set parametrized by $f \in \mathrm{Flag}(\tau)$ of morphisms of schemes such that $\pi_{\partial f} \circ q_f = id_T$

and satisfying the following conditions

(1) for $v \in \mathrm{Vertex}(\tau)$, the degree of $h_v^*(L)$ on each geometric fiber of $\mathcal{C}_v \to S$ is $\beta_\tau(v)$

(2) for $f_1, f_2 \in \mathrm{Flag}(\tau)$ distinct flags with $\partial f_1 = \partial f_2$, q_{f_1} and q_{f_2} are disjoint sections

(3) for $f \in \mathrm{Flag}(\tau)$, we have $h_{\partial f} \circ q_f = h_{\partial \overline{f}} \circ q_{\overline{f}}$.

Convention: For the empty graph, $\tau_\emptyset$ we define a family of strict $\tau_\emptyset$-maps over T to be a morphism $h : T \to X$.

Suppose given two families of strict τ-maps over S, say

$$\eta = ((\pi_v : \mathcal{C}_v \to T), (h_v : \mathcal{C}_v \to X), (q_f : T \to \mathcal{C}_{\partial f})), \tag{35}$$

$$\zeta = ((\pi'_v : \mathcal{C}'_v \to T), (h'_v : \mathcal{C}'_v \to X), (q'_f : T \to \mathcal{C}'_{\partial f})). \tag{36}$$

Definition 3.9. A *morphism* of families of strict τ-maps over S, $\phi : \eta \to \zeta$, is a collection of isomorphisms of S-schemes:

$$\phi = (\phi_v : \mathcal{C}_v \to \mathcal{C}'_v) \tag{37}$$

indexed by $v \in \mathrm{Vertex}(\tau)$ and satisfying

(1) for $v \in \mathrm{Vertex}(\tau)$, $h'_v \circ \phi_v = h_v$

(2) for $f \in \mathrm{Flag}(\tau)$, $\phi_{\partial f} \circ q_f = q'_f$.

One defines composition of morphisms in the obvious way. Notice that every morphism is an isomorphism. Thus the category of families of strict τ-maps over S is a groupoid. Given a morphism $S' \xrightarrow{u} S$ and a family η of strict τ-maps over S, one has the usual pullback $u^*(\eta)$ which is a family of strict τ-maps over S'. In this way we have the notion a category fibered in groupoids over the category of $\mathbb{C}$-schemes along with a *clivage normalisée* in the sense of [5, Exp. VI]. We denote this category by $\mathcal{M}(X, \tau)$. We will occasionally also denote by $\mathcal{M}(X, \tau)$ the associated lax 2-functor from the category of $\mathbb{C}$-schemes to the 2-category of groupoids (with a small skeletal subcategory).

of a functor from the category of $\mathbb{C}$-schemes to the category of groupoids which associates to each S the groupoid of families of strict τ-maps over S. We denote this functor by $\mathcal{M}(X, \tau)$.

In every case it is easy to see that $\mathcal{M}(X, \tau)$ is a stack in groupoids over $\mathbb{C}$. In many cases this is even a Deligne-Mumford stack:

Theorem 3.10. *If X is projective and L is ample, the functor $\mathcal{M}(X, \tau)$ is a Deligne-Mumford stack which is separated and finite type over $\mathbb{C}$.*

Proof. There is a 1-morphism $\mathcal{M}(X,\tau) \to \overline{\mathcal{M}}(X,\tau)$ where $\overline{\mathcal{M}}(X,\tau)$ is the refined functor defined in [2]. In [2] it is proved that $\overline{\mathcal{M}}(X,\tau)$ is a proper Deligne-Mumford stack over $\mathbb{C}$. And it is clear that $\mathcal{M}(X,\tau) \to \overline{\mathcal{M}}(X,\tau)$ is a representable morphism which is an open immersion. Thus $\mathcal{M}(X,\tau)$ is a Deligne-Mumford stack which is separated and finite type over $\mathbb{C}$. $\square$

3.4. **Properties and Related Constructions.**

Notice that with our notation $\overline{\mathcal{M}}(X,\tau_r(e))$ is the moduli stack of Kontsevich stable maps $\overline{\mathcal{M}}_{0,r}(X,e)$, and $\mathcal{M}(X,\tau_r(e))$ simply parametrizes those stable maps such that the domain curve is irreducible.

Definition 3.11. Suppose that τ is a stable A-graph and $f \in \mathrm{Flag}(\tau)$. Then there is a 1-morphism

$$\mathrm{ev}_f : \overline{\mathcal{M}}(X,\tau) \to X \tag{38}$$

defined by sending a family of τ-maps, η (with notation as above), to the morphism $h_{\partial f} \circ q_f$.

If $\alpha = (\alpha_F, \alpha_V) : \sigma \to \tau$ is a combinatorial morphism of graphs, $\tau \hookleftarrow \sigma$, there is an associated 1-morphism

$$\overline{\mathcal{M}}(X,\alpha) : \overline{\mathcal{M}}(X,\tau) \to \overline{\mathcal{M}}(X,\sigma). \tag{39}$$

If α is the inclusion of σ as a subgraph of τ, then $\overline{\mathcal{M}}(X,\alpha)$ is the *forgetful morphism* which "remembers" only those components of τ-maps whose vertex is contained in σ. The reader is referred to [2, Theorem 3.6] for the precise definition. We will refer to the restriction of $\overline{\mathcal{M}}(X,\alpha)$ to $\mathcal{M}(X,\tau)$ as $\mathcal{M}(X,\alpha)$.

If $\phi = (\phi_W, \phi^F) : \tau \to \tau'$ is a contraction of stable A-graphs, there is a corresponding 1-morphism of proper Deligne-Mumford stacks

$$\overline{\mathcal{M}}(X,\phi) : \overline{\mathcal{M}}(X,\tau) \to \overline{\mathcal{M}}(X,\tau'). \tag{40}$$

This morphism "forgets" the labeling of some the individual components of the domain curve. The reader is referred to [2, Theorem 3.6] for the precise definition. We will denote by $\mathcal{M}(X,\phi)$ the restriction of this 1-morphism to the open substack $\mathcal{M}(X,\tau)$ of $\overline{\mathcal{M}}(X,\tau)$.

One important case to understand is when $\beta(\tau) = 0$. We have already defined $\mathcal{M}(X,\tau_\emptyset) = \overline{\mathcal{M}}(X,\tau_\emptyset) = X$ where $\tau_\emptyset$ is the empty graph. For any stable A-graph τ such that $\beta(\tau) = 0$ and such that $\#\mathrm{Tail}(\tau) = r$, we have $\mathcal{M}(X,\tau) = X \times \mathcal{M}(*,\tau)$ where $\mathcal{M}(*,\tau) \subset \overline{\mathcal{M}}_{0,r}$ is the obvious substack.

Consider the case that $\phi : \tau \to \tau'$ is a contraction of stable A-graphs such that $\beta(\tau) = \beta(\tau') = 0$. Then $\mathcal{M}(X,\tau) \to \overline{\mathcal{M}}(X,\tau')$ is simply the product of $\mathrm{id}_X : X \to X$ with the 1-morphism $\mathcal{M}(*,\phi) : \mathcal{M}(*,\tau) \to \overline{\mathcal{M}}(*,\tau')$. In particular, using the notation of [2], consider the case that ϕ is an *isogeny*, i.e. ϕ is the morphism which removes some subset of the set of tails from τ and then stabilizes the resulting (possibly unstable) graph.

Lemma 3.12. *Let τ, τ' be stable A-graphs such that $\beta(\tau) = \beta(\tau') = 0$ and let $\phi : \tau \to \tau'$ be an isogeny. Then $\mathcal{M}(X,\phi) : \mathcal{M}(X,\tau) \to \overline{\mathcal{M}}(X,\tau')$ is smooth of relative dimension $dim(X,\tau) - dim(X,\tau')$ with geometrically connected fibers.*

Proof. Of course it is equivalent to prove that

$$\mathcal{M}(*,\phi) : \mathcal{M}(*,\tau) \to \overline{\mathcal{M}}(*,\tau') \tag{41}$$

is smooth of relative dimension $\dim(X,\tau) - \dim(X,\tau')$ with geometrically connected fibers. Now it follows by Proposition 7.4 of [2] that $\mathcal{M}(*,\tau)$ and $\overline{\mathcal{M}}(*,\tau')$ have the expected dimension. Thus all we really need to show is that $\mathcal{M}(*,\tau) \to \overline{\mathcal{M}}(*,\tau')$ is smooth with geometrically irreducible fibers. Moreover, since every isogeny is a composition of morphisms obtained by stably removing one tail, we may suppose that $\phi : \tau \to \tau'$ corresponds to stably removing one tail $f \in \mathrm{Tail}(\tau)$.

There are two cases. Suppose first of all that when we remove f from τ, the resulting graph is unstable. But then $\partial f = v$ is a vertex with valence 3. Since a rational curve with 3 marked points has no moduli, we conclude that $\mathcal{M}(*,\tau) \to \overline{\mathcal{M}}(*,\tau')$ is an open immersion.

The second case is that when we remove f from τ, the resulting graph is stable, i.e. the resulting graph is just τ'. But then if $v = \phi(\partial f)$, we conclude that $\phi : \mathcal{M}(*,\tau) \to \overline{\mathcal{M}}(*,\tau')$ is simply an open subset of the universal curve over $\overline{\mathcal{M}}(*,\tau')$ corresponding to the vertex v. In both cases we conclude that $\overline{\mathcal{M}}(*,\tau) \to \overline{\mathcal{M}}(*,\tau')$ is smooth with geometrically connected fibers. □

For each stable A-graph τ define $e = \beta(\tau)$ and define $r = \#\mathrm{Tail}(\tau)$. Then there is a contraction $\phi : \tau \to \tau_r(e)$ which is unique up to a labeling of the tails of τ.

Definition 3.13. The contraction ϕ above is the *canonical contraction*. The corresponding 1-morphism (well-defined up to relabeling the tails)

$$\mathcal{M}(X,\phi) : \mathcal{M}(X,\tau) \to \overline{\mathcal{M}}_{0,r}(X,e) \tag{42}$$

will be referred to as the *canonical contraction morphism*.

Notice that the image of $\mathcal{M}(X,\phi)$ as a subset of the set $|\overline{\mathcal{M}}_{0,r}(X,e)|$ is well-defined.

