RESTRICTION OF SECTIONS FOR FAMILIES OF ABELIAN VARIETIES

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Abstract. Given a family of Abelian varieties over a positive-dimensional base, we prove that for a sufficiently general curve in the base, every rational section of the family over the curve is contained in a unique rational section over the entire base.

1. Main results

sec-intro

The starting point for this article is the following theorem.

Theorem 1.1. [GHMS05, Theorem 6.2] *Let B be a smooth, quasi-projective, complex variety, let $A \to B$ be an Abelian scheme over B (i.e. a family of Abelian varieties over B), and let $\pi : T \to B$ be a torsor for $A \to B$. Then π is a trivial torsor if and only if for every curve $C \subset B$, the restriction $T_C \to C$ is a trivial torsor for $C \times_B A \to C$. Stated more succinctly, the following map of étale cohomology groups is injective* thm-GHMS

$$H^1_{\acute{e}t}(B, A) \to \prod_{C \subset B} H^1_{\acute{e}t}(C, C \times_B A).$$

The main result of this article is an analogue for $H^0_{\text{ét}}(B, A)$, together with an extension to positive characteristic.

Let k be an algebraically closed field. To simplify statements, assume k is uncountable. We remind the reader that a subset of a scheme is *general*, resp. *very general*, if it contains a dense, open subset, resp. the intersection of a countable collection of dense, open subsets. A property of points in a scheme holds at a *general point*, resp. a *very general point*, if the set of points where it holds is general, resp. very general.

thm-main

Theorem 1.2. *Let B be a integral, smooth, quasi-projective k-scheme of dimension $b \geq 2$. Let A be an Abelian scheme over B.*

(i) *For every nontrivial A-torsor T over B, for C a very general triangle curve in B, resp. a very general cubic curve in B, the restriction $C \times_B T$ is a nontrivial $C \times_B A$-torsor over C.*

(ii) *For a very general triangle curve C in B, the map*

$$H^0_{\acute{e}t}(B, A) \to H^0_{\acute{e}t}(C, C \times_B A)$$

is a bijection. If $char(k) = 0$, this also holds for a very general cubic curve C in B.

Both parts also hold with C replaced by a very general planar surface Π in B.

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The terms "triangle curve", "planar surface" and "cubic curve" depend on the choice of a quasi-finite, generically unramified morphism $f : B \to \mathbb{P}^b_k$.

defn-tria

Definition 1.3. An *f-linear curve* is an integral, smooth curve C in B of the form $B \times_{\mathbb{P}^b_k} L$ for some line $L \subset \mathbb{P}^b_k$.

An *f-triangle map* is a pair (C, h) of

(i) a reducible, nodal k-curve $C = C_1 \cup C_2 \cup C_3$ whose irreducible components each intersect pairwise in a single node, $C_i \cap C_j = \{q_{i,j}\}$,

(ii) together with a k-morphism $h : C \to B$ whose restriction to every component, $h|_{C_i} : C_i \to B$, is a closed immersion to an f-linear curve in B.

An *f-triangle curve* is the image of an f-triangle map. Equivalently, it is a union of three f-linear curves in B intersecting pairwise in three distinct points (together with other, possibly concurrent, intersections). Stated a third way, it is a curve in B of the form $C = B \times_{\mathbb{P}^b_k} T$ where T is a triangle of lines in $\mathbb{P}^b_k$ (and satisfying the intersection condition above).

An *f-cubic curve* is an integral, smooth curve C in B of the form $B \times_{\mathbb{P}^b_k} E$ where E is a smooth, plane, cubic curve in $\mathbb{P}^b_k$.

Finally, an *f-planar surface* is an integral, normal surface S in B of the form $B \times_{\mathbb{P}^b_k} \Pi$ where Π is a linear 2-plane in $\mathbb{P}^b_k$.

When there is no likelihood of confusion, we drop the prefix f.

rmk-main

Remark 1.4. There are several remarks.

(i) When $k = \mathbb{C}$, Theorem 1.2 (i) is simply Theorem 1.1. By a straightforward descent argument, part (i) follows from part (ii). Thus, the new content of Theorem 1.2 is in part (ii).

(ii) The group $H^1_{\text{ét}}(B, A)$ is a torsion group. The subgroup
$$H^1_{\text{ét}}(B, A)' \subset H^1_{\text{ét}}(B, A)$$
of elements whose order is not divisible by $\text{char}(k)$ is countable (it is the whole group if $\text{char}(k) = 0$). Therefore, Theorem 1.2 (i) implies that for C a very general triangle curve, resp. a very general cubic curve, the restriction map
$$H^1_{\text{ét}}(B, A)' \to H^1_{\text{ét}}(C, C \times_B A)'$$
is injective.

(iii) There is nothing special about triangle curves, resp. cubic curves. The proof works for any type of curve which is at least as "complex" as triangle curves, resp. cubic curves.

(iv) The proof of (ii) uses a partial compactification of the pair (B, A) to a Néron model $(\widetilde{B}, \widetilde{A})$. A general triangle curve, resp. cubic curve, C in $\widetilde{B}$ is projective. Thus, using the Hilbert scheme for instance, the sections of $C \times_{\widetilde{B}} \widetilde{A}$ over C are the k-points of a naturally defined group k-scheme Σ_C. If $\text{char}(k) = 0$, this is a reduced group scheme. But if $\text{char}(k) > 0$, this group scheme is sometimes nonreduced, cf. [MB81, Propostion 3]. This is the reason for the $\text{char}(k) = 0$ hypothesis for cubic curves in part (ii). To extend our argument to positive characteristic, one would need to prove there exists a homomorphism of group schemes over B

$$\Sigma_C \times_k \widetilde{B} \to \widetilde{A}$$

splitting the restriction map. We do not know whether such a splitting always exists.

We also give some examples related to the theorem.

prop-neg

Proposition 1.5. (i) *There exist B and A such that for every cubic curve C in B the map*

$$H^1_{\acute{e}t}(B, A) \to H^1_{\acute{e}t}(C, A)$$

is not surjective.

(ii) *If $char(k) = p$ is positive, there exist B and A such that for every triangle curve C, resp. cubic curve C, the map*

$$H^1_{\acute{e}t}(B, A) \to H^1_{\acute{e}t}(C, A)$$

is not injective. More precisely, there exists an A-torsor T over B (depending on C) whose order equals p and whose restriction $C \times_B T$ is a trivial $C \times_B A$-torsor over C.

(iii) *There exists B and A such that every dense open subset of the parameter space of cubic curves contains a cubic curve C in B for which*

$$H^0_{\acute{e}t}(B, A) \to H^0_{\acute{e}t}(C, C \times_B A)$$

is not surjective.

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2. A Bertini theorem

sec-Bertini

One consequence of the classical Bertini theorem is that sections of a finite, separable cover of a quasi-projective scheme B of dimension ≥ 2 are detected by the restriction of the cover to a general hyperplane section of B. In this section we recall this and extend the result to covers which may not be separable.