Proposition 3.14. *Let $\phi : \tau \to \tau'$ be a contraction of stable A-graphs. The image of the 1-morphism $\mathcal{M}(X,\phi)$ is a locally closed subset of the topological space $|\overline{\mathcal{M}}(X,\tau)|$.*

Proof. For notation's sake let's denote the continuous map of topological spaces

$$|\overline{\mathcal{M}}(X,\phi)| : |\overline{\mathcal{M}}(X,\tau)| \to |\overline{\mathcal{M}}(X,\tau')| \tag{43}$$

by $f : M \to M'$ and let's denote the open substack $\mathcal{M}(X,\tau)$ of $\overline{\mathcal{M}}(X,\tau)$ by M^o. Then $f : M \to M'$ is a closed map. And it is easy to see that $f^{-1}(f(M^o)) = M^o$. Therefore $f(M^o) = f(M) - f(M - M^o)$ is a difference of closed sets and so is locally closed. □

We now fix n and α and consider the set S of all images

$$\{\mathrm{Image}(\mathcal{M}(X,\phi))\} \tag{44}$$

as ϕ ranges over all contractions of stable A-graphs to $\tau_n(\alpha)$. The set of isomorphism classes of such contractions is clearly finite. The previous lemma shows that S forms a *locally closed decomposition* of the topological space $|\overline{\mathcal{M}}_{0,n}(X,\alpha)|$, i.e. a partition of $|\overline{\mathcal{M}}_{0,n}(X,\alpha)|$ into locally closed subsets. This partition is what we call the *Behrend-Manin decomposition*.

4. Flatness and Dimension Results

In this section we consider the dimensions of the stacks $\mathcal{M}(X,\tau)$ and the evaluation morphisms. The main property we are interested in is the following:

Definition 4.1. Given a stable A-graph τ, we say that $\mathcal{D}(X,\tau)$ holds if the dimension of every irreducible component of $\mathcal{M}(X,\tau)$ equals the expected dimension $\dim(X,\tau)$.

By deformation theory there is an a priori lower bound on the dimension of any irreducible component of $\mathcal{M}(X,\tau)$:

Lemma 4.2. *Every irreducible component of $\overline{\mathcal{M}}(X,\tau)$ has dimension at least $\dim(X,\tau)$. In particular, every irreducible component of $\mathcal{M}(X,\tau)$ has dimension at least $\dim(X,\tau)$.*

Proof. This is a standard result. In the case that $\tau = \tau_r(e)$ it follows from [3, section 5.2]. In the general case the theorem follows from [1]. The theorem isn't actually stated in [1], so we show how it follows from results there.

Let $\mathfrak{M}(\tau)$ denote the stack of τ-marked prestable curves as in [1]. There is a forgetful 1-morphism of algebraic (Artin) stacks

$$\overline{\mathcal{M}}(X,\tau) \to \mathfrak{M}(\tau). \tag{45}$$

By [1, Lemma 1], $\mathfrak{M}(\tau)$ is smooth of dimension

$$\#\mathrm{Tail}(\tau) - \#\mathrm{Edge}(\tau) - 3. \tag{46}$$

Let $\mathfrak{C}(\tau) \to \mathfrak{M}(\tau)$ be the universal curve. By [1, Proposition 4], $\overline{\mathcal{M}}(X,\tau)$ is an open substack of the relative morphism-scheme,

$$\mathrm{Mor}_{\mathfrak{M}(\tau)}\left(\mathfrak{C}(\tau), X \times \mathfrak{M}(\tau)\right). \tag{47}$$

By [9, Theorem 2.17.1], it follows that every irreducible component of $\overline{\mathcal{M}}(X,\tau)$ has dimension at least

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) + \#\mathrm{Tail}(\tau) - \#\mathrm{Edge}(\tau). \tag{48}$$

By Riemann-Roch, we have $(h^0 - h^1)(C, f^*T_X) = -K_X.f_*[C] + \dim(X)$. We conclude that every irreducible component of $\overline{\mathcal{M}}(X,\tau)$ has dimension at least

$$-K_X.f_*[C] + \dim(X) - 3 + \#\mathrm{Flag}(\tau) - \#\mathrm{Edge}(\tau) = \dim(X,\tau). \tag{49}$$

□

When $\mathrm{Vertex}(\tau)$ has more than one element, we can try to reduce $\mathcal{D}(X,\tau)$ to $\mathcal{D}(X,\tau_i)$ for some proper subgraphs τ_i of τ, thus giving an inductive proof that $\mathcal{D}(X,\tau)$ holds. To carry out such a proof, we need to know that the evaluation morphisms have constant fiber dimension. So we introduce the following property:

Definition 4.3. Given a stable A-graph τ and a flag $f \in \mathrm{Flag}(\tau)$, we say that $\mathcal{E}(X,\tau,f)$ holds if

$$\mathrm{ev}_f : \mathcal{M}(X,\tau) \to X \tag{50}$$

is dominant and has constant fiber dimension $\dim(X,\tau) - \dim(X)$.

Notice that if there is any flag $f \in \mathrm{Flag}(\tau)$ such that $\mathcal{E}(X,\tau,f)$ holds, then it follows that $\mathcal{D}(X,\tau)$ holds.

In the case that $X \subset \mathbb{P}^N$ is a complete intersection, then the properties $\mathcal{D}$ and $\mathcal{E}$ are equivalent to stronger properties.

Definition 4.4. Given a stable A-graph τ, we say that $\mathcal{LCI}(X,\tau)$ holds if $\mathcal{M}(X,\tau)$ is a local complete intersection and if $\mathcal{D}(X,\tau)$ holds. Given a stable A-graph τ and a flag $f \in \mathrm{Flag}(\tau)$, we say that $\mathcal{FE}(X,\tau,f)$ holds if

$$\mathrm{ev}_f : \mathcal{M}(X,\tau) \to X \tag{51}$$

is flat of relative dimension $\dim(X,\tau) - \dim(X)$.

Lemma 4.5. *If $X \subset \mathbb{P}^N$ is a complete intersection, then $\mathcal{D}(X,\tau)$ holds iff $\mathcal{LCI}(X,\tau)$ holds. Also $\mathcal{E}(X,\tau,f)$ holds iff $\mathcal{FE}(X,\tau,f)$ holds. The same result holds with $\mathcal{M}(X,\tau)$ replaced by $\overline{\mathcal{M}}(X,\tau)$.*

Proof. Suppose that X is a complete intersection of $r = N - n$ hypersurfaces of degrees $d_1, \dots, d_r$. Consider $\mathcal{M}(\mathbb{P}^N,\tau)$ and denote the universal curve by

$$\pi : \mathcal{C} \to \mathcal{M}(\mathbb{P}^N,\tau). \tag{52}$$

Let $h : \mathcal{C} \to \mathbb{P}^N$ denote the universal map. Since

$$\mathcal{O}_{\mathbb{P}^N}(\mathbf{d}) := \mathcal{O}_{\mathbb{P}^N}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^N}(d_r) \tag{53}$$

is generated by global sections, also $h^*\mathcal{O}_{\mathbb{P}^N}(\mathbf{d})$ is generated by global sections. On a genus 0 tree, if F is a sheaf generated by global sections then $H^1(C,F) = 0$, so $R^1\pi_*\left(h^*\mathcal{O}_{\mathbb{P}^N}(\mathbf{d})\right) = 0$.

Now by [2, Proposition 7.4], $\mathcal{M}(\mathbb{P}^N,\tau)$ is smooth of dimension

$$(N+1)\beta(\tau) + (N-3) + \#\mathrm{Flag}(\tau) - \#\mathrm{Edge}(\tau). \tag{54}$$

So by [11, Corollary II.2] the pushforward $E := \pi_*\left(h^*\mathcal{O}_{\mathbb{P}^N}(\mathbf{d})\right)$ is a locally free sheaf of rank

$$\sum_{i=1}^{r} \chi(C, h^*\mathcal{O}_{\mathbb{P}^N}(d_i)) = \sum_{i=1}^{r} (d_i\beta(\tau) + 1) = \left(\sum_i d_i\right)\beta(\tau) + r. \tag{55}$$

Now the defining equations of the hypersurfaces in $\mathbb{P}^N$ give a global section σ of E, and $\mathcal{M}(X,\tau)$ is precisely the zero scheme of σ. Finally notice that the expected codimension, $\dim(\mathbb{P}^n,\tau) - \dim(X,\tau)$, of $\mathcal{M}(X,\tau)$ in $\mathcal{M}(\mathbb{P}^N,\tau)$ is just

$$-K_{\mathbb{P}^N}.f_*[C] + K_X.f_*[C] + \dim(\mathbb{P}^N) - \mathrm{dim}X = \left(\sum_i d_i\right)\beta(\tau) + r. \tag{56}$$

Thus, if $\mathcal{M}(X,\tau)$, then it follows that $\mathcal{M}(X,\tau)$ is a local complete intersection. So if $\mathcal{D}(X,\tau)$ holds, then also $\mathcal{LCI}(X,\tau)$ holds. The opposite inclusion is obvious.

Now suppose that $\mathcal{E}(X,\tau,f)$ holds. In particular $\mathcal{D}(X,\tau)$ holds, so $\mathcal{LCI}(X,\tau)$ holds. But now by [10, Theorem 23.1], ev_f is a dominant morphism from a Cohen-Macaulay scheme to a smooth scheme with constant fiber dimension, therefore it is flat. So $\mathcal{FE}(X,\tau,f)$ holds. The opposite inclusion is obvious.

The same proof works when we replace $\mathcal{M}(X,\tau)$ by $\overline{\mathcal{M}}(X,\tau)$. □

Consider the following diagram:

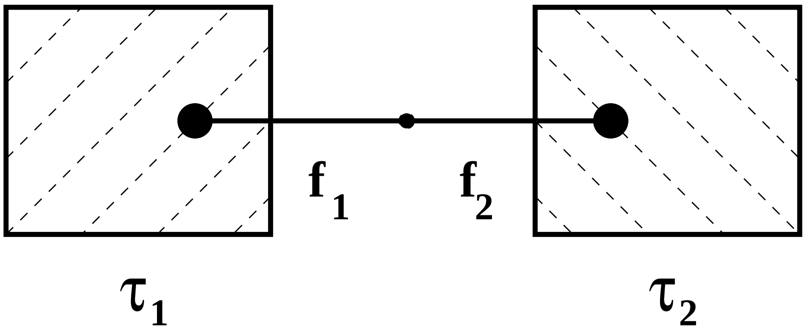

Diagram 1

Here τ is an A-graph which contains the two subgraphs

$$a_1 : \tau \hookleftarrow \tau_1, \quad a_2 : \tau \hookleftarrow \tau_2. \tag{57}$$

The edge $\{f_1, f_2\}$ of τ is made up of the two tails $f_1 \in \mathrm{Tail}(\tau_1), f_2 \in \mathrm{Tail}(\tau_2)$. Let $f \in \mathrm{Flag}(\tau_2)$ be any flag (possibly $f = f_2$).

Lemma 4.6. *If $\mathcal{FE}(X, \tau_1, f_1)$ and $\mathcal{FE}(X, \tau_2, f)$ hold, then $\mathcal{FE}(X, \tau, f)$ holds.*

Proof. The combinatorial morphisms a_1 and a_2 give rise to a 1-morphism

$$\mathcal{M}(X, a_1, a_2) : \mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau_1) \times_{\mathrm{ev}_{f_1}, X, \mathrm{ev}_{f_2}} \mathcal{M}(X, \tau_2). \tag{58}$$

It is clear from the definition of strict τ-maps that $\mathcal{M}(X, a_1, a_2)$ is an open immersion.

Since $\mathrm{ev}_{f_1} : \mathcal{M}(X, \tau_1) \to X$ is flat of relative dimension $\dim(X, \tau_1) - \dim(X)$, it follows that the projection morphism

$$\mathrm{pr}_2 : \mathcal{M}(X, \tau_1) \times_{\mathrm{ev}_{f_1}, X, \mathrm{ev}_{f_2}} \mathcal{M}(X, \tau_2) \to \mathcal{M}(X, \tau_2) \tag{59}$$

is flat of relative dimension $\dim(X, \tau_1) - \dim(X)$. And $\mathrm{ev}_f : \mathcal{M}(X, \tau_2) \to X$ is flat of relative dimension $\dim(X, \tau_2) - \dim(X)$. Thus the composite morphism

$$\mathcal{M}(X, \tau_1) \times_{\mathrm{ev}_{f_1}, X, \mathrm{ev}_{f_2}} \mathcal{M}(X, \tau_2) \xrightarrow{\mathrm{pr}_2} \mathcal{M}(X, \tau_2) \xrightarrow{\mathrm{ev}_f} X \tag{60}$$

is flat of relative dimension $\dim(X, \tau_1) + \dim(X, \tau_2) - 2\dim(X)$. Of course $\mathrm{ev}_f : \mathcal{M}(X, \tau) \to X$ is simply the restriction of the composite morphism, so it is flat of the same relative dimension. But notice that

$$\begin{gathered} \dim(X, \tau_1) + \dim(X, \tau_2) - \dim(X) = \\ (-K_X.\beta(\tau_1) + -K_X.\beta(\tau_2)) + 2\,(\dim(X) - 3) + \\ (\#\mathrm{Flag}(\tau_1) + \#\mathrm{Flag}(\tau_2)) + (\#\mathrm{Edge}(\tau_1) + \#\mathrm{Edge}(\tau_2)) - \dim(X) = \\ -K_X.\beta(\tau) + 2\,(\dim(X) - 3) + (\#\mathrm{Tail}(\tau) + 2) - \\ (\#\mathrm{Edge}(\tau) - 1) - \dim(X) = \dim(X, \tau). \end{gathered} \tag{61}$$

From this it follows that

$$\mathrm{ev}_f : \mathcal{M}(X, \tau) \to X \tag{62}$$

is flat of constant fiber dimension $\dim(X, \tau) - \dim(X)$. Thus $\mathcal{FE}(X, \tau, f)$ holds. $\square$

Definition 4.7. Suppose that τ is a stable A-graph and define the *maximum component degree* of τ to be

$$E(\tau) = \sup_{v \in \mathrm{Vertex}(\tau)} \beta(v). \tag{63}$$

Proposition 4.8. *Suppose that τ is a stable A-graph with $E(\tau) = E$. If for each $e = 0, \ldots, E$ we have $\mathcal{FE}(X, \tau_1(e), f)$ holds, then for each flag $f \in$ Flag(τ), $\mathcal{FE}(X, \tau, f)$ holds.*

Proof. We prove this by induction on $\#\mathrm{Vertex}(\tau)$. Suppose τ has a single vertex. Then $\tau = \tau_r(e)$ for some r and e. If $e = 0$, then $\overline{\mathcal{M}}(X, \tau_r(0)) = \overline{\mathcal{M}}(*, \tau_r(0)) \times X$, and the evaluation morphism is just projection. So ev_f is flat of relative dimension

$$\dim(\overline{\mathcal{M}}(*, \tau_r(0))) = \dim(X, \tau_r(0)) - \dim(X), \tag{64}$$

i.e. $\mathcal{FE}(X, \tau_r(0), f)$ holds.

Next consider the case $\tau = \tau_r(e)$ with $e > 0$. For any flag $f \in \mathrm{Flag}(\tau_r(e))$ there is a combinatorial morphism $a : \tau_r(e) \hookleftarrow \tau_1(e)$ which maps the unique flag $f_1 \in \mathrm{Flag}(\tau_1(e))$ to f. The associated 1-morphism

$$\mathcal{M}(X, a) : \mathcal{M}(X, \tau_r(e)) \to \mathcal{M}(X, \tau_1(e)) \tag{65}$$

is isomorphic to an open subset of the $(e-1)$-fold fiber product of the universal curve over $\mathcal{M}(X, \tau_1(e))$. Since the universal curve is smooth of relative dimension 1 over $\mathcal{M}(X, \tau_1(e))$, we conclude that $\mathcal{M}(X, a)$ is smooth of relative dimension $e - 1$. The evaluation morphism $\mathrm{ev}_f : \mathcal{M}(X, \tau_r(e)) \to X$ factors as the composition

$$\mathcal{M}(X, \tau_r(e)) \xrightarrow{\mathcal{M}(X,a)} \mathcal{M}(X, \tau_1(e)) \xrightarrow{\mathrm{ev}_{f_1}} X. \tag{66}$$

Therefore if $\mathcal{FE}(X, \tau_1(e), f_1)$ holds, then ev_f is flat of relative dimension

$$(e-1) + (\dim(X, \tau_1(e)) - \dim(X)) = \dim(X, \tau_r(e)) - \dim(X), \tag{67}$$

in other words $\mathcal{FE}(X, \tau_r(e), f)$ holds. Thus the Proposition is proved when τ has a single vertex.

Now suppose that τ has more than one vertex and suppose that for all $e \leq E = E(\tau)$, $\mathcal{FE}(\tau_1(e), f)$ holds. By way of induction, assume that the proposition is true for all graphs τ' such that $\#\mathrm{Vertex}(\tau') < \#\mathrm{Vertex}(\tau)$. Let $\{f_1, f_2\}$ be any edge. Define τ_1 and τ_2 to be the graphs obtained by breaking the edge into two tails (see diagram 1). Let $f \in \mathrm{Flag}(\tau)$ be a flag, and without loss of generality suppose that $f \in \mathrm{Flag}(\tau_2)$. Now $E(\tau_2) \leq E(\tau)$ and $\#\mathrm{Vertex}(\tau_2) < \#\mathrm{Vertex}(\tau)$, so by the induction assumption $\mathcal{FE}(X, \tau_2, f)$ holds. Also $E(\tau_1) \leq E(\tau)$ and $\#\mathrm{Vertex}(\tau_1) < \#\mathrm{Vertex}(\tau)$, so by the induction assumption $\mathcal{FE}(X, \tau_1, f_1)$ holds. Then by Lemma 4.6, we conclude that $\mathcal{FE}(X, \tau, f)$ holds. So the proposition is proved by induction. ☐

5. Specializations

In the previous section we reduced the flatness and dimension results for a general stable A-graph τ to flatness and dimension results for the the stable A-graphs $\tau_1(e)$ with $0 \leq e \leq E(\tau)$. In this section we will use specializations to reduce the flatness and dimension results for all $\tau_1(e), e > 1$ to flatness and dimension results for a finite number of cases $\tau_1(e), e = 1, \ldots, E(X)$ where $E(X)$ is the *threshold degree* of X. We define a stable A-graph σ to be *basic* if for each vertex $v \in \mathrm{Vertex}(\sigma)$, we have $\beta(v) \leq E(X)$. The specializations we produce will show that every irreducible component of $\overline{\mathcal{M}}(X, \tau)$ contains a *basic* locally closed subset $\mathcal{M}(X, \sigma)$. Thus to understand the irreducible components of $\overline{\mathcal{M}}(X, \tau)$ it suffices to understand the irreducible components which pass through the general point of a basic locus $\mathcal{M}(X, \sigma)$.

Convention Suppose that we have a contraction $\phi : \sigma \to \tau$. There is an induced morphism

$$\mathcal{M}(X, \phi) : \mathcal{M}(X, \sigma) \to \overline{\mathcal{M}}(X, \tau) \tag{68}$$

which is unramified with locally closed image. We will speak of $\mathcal{M}(X, \sigma)$ as though it is a substack of $\overline{\mathcal{M}}(X, \tau)$. Thus given an irreducible component $M \subset \mathcal{M}(X, \tau)$ and an irreducible component $N \subset \mathcal{M}(X, \sigma)$ we will say $N \subset \overline{M}$ to mean that the image of N is contained in $\overline{M}$.

The basic lemma is the following easy version of Mori's bend-and-break lemma.

Lemma 5.1. *Let $e > 0$. There is no complete curve contained in a fiber of the evaluation morphism*

$$ev_{f_1,f_2} : \mathcal{M}(X, \tau_2(e)) \to X \times X. \tag{69}$$

Proof. Suppose that C is a complete curve and $\zeta : C \to \mathcal{M}(X, \tau_2(e))$ has image in a fiber of ev_{f_1,f_2}. Denote the family ζ of strict $\tau_2(e)$-maps by $(\pi : \Sigma \to C, h, (q_1, q_2))$. Denote $\mathrm{ev}_{f_1,f_2}(C) = (p_1, p_2)$. Let $H \subset X$ be a hyperplane section containing neither p_1 nor p_2. Define $C' = \Sigma \times_X H$, so C' is a finite ramified cover of C. Let B be the normalization of an irreducible component of C' which dominates C. The base-change $\Sigma \times_C B$ now admits the two sections q_1, q_2 as well as a third section q_3 which is everywhere disjoint from both q_1 and q_2. Any $\mathbb{P}^1$-bundle with three everywhere disjoint sections is isomorphic to $\mathbb{P}^1 \times B$ and the three sections are constant sections $\{0\} \times B, \{1\} \times B, \{\infty\} \times B$. But now the morphism $h : \Sigma \times_C B \to X$ contracts the sections q_1 and q_2. By the rigidity lemma, [11, p. 43], we conclude that h factors through the projection $\Sigma \times_C B \cong \mathbb{P}^1 \times B \to \mathbb{P}^1$. So we conclude that $C \to \mathcal{M}(X, \tau)$ is a constant map. $\square$

Corollary 5.2. *Let $M \subset \overline{\mathcal{M}}(X, \tau_2(e))$ be an irreducible, closed substack and suppose that the fibers of $ev_{f_1,f_2} : M \to X \times X$ have dimension at least 1. Then $M \cap \big(\overline{\mathcal{M}}(X, \tau_2(e)) - \mathcal{M}(X, \tau_2(e))\big)$ is either all of M or contains an irreducible component with codimension 1 in M.*