Theorem 2.1. [Jou83, Théorème 4.10, 6.10] *Let B be an integral scheme and let $f : B \to \mathbb{P}^N_k$ be a finite type morphism. If f is generically unramified, then for a general hyperplane H, $B \times_{\mathbb{P}^N_k} H$ is geometrically reduced. If $dim(f(B)) \geq 2$, then for a general hyperplane H, $B \times_{\mathbb{P}^N_k} H$ is geometrically irreducible.* thm-B

A straightforward consequence is the separable case of the following result. We also explain the inseparable case.

cor-B

Corollary 2.2. *Let B be an integral scheme of dimension ≥ 2 and let $f : B \to \mathbb{P}^N_k$ be a generically unramified, finite type morphism. Let $g : X \to B$ be a generically finite morphism. For a general hyperplane $H \subset \mathbb{P}^N_k$, the restriction map from the set of rational sections of g over B to the set of rational sections of*

$$g_H : X \times_{\mathbb{P}^N_k} H$$

over $B \times_{\mathbb{P}^N_k} H$ is a bijection.

Proof. If g has a rational section, its restriction to $B \times_{\mathbb{P}^N_k} H$ is a rational section of g_H for general H. By Noetherian induction, it suffices to consider the case that X is irreducible. If g is not dominant, the result is clear. If g has a rational section, then g is birational and again the result is clear. Thus assume g is dominant and has no rational section. To prove the corollary, we must prove that also g_H has no rational sections for H a general hyperplane.

Because g is dominant, there is an extension of fraction fields

$$g^* : k(B) \to k(X).$$

Because g is generically finite, this is a finite field extension. Because there is no rational section, it is a nontrivial field extension, i.e., it has degree $d > 1$. Let $E/k(B)$ be a nontrivial extension intermediate to $k(X)/k(B)$ such that there is no field strictly intermediate between E and $k(B)$. Denote by X_E the integral closure of B in E and denote by $g_E : X_E \to B$ the induced morphism. Since every rational section of g_H maps to a rational section of $(g_E)_H$, it suffices to prove $(g_E)_H$ has no rational section for general H. Thus, assume finally that $k(X)/k(B)$ has no intermediate subfields.

Case I. $k(X)/k(B)$ is separable. There are two cases depending on whether or not $k(X)/k(B)$ is separable. If $k(X)/k(B)$ is separable, then the composition $g \circ f : X \to \mathbb{P}^N_k$ is generically unramified. By Theorem 2.1 applied to $f \circ g$, for a general H, $X \times_{\mathbb{P}^N_k} H$ is integral. By generic flatness, the morphism

$$g_H : X \times_{\mathbb{P}^N_k} H \to B \times_{\mathbb{P}^N_k} H$$

is generically finite and flat of degree d. Thus $k(B \times_{\mathbb{P}^N_k} H) \to k(X \times_{\mathbb{P}^N_k} H)$ is a finite field extension of degree $d > 1$. Since the extension is not degree 1, it has no splitting. Thus g_H has no rational section.

Case II. $k(X)/k(B)$ is not separable. Let p denote the characteristic of k. Because $k(X)/k(B)$ has no intermediate subfields, for any $z \in k(X) \setminus k(B)$, $k(X) = k(B)[z]$ and the minimal polynomial of z is of the form $z^p - u$ for some $u \in k(B) \setminus k(B)^p$. Thus $k(X) = k(B)[z]/\langle z^p - u\rangle$. Replacing B and X by dense open subsets if necessary, assume B and X are affine and normal, assume f is unramified and factors through $\mathbb{A}^N_k \subset \mathbb{P}^N_k$, assume $u \in k[B]$, and assume $k[X] = k[B][z]/\langle z^p - u\rangle$.

For general H, by Theorem 2.1, $B \times_{\mathbb{A}^N_k} H$ is integral and normal. In this case $X \times_{\mathbb{A}^N_k} H$ is integral and g_H has no rational section precisely if the image of u satisfies

$$\overline{u} \notin k[B \times_{\mathbb{A}^N_k} H]^p$$

which is the same as the condition for $X \times_{\mathbb{A}^N_k} H$ to be generically reduced. If $X \times_{\mathbb{A}^N_k} H$ is generically nonreduced, then for a general codimension $b-1$, linear subspace of $\mathbb{A}^N_k$ contained in H, $\Lambda \subset H$, also $X \times_{\mathbb{A}^N_k} \Lambda$ is nonreduced. Inside the parameter space for subvarieties H, resp. Λ, the set of H, resp. Λ, such that $X \times_{\mathbb{A}^N_k} H$, resp. $X \times_{\mathbb{A}^N_k} H$, is generically reduced is open, [REFERENCE, Jouanolou Theorem 4.10 approx.] Thus, to prove $X \times_{\mathbb{A}^N_k} H$ is generically nonreduced for all H in a dense open subset, it suffices to find a single Λ such that $X \times_{\mathbb{A}^N_k} \Lambda$ is generically nonreduced. Equivalently, it suffices to find a single Λ such that for every minimal prime $\mathfrak{q}$ of $k[B]\otimes_{k[\mathbb{A}^N_k]}k[\Lambda]$, the image of u in $(k[B]\otimes_{k[\mathbb{A}^N_k]}k[\Lambda])_{\mathfrak{q}}$ is not a p^{th} power. This is what we will actually prove.

Denote $b = \dim(B)$. For a general linear projection $p : \mathbb{A}^N_k \to \mathbb{A}^b_k$, the induced morphism $p\circ f : B \to \mathbb{A}^b_k$ is dominant and étale. Because the inverse image in $\mathbb{A}^N_k$ of a linear subspace of $\mathbb{A}^b_k$ of codimension $b-1$ is again a linear subspace of codimension $b-1$, it suffices to prove the result with f replaced by $p\circ f$. Thus, without loss of generality, assume $N = b = \dim(B)$.

Denote by

$$f^* : k[y_1,\dots,y_b] \to k[B]$$

the pullback map on coordinate rings associated to f. Extend this to a morphism of $k[y_1,\dots,y_b]$-algebras,

$$\psi : k[x,y_1,\dots,y_b] \to k[B]$$

by $\psi(x) = u$. Denote $k[A] = \mathrm{Image}(\psi)$. Because A is between $k[y_1,\dots,y_b]$ and $k[B]$

$$b = \text{tr. deg}_k(k(y_1,\dots,y_b)) \leq \text{tr. deg}_k(k(A)) \leq \text{tr. deg}_k(k(B)) = \dim(B) = b.$$

Thus $\dim(k[A]) = \text{tr. deg}_k(k(A)) = b$. So $\mathrm{Ker}(\psi)$ is a height 1 prime ideal. Because $k[x,y_1,\dots,y_b]$ is a UFD, $\mathrm{Ker}(\psi) = \langle g\rangle$ for a nonconstant, irreducible element $g \in k[x,y_1,\dots,y_b]$, i.e.,

$$k[A] = k[x,y_1,\dots,y_b]/\langle g\rangle.$$

The k-algebra homomorphism

$$k[x] \to k[B], \quad x \mapsto u$$

determines a field extension $k(x) \to k(B)$. Because u is not in $k(B)^p$, this is a separably generated field extension. Since $k(A)/k(x)$ is a subextension, it is also separably generated. By the Jacobian criterion, there exists an integer $i = 1,\dots,b$ such that the ideal $\langle g, \partial g/\partial y_i\rangle$ has height 2. Permuting indices if necessary, assume $i = 1$.