Proof. Suppose that M intersects $\mathcal{M}(X, \tau_2(e))$. Define $I \subset X \times X$ to be the image $I = \mathrm{ev}_{f_1,f_2}(M)$. In order to prove that

$$\partial M := M \cap \big(\overline{\mathcal{M}}(X, \tau_2(e)) - \mathcal{M}(X, \tau_2(e))\big) \tag{70}$$

has an irreducible component of codimension 1 in M, it suffices to prove that for every $(p_1, p_2) \in I$, $\mathrm{ev}^{-1}(p_1, p_2) \cap \partial M \subset \mathrm{ev}^{-1}(p_1, p_2) \cap M$ has codimension 1. Suppose that it has codimension at least 2. Then the coarse moduli space $|\mathrm{ev}^{-1}(p_1, p_2) \cap \partial M| \subset |\mathrm{ev}^{-1}(p_1, p_2) \cap M|$ has codimension at least 2. Since the coarse moduli spaces are proper varieties, we can find a complete curve C in $|\mathrm{ev}^{-1}(p_1, p_2) \cap M|$ which does not intersect $|\mathrm{ev}^{-1}(p_1, p_2) \cap \partial M|$. Since $\overline{\mathcal{M}}(X, \tau_2(e))$ is a Deligne-Mumford stack, there exists a finite ramified cover $C' \to C$ such that $C' \to |\overline{\mathcal{M}}(X, \tau_2(e))|$ factors through $\overline{\mathcal{M}}(X, \tau_2(e)) \to |\overline{\mathcal{M}}(X, \tau_2(e))|$. But then by Lemma 5.1, we conclude that $C' \to |\overline{\mathcal{M}}(X, \tau_2(e))|$ is constant, which contradicts the construction of C. Therefore $\partial M \subset M$ has codimension 1. $\square$

The main application is the following:

Proposition 5.3. *Suppose that $X \subset \mathbb{P}^N$ is a complete intersection. Suppose that $\mathcal{FE}(X, \tau_1(e), f_1)$ holds for every $e < E$ and suppose that every irreducible component* of $\mathcal{M}(X,\tau_1(E))$ *has dimension at least* $2dim(X)$*. Then* $\mathcal{FE}(X,\tau_1(E),f_1)$ *holds as well.*

Proof. By Lemma 4.5, to prove that $\mathcal{FE}(X,\tau_1(E),f_1)$ holds, it suffices to prove that $\mathcal{E}(X,\tau_1(E),f_1)$ holds. Let $\zeta \in \mathcal{M}(X,\tau_1(e))$ be a point, denote $p = \mathrm{ev}_{f_1}(\zeta)$, and let $M \subset \mathrm{ev}_{f_1}^{-1}(p)$ be an irreducible component which contains p. We need to prove that $\dim(M) = \dim(X,\tau_1(e)) - \dim(X)$.

Now consider the forgetful morphism

$$\mathcal{M}(X,a) : \mathcal{M}(X,\tau_2(e)) \to \mathcal{M}(X,\tau_1(e)). \tag{71}$$

This is a smooth surjective morphism: let $N \subset \mathcal{M}(X,\tau_2(e))$ be the preimage of M. Then $\dim(N) = \dim(M) + 1$, thus we have to prove that

$$\dim(N) = \dim(X,\tau_1(e)) + 1 - \dim(X). \tag{72}$$

It suffices to prove that the general fiber of $\mathrm{ev}_{f_2} : N \to X$ has dimension at most $\dim(X,\tau_1(e)) + 1 - 2\dim(X)$ (since we already know the dimension is at least this large).

Choose any point $q \neq p$ in $\mathrm{ev}_{f_2}(N)$ and consider $N_q := \mathrm{ev}_{f_2}^{-1}(q) \cap N$. By assumption, $\dim(X,\tau_1(e))+1-2\dim(X) \geq 1$, so $\dim(N_q) \geq 1$. Define $\overline{N} \subset \overline{\mathcal{M}}(X,\tau_2(e))$ to be the closure of N. By Corollary 5.2 we conclude that $\partial\overline{N} \subset \overline{N}$ has codimension 1. In other words, there is a stable A-graph $\sigma' \ncong \tau_2(e)$ whose canonical contraction is $\phi' : \sigma' \to \tau_2(e)$ and such that $\overline{N} \cap \mathrm{Image}(\mathcal{M}(X,\sigma')) \subset \overline{N}$ has an irreducible component of codimension 1.

Now there is precisely one stable A-graph $\sigma_0 \ncong \tau_2(e)$ whose stabilization after removing f_2 equals $\tau_1(e)$, namely:

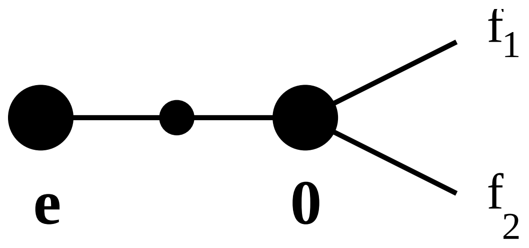

Diagram 2

By the assumption that $q \neq p$, the point $(q,p) \notin \mathrm{ev}_{f_1,f_2}(\mathcal{M}(X,\sigma_0))$. So σ' is not σ_0. We conclude that the image of $\mathcal{M}(X,\sigma')$ in $\overline{\mathcal{M}}(X,\tau_1(e))$ (under the map which stably removes f_2) is again a boundary component $\overline{\mathcal{M}}(X,\sigma)$ for some $\phi : \sigma \to \tau_1(e)$ (not the identity). Therefore $\overline{M} \cap \mathrm{Image}(\mathcal{M}(X,\sigma)) \subset \overline{M}$ is a locally closed substack such that some irreducible component has codimension one in $\overline{M}$.

Since $\sigma \ncong \tau_1(E)$, we have $E(\sigma) < E$. By our assumption and by Proposition 4.8, we conclude that $\mathcal{FE}(X,\sigma,f_1)$ holds. In particular, $\overline{M} \cap \mathrm{Image}(\mathcal{M}(X,\sigma))$ has dimension at most $\dim(X,\sigma) - \dim(X)$. So the dimension of $\overline{M}$ is at most $\dim(X,\sigma) + 1 - \dim(X)$. Since $\dim(X,\sigma) + 1 \leq \dim(X,\tau_1(e))$, we conclude that $\dim(M) \leq \dim(X,\tau_1(e)) - \dim(X)$. So $\mathcal{FE}(X,\tau_1(e),f_1)$ holds. $\square$

Remark The condition that $X \subset \mathbb{P}^N$ be a complete intersection is essentially superfluous. If instead we had worked throughout with the property $\mathcal{E}(X, \tau, f)$ rather than $\mathcal{FE}(X, \tau, f)$ (which is a little trickier), then the argument above proves the analogous result without this condition on X.

Definition 5.4. If $X \subset \mathbb{P}^N$ is a complete intersection of hypersurfaces of degrees $d_1, d_2, \ldots, d_r$, define the *threshold degree* to be

$$E(X) = E(N, (d_1, \ldots, d_r)) = \left\lfloor \frac{N + 2 - r}{N + 1 - (d_1 + \cdots + d_r)} \right\rfloor . \tag{73}$$

In particular, if $X \subset \mathbb{P}^N$ is a hypersurface of degree $d < \frac{N+1}{2}$ then $E(X) = 1$.

Corollary 5.5. *Suppose that $X \subset \mathbb{P}^N$ is a complete intersection of hypersurfaces of degrees $d_1, d_2, \ldots, d_r$. If $\mathcal{FE}(X, \tau_1(e), f_1)$ holds for each $1 \leq e \leq E(X)$, then for every stable A-graph τ and flag $f \in Flag(\tau)$, $\mathcal{FE}(X, \tau, f)$ holds. In particular, if $X \subset \mathbb{P}^N$ is a hypersurface of degree $d < \frac{N+1}{2}$ then it suffices to prove $\mathcal{FE}(X, \tau_1(1), f_1)$.*

Proof. By Proposition 4.8, to prove that $\mathcal{FE}(X, \tau, f)$ always holds it suffices to prove that $\mathcal{FE}(X, \tau_1(e), f_1)$ always holds (for $e > 0$). Now for $e > E(X)$ we have by Lemma 4.2 that every irreducible component of $\mathcal{M}(X, \tau_1(e))$ has dimension at least

$$\begin{aligned} (N + 1 - (d_1 + \cdots + d_r))\, e + (N - r - 3) + 1 \geq \\ (N + 1 - (d_1 + \cdots + d_r))\, \tfrac{N+2-r}{N+1-(d_1+\cdots+d_r)} + (N - r - 3) + 1, \end{aligned} \tag{74}$$

which, of course, is just $2\dim(X)$. So by Proposition 5.3 and induction, to prove $\mathcal{FE}(X, \tau_1(e), f_1)$ for all e, it suffices to prove $\mathcal{FE}(X, \tau_1(e), f_1)$ in the cases $e = 1, \ldots, E(X)$. □

Corollary 5.6. *With the same hypotheses as in Corollary 5.5, suppose that $\mathcal{FE}(X, \tau_1(e), f_1)$ holds for each $1 \leq e \leq E(X)$. Then for every stable A-graph τ, $\mathcal{M}(X, \tau)$ has pure dimension $\dim(X, \tau)$.*

Proof. If $E(\tau) = 0$, then $\mathcal{M}(X, \tau) = \mathcal{M}(*, \tau) \times X$ and the result follows from [2, Proposition 7.4]. Suppose that $E(\tau) > 0$ and let $v \in \text{Vertex}(\tau)$ be such that $\beta(v) > 0$. Define $a : \tau' \hookleftarrow \tau$ to be the combinatorial morphism which adds a new tail f to v. Then $\mathcal{M}(X, a)$ is smooth and surjective of relative dimension 1. By Corollary 5.5, $\mathcal{FE}(X, \tau', f)$ holds. In particular, $\mathcal{M}(X, \tau')$ has pure dimension $\dim(X, \tau')$. It follows that $\mathcal{M}(X, \tau)$ has pure dimension $\dim(X, \tau') - 1 = \dim(X, \tau)$. □

A second application of the proof of Proposition 5.3 is the following:

Proposition 5.7. *Suppose that $X \subset \mathbb{P}^N$ is a complete intersection. Suppose that $\mathcal{FE}(X, \tau_1(e), f_1)$ holds for every $e < E$ and suppose that every irreducible component of $\mathcal{M}(X, \tau_1(E))$ has dimension at least $2\dim(X)$. Then for every irreducible component $M \subset \mathcal{M}(X, \tau_0(E))$ there is a graph $\sigma = \tau_{0,0}(i, j)$, $0 < i, j$ and $i + j = E$, and an irreducible component $N \subset \mathcal{M}(X, \tau_{0,0}(i, j))$ such that $N \subset \overline{M}$ is a codimension 1 subvariety.*

Proof. Let $M' \subset \mathcal{M}(X, \tau_1(e))$ be the irreducible component which dominates M. By the proof of Proposition 5.3 there is a graph $\sigma \ncong \tau_1(e)$ with canonical contraction $\phi : \sigma \to \tau_1(e)$ such that $\overline{M'} \cap \mathcal{M}(X, \sigma) \subset \overline{M}$ has codimension 1. Now $\mathcal{M}(X, \sigma)$ has dimension $\dim(X, \sigma)$ and by Lemma 4.2 $\overline{M'}$ has dimension at least $\dim(X, \tau_1(e))$. But $\dim(X, \sigma) = \dim(X, \tau_1(e)) - \#\text{Edge}(\sigma)$. So σ has exactly one edge, i.e. $\sigma = \tau_{1,0}(i, j)$ for some i, j with $i + j = E$. By stability $i, j > 0$. Moreover, $\overline{M'} \cap \mathcal{M}(X, \sigma)$

has dimension $\dim(X, \sigma)$ so there is an irreducible component $N' \subset \mathcal{M}(X, \sigma)$ such that $N' \subset \overline{M'}$.