For a $(b-1)$-tuple of elements $\mathbf{c} = (c_2,\dots,c_n)$ in k, denote $k[\Lambda] = k[y_1]$ and denote by

$$\chi_{\mathbf{c}} : k[y_1,\dots,y_b] \to k[\Lambda]$$

the unique $k[y_1]$-algebra homomorphism with $\chi_{\mathbf{c}}(y_i) = c_i$ for $i = 2,\dots,b$. The closed scheme of $\mathbb{A}^N_k$ associated to

$$\mathrm{Ker}(\chi_{\mathbf{c}}) = \langle y_2 - c_2,\dots,y_b - c_b\rangle$$

is a codimension $b-1$ linear subspace. For every $k[y_1,\dots,y_b]$-algebra R, denote

$$R_\Lambda = R\otimes_{k[y_1,\dots,y_b]}k[\Lambda].$$

Because $\langle g, \partial g/\partial x_1\rangle \in k[x,y_1,\dots,y_b]$ has height 2, by standard results of dimension theory (cf. [Har77, Exercise II.3.22]), for a general $\mathbf{c}$, the ideal

$$\langle g(x,y_1,c_2,\dots,c_b), \partial g/\partial x_1(x,y_1,c_2,\dots,c_b)\rangle \subset k[x,y_1]$$

also has height 2. Then by the Jacobian criterion, the homomorphism

$$k[x] \mapsto k[A]_\Lambda$$

is generically smooth. Because $k[A] \to k[B]$ is generically étale, for general $\mathbf{c}$ also

$$k[A]_\Lambda \to k[B]_\Lambda$$

is generically étale. Therefore the composition

$$k[x] \mapsto k[B]_\Lambda$$

is generically smooth, i.e., for every minimal prime $\mathfrak{q}$ of $k[B]_\Lambda$, the field extension

$$k(x) \to \kappa(\mathfrak{q}) := (k[B]_\Lambda)_\mathfrak{q}$$

is separably generated. This precisely means that the image

$$\overline{u} \in (k[B] \times_{k[\mathbb{A}^n_k]} k[\Lambda])_\mathfrak{q}$$

is not a p^{th} power for every minimal prime $\mathfrak{q}$. □

sec-triang

3. A triangle lemma

Let B be an integral, normal, quasi-projective k-scheme of dimension $b \geq 2$. Let $f : B \hookrightarrow \mathbb{P}^b_k$ be a quasi-finite, generically unramified morphism. Let $D \subset \text{Grass}(1, \mathbb{P}^b_k)$ be the dense, Zariski open subset of the Grassmannian of lines in $\mathbb{P}^b_k$ parametrizing lines L for which $B \times_{\mathbb{P}^b_k} L$ is an integral, smooth curve. This is the parameter space for f-linear curves in B.

Also denote by $C_D \subset B \times_k D$ the family of f-linear curves in B parametrized by D. Denote by $\rho_D : C_D \to D$ and $h_D : C_D \to D$ the obvious projections.

The following "triangle lemma" is essentially the same as in [GHMS05]. We need to use the result in two slightly different contexts. Rather than repeat the proof twice, we find it convenient to introduce the following terminology which encompasses both contexts.

defn-sec

Definition 3.1. Let $q : \text{Spec}\ \kappa \to B$ be a point. A *family of sections of* π *over* f*-linear curves containing* q is a pair (τ, σ) of a locally finite type morphism

$$\tau : M \to h_D^{-1}(q) = C_D \times_B \text{Spec}\ \kappa$$

and a morphism of B-schemes

$$\sigma : M \times_{\rho_D \circ \tau, D, \rho_D} C_D \to X,$$

where $M \times_D C_D$ is considered as a B-scheme via

$$M \times_D C_D \xrightarrow{\text{pr}_{C_D}} C_D \xrightarrow{h_D} B$$

(rather than via $h_D \circ \tau \circ \text{pr}_M$). The family is *locally generically finite* if for every quasi-compact open U, the restriction

$$\tau|_U : U \to h_D^{-1}(q)$$

is generically finite. The family *avoids rational curves over* q if for a very general curve $[C, q]$ in $h_D^{-1}(q)$, for every m in $\tau^{-1}([C, q])$, the point $\sigma_m(q) \in \pi^{-1}(q)$ is contained in no rational curve in $\pi^{-1}(q)$.

Let M be a family of sections of π over f-linear curves containing q which avoids rational curves. Let C be a triangle such that $q_{1,2} = q$. A stable section of $X_C \to C$ is *in* M *over* C_1 *and* C_2 if for $i = 1, 2$ the associated section of $X_{C_i} \to C_i$ is the section s_m associated to a point $m \in r^{-1}([C_i, q])$.

lem-tri **Lemma 3.2.** [GHMS05, Lemma 3.7] *Let B be an integral, normal, quasi-projective k-scheme of dimension $b \geq 2$. Let $f : B \to \mathbb{P}^b$ be a quasi-finite, generically morphism. Let $\pi : X \to B$ be a projective morphism. Let $q \in B$ be a smooth point. Let*

$$(\tau : M \to h_D^{-1}(q), \sigma : M \times_D C_D \to X)$$

be a locally generically finite family of sections of π over f-linear curves containing q which avoids rational curves over q, cf. Definition 3.1. Let C be a very general triangle curve in B with $q_{1,2} = q$.

(i) *Every stable section of $X_C \to C$ in M over C_1 and C_2 contains a section of $X_C \to C$ in M over C_1 and C_2.*
(ii) *For every section s of $X_C \to C$ in M over C_1 and C_2, there exists a unique irreducible closed subvariety $Z \subset X$ containing $s(C)$ such that $\pi|_Z : Z \to B$ is an isomorphism over q and such that the restriction of Z over a very general linear curve L in B containing q is a section of $X_L \to L$ in M.*

Proof. We may replace B by the integral closure B^{new} of $\mathbb{P}^b$ in the fraction field of B, which contains B as a dense, Zariski open subset. And we may replace X by some projective morphism $X^{\text{new}} \to B^{\text{new}}$ whose restriction over B equals X. Thus, without loss of generality, assume that B is projective. Because g is proper, every rational section of X over a normal curve in B extends uniquely to a regular section (by the valuative criterion of properness).

The subset Ω of X consisting of the images of all sections parametrized by M is a countable union of irreducible, locally closed subvarieties Ω_i^{pre} of X, $i \in \mathbb{Z}$. For each i, denote by Ω_i the normalization of the closure in X of Ω_i^{pre}. Observe that if Z is a subvariety as in (ii), then Z is contained in Ω_i for some i. Thus, to prove the uniqueness in (ii), it suffices to consider subvarieties Z contained in some Ω_i. And the next claim implies each such Z will equal Ω_i.

claim-1

Claim 3.3. *For every $i \in \mathbb{Z}$, the morphism*

$$\pi_{\Omega_i} : \Omega_i \to B$$

is generically finite. For a very general member C_3 of D, for every i, $\Omega_i|_{C_3} \to C_3$ admits a section only if $\Omega_i \to B$ is birational. If $\Omega_i \to B$ is birational, then it is an isomorphism over q and its restriction to every very general linear curve L in B is a section of $X_L \to L$ in M.

First of all, $\Omega_i \to B$ is generically finite if and only if $\Omega_i^{\text{pre}} \to B$ is generically finite. Let M_i be a quasi-compact locally closed subset of M mapping to Ω_i^{pre}. For every $q' \notin f^{-1}(f(q))$, there is a unique line Λ in $\mathbb{P}_k^b$ containing $f(q)$ and $f(q')$. Thus, for q' general, there is a unique f-linear curve C containing q and q'. Because τ is locally generically finite, $M_i \to h_D^{-1}(q)$ is generically finite. Thus, for q' general, there are only finitely many points m in M_i mapping to $[C, q]$. Thus there are only finitely many images $s_m(q') \in \Omega_i^{\text{pre}} \cap \pi^{-1}(q')$. Therefore $\Omega_i \to B$ is generically finite.

If Ω_i does not dominate B, then a general f-linear curve C is not contained in the image of Ω_i. Thus $\Omega_i|_{C_3} \to C_3$ certainly admits no section. Thus assume Ω_i dominates B. By Theorem 2.2, the restriction of Ω_i over a general member C of D has a section if and only if Ω_i has a rational section over B, i.e., if and only if $\Omega_i \to B$ is birational. Finally, by [Kol96, Theorem VI.1.9.3], $\Omega_i \to B$ is an isomorphism over q since otherwise $\Omega_i \cap \pi^{-1}(q)$ is not uniruled (in fact there are no rational curves in $\pi^{-1}(q)$ intersecting Ω_i^{pre}). This proves Claim 3.3.

claim-2

Claim 3.4. *For a very general f-linear curve C_3, for every pair r, s of very general points of C_3, for every i and j, there exists a section of $X_{C_3} \to C_3$ intersecting both $\Omega_i \cap \pi^{-1}(r)$ and $\Omega_j \cap \pi^{-1}(s)$ only if $i = j$ and $\Omega_i \to B$ is birational. In this case, the image of the section is contained in Ω_i.*

Because every $\Omega_i \to B$ is generically finite, for a very general point r of B the subsets $\Omega_i \cap \pi^{-1}(r)$ are all finite and disjoint subsets of $\pi^{-1}(r)$. Because M avoids rational curves in $\pi^{-1}(q)$, for a very general f-linear curve C_1 containing q, for a very general point r of C_1, every section in M over $[C_1, q]$ maps r to a point contained in no rational curve in $\pi^{-1}(r)$. In other words, $\Omega_i \cap \pi^{-1}(r)$ intersects no rational curve in $\pi^{-1}(r)$. Thus, by the rigidity lemma, for every i, every section of $X_{C_3} \to C_3$ intersecting $\Omega_i \cap \pi^{-1}(r)$ is rigid, as a section intersecting $\Omega_i \cap \pi^{-1}(r)$. Thus, there are at most countably many of these rigid sections.

For every section and for every j, either the section is contained in Ω_j or else the section intersects Ω_j in finitely many closed points. Since there are countably many indices i and j, there is a countable set S of closed points of X such that for every section of $X_{C_3} \to C_3$ intersecting $\Omega_i \cap \pi^{-1}(r)$ and not contained in Ω_j, the intersection of the section and Ω_j is contained in S. The image $\pi(S)$ is a countable set of closed points of C_3. Choosing s in the complement of this set, every section of $X_{C_3} \to C_3$ intersecting $\Omega_i \cap \pi^{-1}(r)$ and $\Omega_j \cap \pi^{-1}(s)$ is contained in Ω_j. In this case, firstly, it intersects $\Omega_j \cap \pi^{-1}(r)$. Since $\Omega_i \cap \pi^{-1}(r)$ and $\Omega_j \cap \pi^{-1}(r)$ are disjoint for $i \neq j$, it follows that i equals j. Secondly, since the section of $X_{C_3} \to C_3$ is actually a section of $\Omega_j = \Omega_i$ over C_3, Claim 3.3 implies that $\Omega_i \to B$ is birational. Finally, the image of the section is contained in Ω_j, which equals Ω_i. This proves Claim 3.4.

Let C_1, resp. C_2, be a general f-linear curve containing both q and r, resp. both q and s. Form the f-triangle map $C = C_0 \cup C_1 \cup C_2$ with vertices q, r and s.

claim-3

Claim 3.5. *Every stable section of $X_C \to C$ in M over C_1 and C_2 is regular over q, r and s, thus contains a section of $X_C \to C$ in M over C_1 and C_2. Every section of $X_C \to C$ in M over C_1 and C_2 is contained in an irreducible closed subset Ω_i such that $\Omega_i \to B$ is birational.*

Given a stable section of $X_C \to C$, remove all vertical, rational components not contained in $\pi^{-1}(q)$, $\pi^{-1}(r)$ or $\pi^{-1}(s)$. The remaining components give a stable section of $X_C \to C$ whose vertical, rational components, if any, occur over q, r and s. By hypothesis, the images of the sections over C_1, resp. C_2, intersect no rational curves in $\pi^{-1}(q)$, and then, because r and s are very general, they also intersect no rational curve in $\pi^{-1}(r)$, resp. in $\pi^{-1}(s)$. Thus the stable section has no vertical, rational components over q, r and s, i.e., the stable section is a section. By hypothesis, the image of this section over C_1, resp. over C_2, is contained in a subset Ω_i, resp. Ω_j. The restriction of this section to C_3 is a section of $X_{C_3} \to C_3$ intersecting both $\Omega_i \cap \pi^{-1}(r)$ and $\Omega_j \cap \pi^{-1}(s)$. By Claim 3.4, $i = j$, $\Omega_i \to B$ is birational, and the section over C_3 is contained in Ω_i. Thus the entire section is contained in Ω_i. This proves Claim 3.5, and thus the lemma. ☐

We will use Lemma 3.2 via the following two corollaries.