Since $\mathcal{M}(X, \tau_{1,0}(i,j)) \to \mathcal{M}(X, \tau_{0,0}(i,j))$ is smooth of relative dimension 1, there is an irreducible component $N \subset \mathcal{M}(X, \tau_{0,0}(i,j))$ such that N' is the preimage of N. Thus $N \subset \overline{M}$ is a codimension 1 subvariety. □

Definition 5.8. Let $X \subset \mathbb{P}^N$ be a complete intersection with threshold degree $E(X) = E$. A stable A-graph τ is *basic* for X if its maximal component degree $E(\tau)$ satisfies $E(\tau) \leq E(X)$.

Definition 5.9. For an A-graph τ, define its *degree 0 subgraph* to be the maximal subgraph $\tau \hookleftarrow \tau^0$ such that $E(\tau^0) = 0$, i.e. τ^0 is the (possibly disconnected) subgraph of τ with

$$\mathrm{Vertex}(\tau^0) = \{v \in \mathrm{Vertex}(\tau) | \beta(v) = 0\} \tag{75}$$

and

$$\mathrm{Flag}(\tau^0) = \{f \in \mathrm{Flag}(\tau) | \partial f \in \mathrm{Vertex}(\tau^0)\}. \tag{76}$$

A contraction of A-graphs $\phi : \sigma \to \tau$ is *nice* if ϕ induces an isomorphism $\sigma^0 \cong \tau^0$.

Theorem 5.10. *Let $X \subset \mathbb{P}^N$ be a complete intersection, τ a stable A-graph and $M \subset \mathcal{M}(X, \tau)$ an irreducible component. Suppose that $\mathcal{FE}(X, \tau_1(e), f_1)$ holds for each $1 \leq e \leq E(X)$. Then there exists a nice contraction $\phi : \sigma \to \tau$ and an irreducible component $N \subset \mathcal{M}(X, \sigma)$ such that σ is basic and such that $N \subset \overline{M}$.*

Proof. We will prove this by induction on the maximal component degree $E(\tau)$. If $E(\tau) \leq E(X)$, then we can take ϕ to be the identity $\tau \to \tau$ and $N = M$.

Suppose that $E > E(X)$. By way of induction, assume the theorem is proved for all graphs τ with $E(\tau) < E$. We will deduce the theorem for graphs with $E(\tau) = E$ by induction on $\#\mathrm{Vertex}(\tau)$, and thus establish the theorem by induction on $E(\tau)$.

First we consider the case that $\#\mathrm{Vertex}(\tau) = 1$, i.e. $\tau = \tau_r(E)$ for some r. Define $a : \tau_r(E) \hookleftarrow \tau_0(E)$ be the combinatorial morphism which strips the tails from $\tau_r(E)$. Then $\mathcal{M}(X, a) : \mathcal{M}(X, \tau_r(E)) \to \mathcal{M}(X, \tau_0(E))$ is smooth, surjective with connected fibers of dimension r. So we conclude that M is the preimage of an irreducible component $M' \subset \mathcal{M}(X, \tau_0(E))$. Now by Proposition 5.7 there is a nice contraction $\psi : \rho \to \tau_0(E)$ and an irreducible component $L \subset \mathcal{M}(X, \rho)$ such that $L \subset \overline{M}'$. As $E(\rho) < E$, by assumption there exists a nice contraction $\phi' : \sigma' \to \rho$ such that σ' is basic and there exists an irreducible component $N' \subset \mathcal{M}(X, \sigma)$ such that $N' \subset \overline{L}$. Therefore $\psi \circ \phi' : \sigma' \to \tau_0(E)$ is a nice contraction and $N' \subset \overline{M'}$. Now let $v \in \mathrm{Vertex}(\sigma')$ be any vertex and let $b : \sigma \hookleftarrow \sigma'$ be the graph obtained by attaching r tails to v. Note that $\mathcal{M}(X, b)$ is smooth, surjective with connected fibers of dimension r. Let $\phi : \sigma \to \tau_r(E)$ be the contraction obtained from $\psi \circ \phi'$ by sending the r tails of σ to the r tails of $\tau_r(E)$. Then ϕ is a nice contraction and σ is basic. Define $N \subset \mathcal{M}(X, \sigma)$ to be the preimage of N' under $\mathcal{M}(X, a)$. The morphism $\overline{\mathcal{M}}(X, a)$ is compatible with $\mathcal{M}(X, b)$, i.e. $\overline{\mathcal{M}}(X, a) \circ \mathcal{M}(X, \phi) = \mathcal{M}(X, \psi \circ \phi') \circ \mathcal{M}(X, b)$, and $\overline{\mathcal{M}}(X, a)$ is smooth along the image of $\mathcal{M}(X, \phi)$. Thus we conclude that $N \subset \overline{M}$, the theorem is proved for M.

Now we consider the general case. For each graph τ, define

$$l(\tau) := \#\mathrm{Vertex}(\tau). \tag{77}$$

When $l = 1$, we have the case in the last paragraph. Suppose $l > 1$ and, by way of induction on l, suppose that for all stable A-graphs τ with $E(\tau) = E$ and with $l(\tau) < l$, the theorem is proved. Suppose that τ is a stable A-graph with $E(\tau) = E$

and $l(\tau) = l$. Let $\{f_1, f_2\}$ be an edge of τ. Define $a_1 : \tau \hookleftarrow \tau_1$ and $a_2 : \tau \hookleftarrow \tau_2$ be the two subgraphs obtained by breaking the edge (see diagram 1).

Now the morphism $\mathcal{M}(X, a_1) : \mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau_1)$ is the composition of an open immersion and the projection of the fiber product

$$\mathcal{M}(X, \tau_1) \times_{\mathrm{ev}_{f_1}, X, \mathrm{ev}_{f_2}} \mathcal{M}(X, \tau_2) \to \mathcal{M}(X, \tau_1). \tag{78}$$

By Proposition 5.3 the morphism ev_{f_2} is flat. Therefore $\mathcal{M}(X, a_1)$ is flat. So M dominates an irreducible component M_1 of $\mathcal{M}(X, a_1)$. Since $l(\tau_1) = l(\tau) - l(\tau_2)$ and $l(\tau_2) > 0$ by the assumption that $E(\tau_2) = E$, we have $l(\tau_1) < l(\tau)$. So by the induction assumption, there exists a nice contraction $\phi_1 : \rho_1 \to \tau_1$ and an irreducible component $L_1 \subset \mathcal{M}(X, \rho_1)$ such that ρ_1 is basic and such that $L_1 \subset \overline{M_1}$.

Since $\overline{\mathcal{M}}(X, a_1)$ is proper, there exists an irreducible subvariety $L \subset \overline{M}$ such that $\overline{\mathcal{M}}(X, a_1)(L) = L_1$ and such that the fiber dimension of $L \to L_1$ is at least the fiber dimension of $\mathcal{M}(X, a_1)$. Up to replacing L by an open subset, we may suppose L is contained in one of the locally closed substacks $\mathcal{M}(X, \rho)$. Since $\overline{\mathcal{M}}(X, a_1)$ maps L into $\mathcal{M}(X, \rho_1)$, we must have that ρ is glued from ρ_1 and a graph ρ_2 by making an edge out of f_1 and f_2. Moreover ρ_2 must contract to τ_2. But then the dimension of L is at most

$$\begin{gathered} \dim(X, \rho_1) + \dim(X, \rho_2) - \dim(X) = \\ \dim(X, \rho_1) + (\dim(X, \tau) - \dim(X, \tau_1)) - (\#\mathrm{Edge}(\rho_2) - \#\mathrm{Edge}(\tau_2)) . \end{gathered} \tag{79}$$

Thus we conclude that $\#\mathrm{Edge}(\rho_2) = \#\mathrm{Edge}(\tau_2)$, i.e. $\rho_2 = \tau_2$. So $\psi : \rho \to \tau$ is a nice contraction and $L \subset \mathcal{M}(X, \rho)$ is an irreducible component such that $L \subset \overline{M}$. Moreover, $l(\rho) = l(\rho_2) = l(\tau_2) < l(\tau)$. By the induction assumption, there exists a nice contraction $\phi : \sigma \to \rho$ and an irreducible component $N \subset \mathcal{M}(X, \sigma)$ such that σ is basic and such that $N \subset \overline{L}$. But then $\psi \circ \phi : \sigma \to \tau$ is nice and $N \subset \overline{M}$, i.e. the theorem is proved for M. So the theorem is proved by induction on E and l. $\square$

The previous theorem suggests a strategy for proving that any given $\mathcal{M}(X, \tau)$ is irreducible:

1. Determine all nice contractions $\phi : \sigma \to \tau$ such that σ is basic.
2. Determine all irreducible components $N \subset \mathcal{M}(X, \sigma)$.
3. Show that for each N, there is a unique irreducible component $M(N) \subset \mathcal{M}(X, \sigma)$ which contains N.
4. Prove that all of the putative irreducible components $M(N)$ are actually equal.

The first step (1) is a combinatorial problem. The simplest case for (2) is when $\mathcal{M}(X, \sigma)$ is itself irreducible for each basic σ. One can try to prove (3) by a deformation theory argument; if one proves that the general point of M is a smooth point of the stack $\mathcal{M}(X, \tau)$, then it follows that there is a unique irreducible component $M(N)$ which contains N. We will prove (4) by linking up basic graphs using *almost basic graphs* (we will explain this further below). Although one should be able to carry out this strategy in the case of complete intersections (and perhaps even more general varieties), in the remainder of this paper we will restrict ourselves to hypersurfaces $X \subset \mathbb{P}^N$ with $d < \frac{N+1}{2}$. Then the steps above all reduce to questions regarding lines on X.

6. Properties of evaluation morphisms

In the last two sections we investigated when an evaluation morphism $\mathrm{ev}_f : \mathcal{M}(X, \tau) \to X$ is flat of the expected dimension. In this section we also investigate when the general fiber is irreducible, and when the morphism ev_f is *unobstructed* at a general point of $\mathcal{M}(X, \tau)$. By the same techniques in the last two sections, we reduce these properties for a general basic A-graph τ to the property for A-graphs of the form $\tau_1(e)$ with $e = 1, \dots, E(X)$. Using this result, we carry out Steps (2) and (3) of the strategy of proof in the previous sections.

The new property of evaluation morphisms we want to consider is the following.

Definition 6.1. Suppose $X \subset \mathbb{P}^N$ is a smooth subvariety, τ is a stable A-graph and $f \in \mathrm{Flag}(\tau)$. We say that $\mathcal{B}(X, \tau, f)$ holds if we have

(1) $\mathcal{FE}(X, \tau, f)$ holds,
(2) the general fiber of ev_f is geometrically irreducible,
(3) in $\mathcal{M}(X, \tau)$ there is a point $[h : C \to X]$ which is *free*, i.e. h^*T_X is generated by global sections.