Corollary 3.6. *Let B be an integral, normal, quasi-projective k-scheme of dimension $b \geq 2$. Let $f : B \to \mathbb{P}^b$ be a generically unramified, quasi-finite morphism. Let $\pi : X \to B$ be a projective morphism. Let $q \in B$ be a smooth point and let $r \in \pi^{-1}(q)$ be a point contained in no rational curve in $\pi^{-1}(q)$. Let C be a very general f-triangle curve in B with $q_{1,2} = q$.* cor-tri1

(i) *Every stable section of $X_C \to C$ mapping $q_{1,2}$ to r contains a section of $X_C \to C$ mapping $q_{1,2}$ to r.*
(ii) *For every section of $X_C \to C$ mapping $q_{1,2}$ to r, there exists a unique irreducible closed subvariety $Z \subset X$ containing the image of the section and such that $\pi|_Z : Z \to B$ is an isomorphism over q.*

Proof. Let $(\tau : M \to h_D^{-1}(q), \sigma)$ be the family of all sections over f-linear curves C containing q which map q to r. Since r is contained in no rational curve in $\pi^{-1}(q)$, this family avoids rational curves over q. And by the rigidity lemma, since there is no rational curve in $\pi^{-1}(q)$ containing r, every section over C mapping q to r is rigid (as a section mapping q to r). Thus the family is locally generically finite. So the family satisfies the hypotheses of Lemma 3.2, which gives the corollary. □

cor-tri2

Corollary 3.7. *Let B be an integral, normal, quasi-projective k-scheme of dimension $b \geq 2$. Let $f : B \to \mathbb{P}^b$ be a generically unramified, quasi-finite morphism. Let $\pi : X \to B$ be a projective morphism. Let $q \in B$ be a smooth point. Assume that for a very general f-linear curve C' containing q, every section of $X_{C'} \to C'$ is rigid and maps q to a point contained in no rational curve in $\pi^{-1}(q)$. Let C be a very general triangle curve in B with $q_{1,2} = q$.*

(i) *Every stable section of $X_C \to C$ contains a section of $X_C \to C$.*
(ii) *For every section of $X_C \to C$, there exists a unique irreducible closed subvariety $Z \subset X$ containing the image of the section and such that $\pi|_Z : Z \to B$ is birational.*

Proof. As in the proof of Lemma 3.2, we may reduce to the case where B is projective. Then, by the existence of the Chow variety, etc., there exists a universal family $(\tau_U : U \to h_D^{-1}(q), \sigma_U)$ of sections over f-linear curves containing q. Moreover, U has countably many connected components, each of which is quasi-projective. Denote by M the closed subvariety of U which is the union of all irreducible components U_i such that $\tau_U(U_i)$ is dense in $h_D^{-1}(q)$. In particular, for a very general D-curve C' containing q, every irreducible component of M intersects $\tau^{-1}([C', q])$. By hypothesis, every section of $X_{C'} \to C'$ is rigid and avoids rational curves in $\pi^{-1}(q)$. Therefore the family

$$(\tau_U|_M : M \to h_D^{-1}(q), \sigma_U|_M)$$

is locally generically finite and avoids rational curves over q. So the family satisfies the hypotheses of Lemma 3.2. By definition of M, for a very general triangle C with $q_{1,2} = q$, since both C_1 and C_2 are very general f-linear curves containing q, every section of $X_{C_1} \to C_1$, resp. of $X_{C_2} \to C_2$, is in M. Thus Lemma 3.2 gives the corollary. □

4. Elementary reductions

sec-red

In this section, we prove some elementary lemmas useful in the proof of Theorem 1.2. The next definition simplifies the statements of the lemmas. As always, let k be an algebraically closed field.

defn-P

Definition 4.1. Let B be an integral, smooth, quasi-projective scheme over an algebraically closed field k. Let $f : B \to \mathbb{P}^b_k$ be a quasi-finite, generically unramified morphism. Let A be an Abelian scheme over B. We say *property* $\mathcal{P}(B, f, A)$ *holds* for triangle curves, resp. for cubic curves, if

$$H^0_{\text{ét}}(B, A) \to H^0_{\text{ét}}(C, C \times_B A)$$

is a bijection for a very general f-triangle curve, resp. for a very general f-cubic curve.

lem-red0

Lemma 4.2. *For every dense, open immersion $U \to B$, $\mathcal{P}(U, f|_U, U \times_B A)$ holds if and only if $\mathcal{P}(B, f, A)$ holds.*

Proof. **(i)** By Weil's extension theorem, cf. [BLR90, Theorem 1, p.109] and [Kol96, Theorem VI.1.9.3], sections of $U \times_B A$ over U extend uniquely to sections of A over B. Assuming C is smooth at every point of $B \setminus U$, also sections of $U \times_B C \times_B A$ over $U \times_B C$ extend uniquely to sections of $C \times_B A$ over C. ☐

lem-red1

Lemma 4.3. *Let B' be an integral scheme and let $h : B' \to B$ be a finite, étale, Galois morphism with Galois group G. Denote $f' = f \circ h$ and $A' = A \times_B B'$. If $\mathcal{P}(B', f', A')$ holds, then $\mathcal{P}(B, f, A)$ holds.*

Proof. Let C be a very general triangle curve in B, resp. cubic curve in B. The restriction map

$$H^0_{\text{ét}}(B', A') \to H^0_{\text{ét}}(B' \times_B C, (B' \times_B C) \times_{B'} A')$$

is a homomorphism of Galois modules. By étale descent, the induced map of Galois invariants is canonically isomorphic to the restriction map

$$H^0_{\text{ét}}(B, A) \to H^0_{\text{ét}}(C, C \times_B A).$$

Thus if the first map is an isomorphism, also the second map is an isomorphism. ☐

The next reduction uses part of the theory of Chow's K/k-trace. We review the part that we need.

defn-Chow

Definition 4.4. Let B be an integral, smooth, quasi-projective scheme over an algebraically closed field k. Let A be an Abelian scheme over B. A *Chow B/k-trace of A* is an initial pair $(\mathrm{Tr}_{B/k}(A), u)$ of an Abelian k-scheme $\mathrm{Tr}_{B/k}(A)$ and a morphism of Abelian schemes over B,

$$u : B \times_k \mathrm{Tr}_{B/k}(A) \to A,$$

i.e., for every pair (A_0, v) of an Abelian k-scheme A_0 and a morphism of Abelian schemes over B,

$$v : B \times_k A_0 \to A,$$

there exists a unique morphism of Abelian k-schemes

$$w : A_0 \to \mathrm{Tr}_{B/k}(A)$$

such that $v = u \circ (\mathrm{Id}_B \times w)$.

The basic result concerning the Chow trace is the following.

Theorem 4.5. [Lan83, §VIII.3], [Con06] thm-Chow

(i) *For every integral, smooth, quasi-projective k-scheme B and every Abelian scheme A over B, there exists a Chow B/k-trace of A.*

(ii) *Let E be an integral, smooth, quasi-projective k-scheme and let $E \to B$ be a dominant k-morphism such that $k(B)$ is separably closed in $k(E)$. The induced morphism of Abelian k-schemes*

$$w : Tr_{B/k}(A) \to Tr_{E/k}(E \times_B A)$$

is an isomorphism.

rmk-Chow

Remark 4.6. A dense open immersion $U \to B$ satisfies the condition in (ii). Taking the limit over all dense open subsets of B, the Chow trace depends only on the field extension $k(B)/k$ and the Abelian $k(B)$-scheme $A \otimes_{\mathcal{O}_B} k(B)$. Usually the Chow trace is formulated for pairs $(K/k, A_K)$ of a field extension K/k and an Abelian K-scheme. It is more useful for us to formulate it as above.