The difficult item to check is still (1). We return to this point at the end of this section. First we give a reformulation of item (3) of the definition above.

Lemma 6.2. *Suppose that (C, q_{f_i}) is a genus 0 prestable curve with dual graph τ, and suppose that E is a locally free sheaf on C. The following are equivalent:*

(1) *E is generated by global sections.*
(2) *For each irreducible component C_v of C, the restriction E_v of E to C_v is generated by global sections.*
(3) *For each irreducible component C_v of C and each point $q \in C_v$, we have $H^1(C_v, E_v(-q)) = 0$.*

Proof. Clearly (1) implies (2). By Grothendieck's Lemma [7, exercise V.2.4], E_v splits as a direct sum of line bundles $L_1 \oplus \dots \oplus L_r$. If (2) is satisfied, then each L_v is generated by global sections, i.e. if we identify C_v with $\mathbb{P}^1$, then $L_i = \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 0$. Since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a-1)) = 0$ for $a \geq 0$, we conclude that $H^1(C_v, E_v(-q)) = 0$. So (2) implies (3).

Finally suppose that (3) is satisfied. We shall prove that E is generated by global sections by induction on the number of vertices of τ, i.e. on the number of irreducible components of C. If C has a single irreducible component, then C is isomorphic to $\mathbb{P}^1$. By Grothendieck's Lemma we know $E = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ for some integers a_i. Since $H^1(\mathbb{P}^1, E(-q)) = 0$, we conclude that each $a - 1 \geq -1$, i.e $a \geq 0$. So E is generated by global sections.

Now suppose that τ has more than one vertex and let v_1 be any leaf of τ, i.e. v_1 is adjacent to exactly one other vertex. Let $i_1 : C_1 \to C$ be the irreducible component associated to v_1, let $i_2 : C_2 \to C$ be the union of all the other irreducible components of C and let q be the unique point of intersection of C_1 and C_2. Let E_1 denote the restriction of E to C_1 and let E_2 denote the restriction of E to C_2. By the induction assumption, we may assume that E_2 is generated by global sections. But now we have an exact sequence of sheaves on C:

$$0 \to (i_1)_*(E_1(-q)) \to E \to (i_2)_*(E_2) \to 0. \tag{80}$$

The obstruction to lifting the global sections of E_2 to global sections of E is an element of $H^1(C_1, E_1(-q))$, which is zero by assumption. So every global section of E_2 is the restriction of a global section of E. Thus the locus where E isn't generated by global sections (i.e. the cokernel of the morphism $H^0(C, E) \otimes_{\mathbb{C}} \mathcal{O}_E \to E$) is a closed subset of $C_1 - C_2$. Since τ has more than one vertex, we can find a second leaf v_2 of τ. Repeating the argument with v_2 we conclude that E is generated by global sections. $\square$

Lemma 6.3. *With the same notation as in Lemma 6.2, if E satisfies any of the three equivalent conditions above, and if $p \in C$ is any smooth point, then $H^1(C, E(-p)) = 0$.*

Proof. If C has a single irreducible component, this follows from the equivalent condition (3) in Lemma 6.2. Suppose that C has $l > 1$ irreducible components. By way of induction, suppose that the lemma has been proved for all curves with fewer than l irreducible components. We can find a leaf v_1 of C such that p is not contained in the corresponding irreducible component C_2. Let C_2, i_1, i_2, and q be as in the proof of Lemma 6.2. Then we have a short exact sequence:

$$0 \to (i_1)_*(E_1(-q)) \to E(-p) \to (i_2)_*(E_2(-p)) \to 0. \tag{81}$$

By the induction assumption, both $H^1(C_1, E_1(-q)) = 0$ and $H^1(C_2, E_2(-p)) = 0$. So by the long exact sequence in cohomology associated to the short exact sequence above, we conclude that $H^1(C, E(-p)) = 0$. So the lemma is proved by induction on l. $\square$

Lemma 6.4. *Suppose $n > 2$ and $X \subset \mathbb{P}^n$ is a general hypersurface of degree $d < \frac{n+1}{2}$. Then $\mathcal{B}(X, \tau_1(1), f_1)$ holds.*

Proof. By Theorem 2.1, for general X we have that $\mathcal{FE}(X, \tau_1(1), f_1)$ holds, i.e. we have (1). By Lemma 2.6, for general X there is a free line on X, i.e. we have (3).

By Theorem 2.7, for general X, $F_{0,1}(X)$ is smooth. Thus by generic smoothness, the general fiber of ev_{f_1} is smooth. By Lemma 2.3, the general fiber of ev_{f_1} is a complete intersection in $\mathbb{P}^{n-1}$ of dimension $n - d - 1 > 1$. Thus the general fiber is geometrically connected by repeated application of [4, corollaire 3.5, exp. XII]. Since a smooth, geometrically connected scheme is geometrically irreducible, we have (2). $\square$

The main theorem of this section is the following:

Proposition 6.5. *Suppose $X \subset \mathbb{P}^N$ is a smooth subvariety which satisfies $\mathcal{B}(X, \tau_1(e), f_1)$ for $e = 1, \ldots, E$. Let τ be an A-graph such that $E(\tau) \leq E$. Then we have the following:*

1. *For each $f \in Flag(\tau)$, we have $\mathcal{B}(X, \tau, f)$.*
2. *$\mathcal{M}(X, \tau)$ is an irreducible stack.*

Proof. Both statements are trivial in case τ is empty, so assume τ is nonempty. Observe that (1) implies (2): given $v \in \mathrm{Vertex}(\tau)$, define a new A-graph τ' and a combinatorial morphism $\alpha : \tau' \hookleftarrow \tau$ which attaches a new flag f' to τ at v. Then $\mathcal{M}(X, \alpha) : \mathcal{M}(X, \tau') \to \mathcal{M}(X, \tau)$ is smooth, surjective with geometrically irreducible fibers. So $\mathcal{M}(X, \tau')$ is irreducible iff $\mathcal{M}(X, \tau)$ is irreducible. By (1), $\mathrm{ev}_{f'} : \mathcal{M}(X, \tau') \to X$ is flat and the general fiber is geometrically irreducible. Since X is irreducible, it follows that $\mathcal{M}(X, \tau')$ is irreducible. So it remains to prove (1).

First of all, suppose that $\beta(\tau) = 0$. Let $\alpha : \tau \hookleftarrow \emptyset$ be the unique morphism. Then ev_f coincides with $\mathcal{M}(X, \alpha) : \mathcal{M}(X, \tau) \to \mathcal{M}(X, \emptyset) = X$. Thus (1) follows from Lemma 3.12. So we are reduced to the case that $\beta(\tau) > 0$.

We prove (1) by induction on the number of vertices of τ. Suppose that τ has a single vertex, i.e. $\tau = \tau_r(e)$ for some $r > 0$. Let $\alpha : \tau_r(e) \hookleftarrow \tau_1(e)$ be the unique combinatorial morphism which maps $f_1 \in \text{Flag}(\tau_1(e))$ to $f \in \text{Flag}(\tau_r(e))$. Then ev_f factors as the composition

$$\mathcal{M}(X, \tau_r(e)) \xrightarrow{\mathcal{M}(X,\alpha)} \mathcal{M}(X, \tau_1(e)) \xrightarrow{\text{ev}_{f_1}} X. \tag{82}$$

Of course $\mathcal{M}(X, \alpha)$ is an open immersion into the $(r-1)$-fold fiber product of the universal curve, so $\mathcal{M}(X, \alpha)$ is smooth with geometrically irreducible fibers. And by $\mathcal{B}$, ev_{f_1} is flat and the general fiber is geometrically irreducible. Therefore the composition is flat and the general fiber is geometrically irreducible, i.e. items (1) and (2) of condition $\mathcal{B}(X, \tau_r(e), f)$ hold. The condition that h^*T_X is generated by global sections is independent of the number of vertices, so item (3) of $\mathcal{B}(X, \tau_r(e), f)$ holds as well, i.e. $\mathcal{B}(X, \tau_r(e), f)$ holds.

Now suppose that there is more than one vertex, say $\#\text{Vertex}(\tau) = l > 1$. By way of assumption, suppose that $\mathcal{B}(X, \sigma, f)$ has been proved for all A-graphs σ such that $\#\text{Vertex}(\sigma) < l$. Let $\{f_1, f_2\}$ be any edge and consider the subgraphs $\alpha_1 : \tau \hookleftarrow \tau_1$ and $\alpha_2 : \tau \hookleftarrow \tau_2$ as in diagram 1. Without loss of generality, suppose that f is in τ_1. Then ev_f factors as the composition:

$$\mathcal{M}(X, \tau) \xrightarrow{\mathcal{M}(X,\alpha_1)} \mathcal{M}(X, \tau_1) \xrightarrow{\text{ev}_f} X. \tag{83}$$

Now $\mathcal{M}(X, \alpha_1)$ factors as the composition of an open immersion and the projection

$$\pi_1 : \mathcal{M}(X, \tau_1) \times_{\text{ev}_{f_1}, X, \text{ev}_{f_2}} \mathcal{M}(X, \tau_2) \to \mathcal{M}(X, \tau_1). \tag{84}$$

Since $\#\text{Vertex}(\tau_i) < l$ for $i = 1, 2$, the induction assumption says that (2) holds for $\text{ev}_{f_i} : \mathcal{M}(X, \tau_i) \to X$. Since ev_{f_1} is open, the general fiber of π_1 dominates the general fiber of ev_{f_2}. So π_1 is flat and the general fiber is geometrically irreducible. Thus the same is true of $\mathcal{M}(X, \alpha_1)$. Since $\#\text{Vertex}(\tau_1) < l$, $\text{ev}_f : \mathcal{M}(X, \tau_1) \to X$ is flat and the general fiber is geometrically irreducible. Thus the composition is flat and the general fiber is geometrically irreducible, i.e. items (1) and (2) of $\mathcal{B}(X, \tau, f)$ hold.

Finally we consider item (3) of $\mathcal{B}(X, \tau, f)$. Each of the two projections $\mathcal{M}(X, \alpha_1)$ and $\mathcal{M}(X, \alpha_2)$ are dominant. By the induction assumption, for $i = 1, 2$ the set of points in $\mathcal{M}(X, \tau_1)$ which parametrize stable maps with $h * T_X$ generated by global sections is an open, dense set U_i. The preimage of each U_i in $\mathcal{M}(X, \tau)$ is an open dense set, and the intersection of these two open dense sets is an open dense set. For a point in this intersection – using the equivalent condition (2) of Lemma 6.2 – we have that the restriction of h^*T_X to each irreducible component with vertex $v \in \tau_1$ is generated by global sections, and also the restriction of h^*T_X to each irreducible component with vertex $v \in \tau_2$ is generated by global sections. So by the equivalent condition (2) of Lemma 6.2, we conclude that h^*T_X is generated by global sections. Thus item (3) of $\mathcal{B}(X, \tau, f)$ is satisfied. This completes the proof that $\mathcal{B}(X, \tau, f)$ holds, and the proposition is proved by induction. □

Propsition 6.5 simplifies Step 2 in the strategy of the last section to checking that $\mathcal{B}(X, \tau_1(e), f_1)$ holds for all $e = 1, \ldots, E(X)$. Next we reduce Step 3 to checking that $\mathcal{B}(X, \tau_1(e), f_1)$ holds for all $e = 1, \ldots, E(X)$.

Proposition 6.6. *Suppose that τ is a stable A-graph, $f \in$ Tail(τ) and suppose that $\mathcal{B}(X, \tau, f)$ holds. Suppose that $\alpha : \tau \to \sigma$ is a contraction. The morphism $\mathcal{M}(X, \alpha)$ maps a general point of $\mathcal{M}(X, \tau)$ to a point in the smooth locus of the morphism ev$_f : \overline{\mathcal{M}}(X, \sigma) \to X$.*

Proof. As in the proof of Proposition 6.5, the case that $\beta(\sigma) = 0$ follows from Lemma 3.12. So we are reduced to the case $\beta(\sigma) > 0$.

Let $\mathfrak{M}(\sigma)$ denote the (non-separated) Artin stack of prestable σ-curves as in [2, definition 2.6]. There is a 1-morphism $\overline{\mathcal{M}}(X,\sigma) \to \mathfrak{M}(\sigma)$ given by forgetting the map to X. "Remembering" the map to X gives an isomorphism of $\overline{\mathcal{M}}(X,\sigma)$ with the relative scheme of morphisms $\mathrm{Mor}_{\mathfrak{M}}(\sigma)(\sigma, X)$, by [1, prop. 4]. Since $\mathfrak{M}(\sigma)$ is smooth by [1, prop. 2], to prove that ev_f is smooth at a point, it suffices to prove the following morphism is smooth at this point:

$$(\pi, \mathrm{ev}_f) : \overline{\mathcal{M}}(X,\sigma) \to \mathfrak{M} \times X. \tag{85}$$

By [9, Theorem II.1.7], to check that a point $\zeta = ((C_v),(q_{f'}),(h_v))$ of $\overline{\mathcal{M}}(X,\sigma)$ is in the smooth locus of (π, ev_f), it suffices to check that $H^1(C, h^*T_X(-q_f)) = 0$. For a general point in the image of $\mathcal{M}(X,\alpha)$, this follows from item (3) of $\mathcal{B}(X,\tau,f)$ along with Lemma 6.3. □

Corollary 6.7. *Suppose $X \subset \mathbb{P}^N$ is a smooth subvariety which satisfies $\mathcal{B}(X,\tau_1(e),f_1)$ for all $e = 1,\dots,E$. Let τ be an A-graph with $E(\tau) \leq E$ and suppose that $\alpha : \tau \to \sigma$ is a contraction. The morphism $\mathcal{M}(X,\alpha)$ maps a general point of $\mathcal{M}(X,\tau)$ to a smooth point of $\overline{\mathcal{M}}(X,\sigma)$.*

Proof. This is trivial if τ is empty. Suppose τ is not empty and let v be a vertex of τ. Let $\tau \hookleftarrow \tau'$ be the combinatorial morphism which attaches a new tail, f, at the vertex v. Let $\sigma \hookleftarrow \sigma'$ be the combinatorial morphism which attaches a new tail, f, at the vertex of σ which is the image of v. Let $\alpha' : \tau' \to \sigma'$ be the contraction which restricts to α and which maps f to f. By Proposition 6.5, $\mathcal{B}(X,\tau',f)$ holds. By Proposition 6.6, $\mathcal{M}(X,\alpha')$ maps a general point of $\mathcal{M}(X,\tau')$ to a point in the smooth locus of $\mathrm{ev}_f : \overline{\mathcal{M}}(X,\sigma') \to X$. A point in the smooth locus of ev_f is a smooth point of $\overline{\mathcal{M}}(X,\sigma')$. The image of this point in $\overline{\mathcal{M}}(X,\sigma)$ is also a smooth point. Since $\mathcal{M}(X,\tau')$ surjects onto $\mathcal{M}(X,\tau)$, a general point of $\mathcal{M}(X,\tau')$ maps to a general point of $\mathcal{M}(X,\tau)$. Thus $\mathcal{M}(X,\alpha)$ maps a general point of $\mathcal{M}(X,\tau)$ to a smooth point of $\overline{\mathcal{M}}(X,\sigma)$. □

Remark: Now suppose $\mathcal{B}(X,\tau_1(e),f_1)$ holds for all $e = 1,\dots,E$, suppose that τ is a stable A-graph with $E(\tau) \leq E$ and suppose that $\alpha : \tau \to \sigma$ is a contraction. By Corollary 6.7 and (2) of Proposition 6.5, we conclude that there is a unique irreducible component $M(\alpha)$ of $\overline{\mathcal{M}}(X,\sigma)$ which contains the image of $\mathcal{M}(X,\alpha)$, and $M(\alpha)$ is smooth of the expected dimension at a general point. So Steps (2) and (3) of the strategy in the last section are successful.

Finally we give a simpler criterion for when $\mathcal{B}(X,\tau,f)$ holds for all τ with $E(\tau) \leq E$, where E is some fixed integer, and also reduce the number of components $M(\alpha)$ we have to deal with in Step (4) of our strategy.

Proposition 6.8. *Suppose that $X \subset \mathbb{P}^N$ is a smooth subvariety satisfying*

1. *$\mathcal{B}(X,\tau_1(1),f_1)$ holds,*
2. *$\mathcal{FE}(X,\tau_1(e),f_1)$ holds for $e = 1,\dots,E$, and*
3. *$\mathcal{M}(X,\tau_0(e))$ is irreducible for $e = 1,\dots,E$.*

Then for each stable A-graph τ with $E(\tau) \leq E$ and each flag $f \in Flag(\tau)$, $\mathcal{B}(X,\tau,f)$ holds and there is a nice contraction $\alpha : \sigma \to \tau$ such that $E(\sigma) \leq 1$ and such that $\mathcal{M}(X,\alpha)$ maps the general point of $\mathcal{M}(X,\sigma)$ to a smooth point of $\overline{\mathcal{M}}(X,\tau)$.

Proof. It is easy to see that there is always a nice contraction $\alpha : \sigma \to \tau$ such that $E(\sigma) \leq 1$. By Proposition 6.6, we know that $\mathcal{M}(X, \alpha)$ maps a general point of $\mathcal{M}(X, \sigma)$ to a smooth point of $\overline{\mathcal{M}}(X, \tau)$.

By Proposition 6.5, to prove that $\mathcal{B}(X, \tau, f)$ holds for all τ with $E(\tau) \leq E$, it suffices to prove that $\mathcal{B}(X, \tau_1(e), f_1)$ holds for all $e = 1, \dots, E$. Let $\overline{\mathcal{M}}(X, \tau_1(e))'$ denote the normalization of $\overline{\mathcal{M}}(X, \tau_1(e))$. Consider the Stein factorization $\overline{\mathcal{M}}(X, \tau_1(e))' \to Z \to X$ of ev_{f_1}. By Proposition 6.6, there is a open, dense subset $U \subset \mathcal{M}(X, \sigma)$ such that $\mathcal{M}(X, \alpha)$ maps U into the smooth locus of ev_{f_1}. So $\mathcal{M}(X, \alpha)|_U : U \to \overline{\mathcal{M}}(X, \tau_1(e))$ factors through $\overline{\mathcal{M}}(X, \tau_1(e))'$. Now consider the image V of U in Z. By Proposition 6.5, the general fiber of $\mathrm{ev}_{f_1}|_U : U \to X$ is geometrically irreducible. Therefore $V \to X$ is generically injective.

Let $\overline{V} \subset Z$ be the Zariski closure of V with the induced, reduced scheme structure. Then $\overline{V}$ is an irreducible stack and $\overline{V} \to X$ is surjective and generically injective. In particular, $\overline{V}$ is a nonempty irreducible component of Z. So the preimage of V in $\overline{\mathcal{M}}(X, \tau_1(e))$ is a nonempty irreducible component of $\overline{\mathcal{M}}(X, \tau_1(e))$. By Corollary 5.6, every stratum in the Behrend-Manin decomposition of $\overline{\mathcal{M}}(X, \tau_1(e))$ has the expected dimension. Thus $\mathcal{M}(X, \tau_1(e))$ is Zariski dense in $\overline{\mathcal{M}}(X, \tau_1(e))$. By assumption, $\mathcal{M}(X, \tau_0(e))$ is irreducible. Since $\mathcal{M}(X, \tau_1(e)) \to \mathcal{M}(X, \tau_1(e))$ is smooth with geometrically connected fibers, also $\mathcal{M}(X, \tau_1(e))$ is irreducible. Therefore $\overline{\mathcal{M}}(X, \tau_1(e))$ is irreducible. Therefore $\overline{V} = Z$ and we conclude that the general fiber of $\overline{\mathcal{M}}(X, \tau_1(e))' \to X$ is normal and geometrically connected, thus geometrically irreducible. It follows that the general fiber of $\mathcal{M}(X, \tau_1(e)) \to X$ is geometrically connected, i.e. we have established item (2) of the definition of $\mathcal{B}(X, \tau_1(e), f_1)$.

To establish item (3) of the definition of $\mathcal{B}(X, \tau_1(e), f_1)$, observe that the locus of points in $\overline{\mathcal{M}}(X, \tau_1(e))$ parametrizing stable maps for which h^*T_X is generated by global sections is an open locus. By Lemma 6.2 and the assumption $\mathcal{B}(X, \tau_1(1), f_1)$, for a general point of $\mathcal{M}(X, \sigma)$, we have that h^*T_X is generated by global sections. So this open set intersects the general point of the image of $\mathcal{M}(X, \alpha)$, so it is nonempty. Therefore it intersects $\mathcal{M}(X, \tau)$ and item (3) follows. ☐

We summarize the results of this section for the case of complete intersections in the following corollary.

Corollary 6.9. *Suppose that $X \subset \mathbb{P}^N$ is a smooth complete intersection of threshold degree $E(X)$ which satisfies:*

(1) $\mathcal{B}(X, \tau_1(1), f_1)$ *holds,*
(2) $\mathcal{FE}(X, \tau_1(e), f_1)$ *holds for* $e = 1, \dots, E(X)$*, and*
(3) $\mathcal{M}(X, \tau_0(e))$ *holds for* $e = 1, \dots, E(X)$.