The Chow trace is closely related to the property of isotriviality of an Abelian scheme.

defn-noniso

Definition 4.7. Let B be an integral, smooth, quasi-projective k-scheme. An Abelian scheme A over B is *non-isotrivial*, resp. *strongly non-isotrivial*, if for the geometric generic point of $B \times_k B$,

$$(p, q) : \text{Spec } \kappa \to B \times_k B$$

the Abelian κ-schemes

$$A_p := \text{Spec } \kappa \times_{p,B} A, \quad A_q := \text{Spec } \kappa \times_{q,B} A$$

are not isomorphic, resp. there is no nonzero morphism of Abelian κ-schemes

$$A_p \to A_q.$$

lem-Chow2

Lemma 4.8. *Let B be an integral, smooth, quasi-projective k-scheme. An Abelian scheme A over B is strongly non-isotrivial if and only if $Tr_{k(B)^{sep}/k}(A \otimes_{\mathcal{O}_B} k(B)^{sep})$ is zero.*

Proof. First of all, dual to the morphism of Abelian varieties

$$u : \text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}}) \otimes_k k(B)^{\text{sep}} \to A_{k(B)^{\text{sep}}},$$

there is a morphism

$$v : A_{k(B)^{\text{sep}}} \to \text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}}) \otimes_k k(B)^{\text{sep}}$$

such that $v \circ u$ is multiplication by some positive integer N. Thus, if $\text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}})$ is nonzero, the composition

$$A_p \xrightarrow{p^* v} \text{Tr}_{k(B)^{\text{sep}}/k}(A_{k(B)^{\text{sep}}}) \otimes_k \kappa \xrightarrow{q^* u} A_q$$

is a nonzero homomorphism.

Denote by $e_p, e_q : k(B)^{\text{sep}} \to \kappa$ the field monomorphism associated to projection p, resp. q. Then, conversely, every nonzero homomorphism

$$A_p \to A_q$$

factors through

$$\mathrm{Tr}_{e_p}(A_q) \otimes_{k(B)^{\mathrm{sep}}, e_p} \kappa \to A_q.$$

By Theorem 4.5, this second map is precisely

$$e_q^* u : \mathrm{Tr}_{k(B)^{\mathrm{sep}}/k}(A_{k(B)^{\mathrm{sep}}}) \otimes_k \kappa \to A_q.$$

So if there is a nonzero homomorphism $A_p \to A_q$ (or symmetrically $A_q \to A_p$), then $\mathrm{Tr}_{k(B)^{\mathrm{sep}}/k}(A_{k(B)^{\mathrm{sep}}})$ is nonzero. □

lem-rigid

Lemma 4.9. *Let B be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 1$. Let Q be an Abelian scheme over B. If $Tr_{B/k}(Q) = 0$, then there are at most countably many sections of Q over B.*

Proof. Let $\overline{B}$ be a normal, projective scheme containing B as a dense, open subscheme. Let $\overline{Q} \to \overline{B}$ be a projective morphism whose restriction over B equals Q. There is a Chow variety parametrizing cycles in $\overline{Q}$. The Chow variety has countably many irreducible components, for the usual reason (countably many Hilbert polynomials, etc.). The claim is that every cycle $Z_0 \subset \overline{Q}$ for which $Z \to B$ is birational gives an isolated point of the Chow variety.

Towards this claim, let T be an irreducible, quasi-projective curve and let $Z \subset \overline{Q} \times_k T$ be a cycle such that $Z \to \overline{B} \times_k T$ is birational. Replacing T by a dense open subset if necessary, assume T is smooth. Then $Z \cap B \times_k T$ is the graph of a B-rational transformation,

$$F : B \times_k T \dashrightarrow Q.$$

By [Kol96, Theorem VI.1.9.3], this rational transformation is regular. Fix a point $t_0 \in T$ and denote

$$G : B \times_k T \to Q, \quad F^{\mathrm{new}}(q,t) = F(q,t) - F(q,t_0).$$

Denote by

$$e : T \to \mathrm{Alb}(T)$$

the Albanese morphism sending t_0 to 0. Then, by the universal property of the Albanese, the morphism G factors through

$$\mathrm{Id}_B \times e : B \times_k T \to B \times_k \mathrm{Alb}(T).$$

Because $\mathrm{Tr}_{B/k}(Q)$ is trivial, by the universal property of the trace, the induced homomorphism of Abelian schemes over B,

$$\widetilde{G} : B \times_k \mathrm{Alb}(T) \to Q$$

is the zero homomorphism. Thus G is the zero map, i.e.,

$$F(q,t) = F(q,t_0)$$

for every $(q,t) \in B \times_k T$. Thus $Z \cap (B \times_k T)$ is independent of $t \in T$. Since Z is the closure of $Z \cap (B \times_k T)$, the same holds for Z, i.e., $Z = Z_0 \times_k T$ for a cycle $Z_0 \subset \overline{Q}$. Therefore every k-morphism from an irreducible, quasi-projective curve T to the Chow variety parametrizing rational transformations Z is constant. In other words, every cycle $Z \subset \overline{Q}$ with $Z \to \overline{B}$ birational gives an isolated point of the Chow variety. □

lem-trick

Lemma 4.10. *Let B be an integral, smooth, quasi-projective k-scheme and let A be an Abelian scheme over B. Let A_0 be an Abelian k-variety and let $v_0 : A \to B \times_k A_0$ be a surjective homomorphism of Abelian schemes. Let C be a curve in B and let q be a closed point in C. To prove that every section s of $C \times_B A$ over C is the restriction of a unique section S of A over B, it is necessary and sufficient to prove this for sections s with $v_0(s(q)) = (0, q)$.*

Proof. Necessity is obvious. To prove sufficiency, assume the result holds for sections with $v_0(s(q)) = (0, q)$. By existence of polarizations, dual homomorphisms of Abelian schemes, etc., there exists a homomorphism of Abelian schemes

$$u_0 : B \times_k A_0 \to A$$

such that $v_0 \circ u_0$ is multiplication by N for some positive integer N. Let s be a section of $C \times_B A$, not necessarily satisfying the condition. Let $a_0 \in A_0$ be the unique closed point with $v_0(s(q)) = (a_0, q)$. There exists an element $a_1 \in A_0$ such that $a_0 = N \cdot a_1$. Denote by $\tilde{a_1} : B \to B \times_k A_0$ the section whose projection to A_0 is the constant morphism with value a_1. Then $s^{\text{new}} = s - u_0 \circ \tilde{a_1}|_C$ is a section of $C \times_B A$ over C such that $v_0(s^{\text{new}}(q)) = (0, q)$. By hypothesis, there exists a unique section S^{new} of A over B whose restriction to C equals s^{new}. Then $S = S^{\text{new}} + u_0 \circ \tilde{a_1}$ is the unique section of A over B whose restriction to C is s. □

cor-Chow2

Corollary 4.11. *Let B be an integral, smooth, quasi-projective k-scheme and let A be an Abelian scheme over B. There exists a dense open subscheme $U \subset B$, a finite, étale, Galois morphism $U' \to U$, an Abelian k-scheme A_0, an Abelian scheme Q over U' and an isogeny of Abelian schemes over U',*

$$u = u_0 \oplus u_Q : (U' \times_k A_0) \times_{U'} Q \to U' \times_B A,$$

with the following properties.