Then we have

(1) *For each basic A-graph τ and each flag $f \in Flag(\tau)$, $\mathcal{B}(X, \tau, f)$ holds.*
(2) *For each stable A-graph τ and each contraction $\alpha : \sigma \to \tau$ of a basic A-graph σ to τ, there is a unique irreducible component $M(\alpha)$ of $\overline{\mathcal{M}}(X, \tau)$ which contains the image of $\mathcal{M}(X, \alpha)$. Moreover $M(\alpha)$ is smooth of the expected dimension at a general point of the image of $\mathcal{M}(X, \alpha)$.*
(3) *$\overline{\mathcal{M}}(X, \tau)$ is the union of the irreducible components $M(\alpha)$ as $\alpha : \sigma \to \tau$ ranges over nice contractions such that $E(\sigma) \leq 1$.*

Proof. The only new statement is (3). By Theorem 5.10, we know that each irreducible component of $\overline{\mathcal{M}}(X, \tau)$ is one of the irreducible components $M(\alpha')$ for a nice contraction $\alpha' : \sigma' \to \tau$ with σ a basic A-graph. Of course we can find a nice contraction $\beta : \sigma \to \sigma'$ such that $E(\sigma) \leq 1$. Let $\alpha : \sigma \to \tau$ be the composition of β and α'. Then $M(\alpha')$ is an irreducible component which contains the image of $\mathcal{M}(X, \alpha)$. So $M(\alpha') = M(\alpha)$, i.e. we have proved (3). $\square$

7. Equating Irreducible Components

Suppose that $X \subset \mathbb{P}^N$ is a complete intersection which satisfies the hypotheses of Corollary 6.9. Then for each stable A-graph τ, we know that $\overline{\mathcal{M}}(X, \tau)$ has the expected dimension and is a union of irreducible components $M(\alpha)$ as $\alpha : \sigma \to \tau$ ranges over nice contractions with $E(\sigma) \leq 1$. To prove that $\overline{\mathcal{M}}(X, \tau)$ (and hence $\mathcal{M}(X, \tau)$) is irreducible, we are reduced to proving that the irreducible components $M(\alpha)$ are all equal.

Suppose that $\mathcal{B}(X, \tau_1(e), f_1)$ holds for all $e = 1, \ldots, E$ where E is some integer with $E \geq E(X)$. Fix a stable A-graph τ and let $S_E(\tau)$ be the set of (isomorphism classes of) nice contractions $\alpha : \sigma \to \tau$ with $E(\sigma) \leq E$. Define a relation $\alpha \leq \alpha'$ if there exists a contraction $\epsilon : \sigma \to \sigma'$ such that $\alpha = \alpha' \circ \epsilon$. If $\alpha \leq \alpha'$, then observe $M(\alpha) = M(\alpha')$. Form the equivalence relation $\cong$ on $S_E(\tau)$ generated by $\leq$. Notice conclusion (3) of Corollary 6.9 implies that every equivalence class contains a contraction $\alpha : \sigma \to \tau$ such that $E(\sigma) \leq 1$. Since $M(\alpha) = M(\alpha')$ if $\alpha \cong \alpha'$, we see that the number of irreducible components of $\overline{\mathcal{M}}(X, \tau)$ is bounded by the number of equivalence classes of $\cong$ on $S_E(\tau)$. So to prove that $\overline{\mathcal{M}}(X, \tau)$ is irreducible, it suffices to prove that every two elements of $S_E(\tau)$ are equivalent.

Definition 7.1. Given $X \subset \mathbb{P}^N$ a smooth complete intersection, define the *modified threshold degree* of X to be $E'(X) = \max(E(X), 2)$.

Proposition 7.2. *Suppose that $X \subset \mathbb{P}^N$ is a smooth complete intersection such that for $e = 1, \ldots, E'(X)$, we have $\mathcal{B}(X, \tau_1(e), f_1)$ holds. Then for each positive integer e, every two elements of $S_{E'(X)}(\tau_0(e))$ are equivalent. In particular $\overline{\mathcal{M}}(X, \tau_0(e))$ is irreducible.*

Proof. Recall a connected tree τ is called an *path* if τ has precisely one or two vertices (so no vertex has valence greater than 2). The number of vertices in a *path* is the *diameter* of the path. Given any connected tree τ, the *diameter* of τ, $\mathrm{diam}(\tau)$, is defined to be the maximum diameter of a subgraph which is a path. If $\alpha : \sigma \to \tau_0(e)$ is a nice contraction, then there are at most e vertices in σ. So the diameter of σ is at most e. Moreover, there is a unique contraction $\alpha_e : \sigma_e \to \tau_0(e)$ with $\mathrm{diam}(\sigma_e) = e$. Here σ_e is the A-graph whose underlying graph is the path of length e, and for each vertex $v \in \sigma_e$ we have $\beta(v) = 1$.

To prove that any two elements in $S_{E'(X)}(\tau_0(e))$ are equivalent, it suffices to prove that any two nice contractions $\alpha : \sigma \to \tau$ with $E(\sigma) = 1$ are equivalent. We will prove that for each such $\alpha : \sigma \to \tau$ with $\mathrm{diam}(\sigma) < e$, there is a nice contraction $\alpha' : \sigma' \to \tau$ such that $\sigma \cong \sigma'$, $E(\sigma') = 1$ and $\mathrm{diam}(\sigma') \geq \mathrm{diam}(\sigma)$. From this it follows by induction that all such contractions are equivalent to $\alpha_e : \sigma_e \to \tau$.

Suppose that $\alpha : \sigma \to \tau$ is a nice contraction with $E(\sigma) = 1$ and $\mathrm{diam}(\sigma) < e$. Let $\gamma \hookrightarrow \sigma$ be a subgraph which is a path such that $\mathrm{diam}(\gamma) = \mathrm{diam}(\sigma)$. Since γ does not equal σ, there exists a vertex v_1 of γ such that the valence of v_1 is at least 3. Let f_1, f_2 be an edge of σ not contained in γ such that $\partial f_1 = v_1$. Let $v_2 = \partial f_2$. Form the nice contraction $\epsilon : \sigma \to \rho$ which contracts v_1 and v_2 to a common vertex v of σ with $\beta(v) = 2$. The nice contraction $\alpha : \sigma \to \tau_0(e)$ factors through a nice contraction $\alpha_\rho : \rho \to \tau_0(e)$.

The image of γ in ρ is a path γ_ρ which contains v. Now let $\gamma' \to \gamma_\rho$ be a contraction of a path of length $\mathrm{diam}(\gamma) + 1$ which contracts two adjacent vertices w_1 and w_2 to v (where $\beta(w_1) = \beta(w_2) = 1$). There is a unique nice contraction $\epsilon' : \sigma' \to \rho$ such that γ' is a path in σ', such that the restriction of ϵ' to γ' is just $\gamma' \to \rho$, such that every flag of v not contained in γ_ρ is the image of a flag of w_1, and which is an isomorphism from $\sigma' - \gamma' \to \rho - \gamma_\rho$. Define $\alpha' = \alpha_\rho \circ \epsilon$. Then $\alpha' : \sigma' \to \tau_0(e)$ is a nice contraction, $E(\sigma') = 1$, $\alpha \cong \alpha'$ and $\mathrm{diam}(\sigma') = \mathrm{diam}(\sigma)+1$. This proves the claim.

So by induction on the diameter of σ, every element of $S_{E'(X)}(\tau_0(e))$ is equivalent to $\alpha_e : \sigma_e \to \tau_0(e)$. In particular $\overline{\mathcal{M}}(X, \tau_0(e)) = \overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible. □

Corollary 7.3. *With the same hypotheses as in Proposition 7.2, for each stable A-graph τ we have*

1. *$\overline{\mathcal{M}}(X, \tau)$ is an integral, local complete intersection stack of the expected dimension $\dim(X, \tau)$, and $\mathcal{M}(X, \tau)$ is the unique dense stratum in the Behrend-Manin decomposition.*
2. *For each flag $f \in \mathrm{Flag}(\tau)$, $\mathcal{B}(X, \tau, f)$ holds.*
3. *For each contraction $\alpha : \sigma \to \tau$, $\overline{\mathcal{M}}(X, \tau)$ is smooth at the general point of the image of $\mathcal{M}(X, \alpha) : \mathcal{M}(X, \sigma) \to \overline{\mathcal{M}}(X, \tau)$.*

Proof. By Corollary 5.5, $\mathcal{FE}(X, \tau_1(e), f_1)$ holds for all integers $e > 0$. By assumption, $\mathcal{B}(X, \tau_1(1), f_1)$ holds. And by Proposition 7.2, $\mathcal{M}(X, \tau_0(e))$ is irreducible for each integer $e > 0$. Thus by Proposition 6.8, for every stable A-graph τ and every flag $f \in \mathrm{Flag}(\tau)$, we have that $\mathcal{B}(X, \tau, f)$ holds. This establishes (2).

As in the proof of Proposition 6.5, (2) implies that for every stable A-graph τ, $\mathcal{M}(X, \tau)$ is irreducible of the expected dimension. By a parameter count, we conclude that $\mathcal{M}(X, \tau)$ is the unique dense stratum of the Behrend-Manin decomposition of $\overline{\mathcal{M}}(X, \tau)$. So $\overline{\mathcal{M}}(X, \tau)$ is also irreducible of the expected dimension, and generically smooth. So $\mathcal{D}(X, \tau)$ holds. By Lemma 4.5, we conclude that $\mathcal{LCI}(X, \tau)$ holds, i.e. $\overline{\mathcal{M}}(X, \tau)$ is a local complete intersection stack. Since it is generically smooth, and thus generically reduced, it is reduced. So $\overline{\mathcal{M}}(X, \tau)$ is an integral, local complete intersection stack of the expected dimension $\dim(X, \tau)$ and $\mathcal{M}(X, \tau)$ is the unique dense stratum in the Behrend-Manin decomposition. This establishes (1).

Finally (3) follows from (1) and Corollary 6.7. □

Finally, we prove that a general hypersurface $X \subset \mathbb{P}^n$ of degree $d < \frac{n+1}{2}$ satisfies the hypotheses of Proposition 7.2.

Proposition 7.4. *Suppose $n > 2$, $d \leq \frac{n+1}{2}$ and suppose $X \subset \mathbb{P}^n$ is a hypersurface of degree d, so $E'(X) = 2$. If $\mathcal{B}(X, \tau_1(1), f_1)$ holds for X (recall from Lemma 6.4 that $\mathcal{B}(X, \tau_1(1), f_1)$ holds for a general X) then also $\mathcal{B}(X, \tau_1(2), f_1)$ holds. For such an X, the results of Corollary 7.3 hold.*

Proof. By Corollary 5.5, $\mathcal{LCI}(X, \tau_1(2), f_1)$ holds. Since $\mathcal{M}(X, \tau_0(2))$ is the unique dense stratum in the Behrend-Manin decomposition, to prove that $\mathcal{M}(X, \tau_0(2))$ is irreducible, it is equivalent to prove that $\overline{\mathcal{M}}(X, \tau_0(2))$ is irreducible. To see that $\overline{\mathcal{M}}(X, \tau_0(2))$ is irreducible, observe by Theorem 5.10 that every irreducible component of $\overline{\mathcal{M}}(X, \tau)$ is of the form $M(\alpha)$ for a nice contraction $\alpha : \sigma \to \tau_0(2)$ with $E(\alpha) = 1$. But there is a unique such contraction, namely $\alpha_2 : \sigma_2 \to \tau_0(2)$. So $\mathcal{M}(X, \tau_0(2))$ is irreducible.

Now by property 6.8, $\mathcal{B}(X, \tau_1(2), f_1)$ holds. □

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