(i) *The pair (A_0, u_0) is a Chow U'/k-trace of $U' \times_B A$.*
(ii) *For every dense open subset V of U and every finite, étale, Galois morphism $V' \to V \times_U U'$, the induced map of Abelian k-schemes*

$$A_0 = Tr_{U'/k}(U' \times_B A) \to Tr_{V'/k}(V' \times_B A)$$

is an isomorphism.
(iii) *The Abelian scheme Q is strongly non-isotrivial.*
(iv) *The quotient of $U' \times_B A$ by $U' \times_k A_0$ is isomorphic to Q in such a way that the composition*

$$Q \xrightarrow{u_Q} U' \times_B A \xrightarrow{quotient} Q$$

is the multiplication by N isogeny for some positive integer N.

lem-Chow3

Lemma 4.12. *Let B be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 2$. Let $f : B \to \mathbb{P}^b_k$ be a quasi-finite, generically unramified morphism. Let A_0 be an Abelian k-scheme, let A and Q be Abelian schemes over B, and let*

$$u = u_0 \oplus u_Q : (B \times_k A_0) \times_B Q \to A$$

be an isogeny of Abelian schemes over B. Assuming $Tr_{B/k}(Q)$ is zero, $\mathcal{P}(B, f, A)$ holds for triangle curves, resp. cubic curves, if and only if both $\mathcal{P}(B, f, B \times_k A_0)$ and $\mathcal{P}(B, f, Q)$ hold for triangle curves, resp. cubic curves.

Proof. Denote
$$A' := (B \times_k A_0) \times_B Q.$$
Then $\mathcal{P}(B, f, A')$ holds if and only if both $\mathcal{P}(B, f, B \times_k A_0)$ and $\mathcal{P}(B, f, Q)$ hold. There exists an isogeny
$$v = (v_0, v_Q) : A \to A' = (B \times_k A_0) \times_B Q$$
such that both $u \circ v$ and $v \circ u$ are multiplication by a positive integer N.

Let q be a very general k-point of B. Let C be a very general triangle curve, resp. cubic curve, containing q. Property $\mathcal{P}(B, f, A)$ says that every section s of $C \times_B A$ over C is the restriction of a unique section S of A over B. Let S, resp. s, be a section of A over B, resp. of $C \times_B A$ over C. By Lemma 4.10, it suffices to prove this for sections s with $v_0(s(q)) = (0, q)$.

Consider the collection of all sections S' of A' over B with $\mathrm{pr}_{A_0}(S'(q)) = 0$. This is the same as the collection of pairs (S'_0, S'_Q) of a section S'_Q of Q over B and a morphism $S'_0 : B \to A_0$ such that $S'_0(q) = 0$. By the rigidity lemma, every morphism S'_0 as above is rigid, and thus there are only countably many such morphisms. By Lemma 4.9, also there are only countably many sections S'_Q of Q over B. Altogether, there are only countably many sections S' of A' over B.

For each section S' of A' over B, consider the scheme
$$X = B \times_{S', A', v} A.$$
The projection $X \to A$ is a finite, flat morphism; in fact a torsor for $\mathrm{Ker}(v)$. By Corollary 2.2, for C a general curve, the restriction map from the set of sections of X over B to the set of sections of $C \times_B X$ over C is a bijection. To spell this out a bit, inductively applying Corollary 2.2 implies the result for a general f-linear curve C_3 in B. Then the result also holds for a triangle curve with $q_{1,2} = q$ whose component C_3 is a general f-linear curve as above. Finally the result follows for cubic curves since cubic curves specialize to triangle curves.

Said differently, the conclusion is that the restriction map from the
$$\{S : B \to A | S \text{ a section}, v \circ S = S'\} \to \{s : C \to C \times_B A | s \text{ a section}, v \circ s = S'|_C\}$$
is a bijection if C is general. By the argument above, there are at most countably many sections S' with $\mathrm{pr}_{A_0}(S'(q)) = 0$. Therefore, for C very general, for every section S' with $\mathrm{pr}_{A_0}(S'(q)) = 0$, the restriction map is a bijection.

Let s be a section of $C \times_B A$ over C with $v_0(s(q)) = 0$. Assuming $\mathcal{P}(B, f, A')$, there exists a unique section S' of A' over B such that $S'|_C = v \circ s$. In particular, for every section S of A over B with $S|_C = s$ (assuming such exist), the composition $v \circ S$ must equal S'. Since $v_0(s(q)) = 0$, also $\mathrm{pr}_{A_0}(S'(q)) = 0$. By the last paragraph there exists a unique section S of A over B such that $S|_C = s$. Thus $\mathcal{P}(B, f, A)$ follows from $\mathcal{P}(B, f, A')$.

The argument in the reverse direction is similar. Again by Lemma 4.10, to prove $\mathcal{P}(B, f, A')$, it suffices to prove every section s' of $C \times_B A'$ over C such that $\mathrm{pr}_{A_0}(s'(q)) = 0$ is the restriction of a unique section S' of A' over B. Assuming $\mathcal{P}(B, f, A)$, there exists a unique section S of A over B such that $S|_C = u \circ s'$. In particular, $v_0(S(q)) = 0$. By the argument above, there are countably many sections S of A over B with $v_0(S(q)) = 0$. Thus, assuming C is very general, the argument above proves the restriction map from sections S' of A' over B for which

$u \circ S' = S$ to sections s' of $C \times_B A'$ over C for which $u \circ s' = S|_C$ is a bijection. Therefore there exists a unique sections S' such that $S'|_C = s'$. $\square$

5. Proof of the theorem

sec-proof

We are ready to prove the theorem. We begin by proving the theorem in the special cases of a constant family of Abelian varieties and of a strongly non-isotrivial family of Abelian varieties.

lem-const

Lemma 5.1. *Let B be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 2$. Let A_0 be an Abelian k-variety. For a very general triangle curve C in B, the map*

$$H^0_{\acute{e}t}(B, B \times_k A_0) \to H^0_{\acute{e}t}(C, C \times_k A_0)$$

is a bijection.

Proof. Let q be a very general k-point of B. Let C be a very general triangle curve in B such that $q_{1,2} = q$. Set $r = (q, 0)$ in $B \times_k A_0$. By Corollary 3.6, for every section s of $C \times_k A_0$ over C with $s(q) = r$, there exists a unique section S of $B \times_k A_0$ over B with $S|_C = s$. By Lemma 4.10, this holds for all sections s, whether or not $s(q) = r$. $\square$

lem-const2

Lemma 5.2. *Let B be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 2$. Let A_0 be an Abelian k-variety. For a very general cubic curve C in B, the map*

$$H^0_{\acute{e}t}(B, B \times_k A_0) \to H^0_{\acute{e}t}(C, C \times_k A_0)$$

is a bijection.

Proof. Cubic curves are deformations of triangle curves. The proof of the result for cubic curves uses Lemma 5.1 and a deformation-specialization argument. There are two parts of the argument. First, sections over cubic curves specialize to sections over triangle curves. Second, sections over triangle curves deform uniquely.

Let $\overline{B}$ be the integral closure of $\mathbb{P}^b_k$ in the fraction field of B. Thus $\overline{B}$ is an integral, normal, projective k-scheme containing B as a dense, open subscheme. Replace B by the smooth locus of $\overline{B}$. In particular, $\overline{B} \setminus B$ has codimension ≥ 2 in $\overline{B}$. Therefore every general triangle curve in $\overline{B}$, resp. every general cubic curve in $\overline{B}$, is actually contained in B.

Let q be a very general k-point of B. Let r be the point $(q, 0) \in B \times_k A_0$. Let C be a very general triangle curve in B containing q. Let (T, t_0) be an irreducible pointed variety and let $\mathcal{C} \subset T \times B$ be a flat family of curves in B containing $T \times \{q\}$ and with $\mathcal{C}_{t_0} = C$. By the valuative criterion for stable maps, stable sections s_t of $\mathcal{C}_t \times_k A_0$ over $\mathcal{C}_t$ containing r specialize to stable sections s of $C \times_k A_0$ over C containing r as t specializes to t_0. Since there are no rational curves in A_0, stable sections are always sections. This observation is an elementary special case of [Kol96, Theorem VI.1.9.3].

Next we claim that s_t specializes *uniquely* to a section over C. This is the same as the claim that the relative Hom scheme over T, $\mathrm{Hom}_T(\mathcal{C}, \mathcal{C} \times_k A_0, q \mapsto r)$ is unramified over T, cf. [Kol96, Definition II.1.4]. To see this, observe that the Zariski tangent space of the fiber of the relative Hom scheme over T at a section

$$(\mathrm{Id}, s) : \mathcal{C}_t \to \mathcal{C}_t \times_k A_0$$

equals

$$H^0(\mathcal{C}_t, s^*T_{A_0} \otimes \mathcal{I})$$

where T_{A_0} is the tangent bundle of A_0 and $\mathcal{I}$ is the ideal sheaf of q in $\mathcal{C}_t$, cf. [Kol96, Theorem II.1.7]. Because A_0 is an Abelian variety, $T_{A_0} = T_{A_0,0} \otimes_k \mathcal{O}_{A_0}$. Thus the Zariski tangent space of the fiber is

$$T_{A_0,0} \otimes_k H^0(\mathcal{C}_t, \mathcal{I}) = T_{A_0,0} \otimes_k 0 = 0,$$

i.e., the relative Hom scheme is unramified over T.

Now let t be a very general point of T and let s_t be a section of $\mathcal{C}_t \times_k A_0$ over $\mathcal{C}_t$ sending q to r. By the last paragraph, s_t specializes uniquely to a section s of $C \times_k A_0$ over C sending q to r. By Lemma 5.1, each such section s is the restriction of a unique section S over all of B. Since $S|_{\mathcal{C}_t}$ is also a section over $\mathcal{C}_t$ specializing to s, uniqueness implies $s_t = S|_{\mathcal{C}_t}$. Therefore, for t a very general point of T, every section of $\mathcal{C}_t \times_k A_0$ over $\mathcal{C}_t$ sending q to r is the restriction of a unique section of $B \times_k A_0$ over B. Finally, Lemma 4.10 implies the result for all sections s_t of $\mathcal{C}_t \times_k A_0$ over $\mathcal{C}_t$, whether or not they map q to r. □

lem-noniso

Lemma 5.3. *Let B be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 2$. Let Q be an Abelian scheme over B. Assume that Q is strongly non-isotrivial, cf. Definition 4.7. For a very general triangle curve C in B, the map*

$$H^0_{\acute{e}t}(B, Q) \to H^0_{\acute{e}t}(C, C \times_B Q)$$

is a bijection.

Proof. Because Q is strongly non-isotrivial, for a very general pair of closed points $p, q \in B$, there is only the zero homomorphism of Abelian varieties $Q_p \to Q_q$. Since a very general pair of closed points is contained in an f-linear curve C', the same conclusion holds for the Abelian scheme $C' \times_B Q$ over C'. Thus $C' \times_B Q$ is strongly non-isotrivial. By Lemma 4.8, $\mathrm{Tr}_{C'/k}(C' \times_B Q) = 0$. Thus, by Lemma 4.9, every section of $C' \times_B Q$ over Q is rigid and there are only countably many such sections. By Corollary 3.7, for a very general triangle curve C in B, every section of $C \times_B Q$ over C is contained in a unique "rational section" of Q over B. By [Kol96, Theorem VI.1.9.3], these rational sections are all regular. Therefore

$$H^0_{\text{ét}}(B, Q) \to H^0_{\text{ét}}(C, C \times_B Q)$$

is a bijection. □

The most involved step in the proof of Theorem 1.2 is the analogue of Lemma 5.3 for very general cubic curves C in B. As with Lemma 5.2, the argument is a deformation-specialization argument. However, both steps are more involved than in Lemma 5.2. The first step proves that sections over general cubic curves specialize to sections over triangle curves. This no longer follows from Abhyankar's lemma, [Kol96, Theorem VI.1.9.3], as the codimension 1 fibers of a compactification of Q are typically uniruled. Instead we use the Néron extension property. We recall the definition in the context which we use it.

defn-Neron

Definition 5.4. Let T be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 1$. A smooth, finite type, separated morphism $X \to T$ has the *Néron extension property* if for every triple $(Y \to T, U, s_U)$ of

(i) a smooth morphism $Y \to T$,

(ii) a dense, open subset $U \subset T$,
(iii) and a T-morphism $s_U : Y \times_T U \to X$,

there exists a pair (V, s_V) of

(i) an open subset $V \subset T$ containing U and all codimension 1 points of T
(ii) and a T-morphism $s_V : Y \times_T V \to X$ whose restriction to $Y \times_T U$ equals s_U.

If $X \to T$ has the Néron extension property, then it is called a *Néron model*, or a *Néron model of its generic fiber.*

Let $X_1 \to T$ and $X_2 \to T$ be Néron models. Let $U \subset T$ be a dense, open subset. And let $U \times_T X_1 \cong U \times_T X_2$ be a T-isomorphism. By the Néron extension property, there exists an open subset $V \subset T$ containing U and all codimension 1 points and a T-isomorphism $V \times_T X_1 \cong V \times_T X_2$ extending the isomorphism over U. Thus, Néron models are unique in codimension 1.

The basic result about existence of Néron models is the following.

Lemma 5.5. [BLR90, Theorem 3, p.19] *Let W be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 1$. Let B be a dense open subset of W and let A be an Abelian scheme over B. There exists an open subset $\widetilde{B}$ of W containing B and all codimension 1 points and a Néron model $\widetilde{A}$ over $\widetilde{B}$ whose restriction over B equals A.* lem-neron

Let $\overline{B}$ be an integral, normal, projective k-scheme containing B as a dense, open subscheme. Let W be the smooth locus of $\overline{B}$. By Lemma 5.5, there exists an open subset $\widetilde{B}$ of W containing B and all codimension 1 points and a Néron model $\widetilde{Q}$ over $\widetilde{B}$ whose restriction over B equals Q. By the uniqueness part of Néron models, or by the construction, the Néron model $\widetilde{Q}$ can be chosen to be a smooth group scheme over $\widetilde{B}$.

The parameter space for both cubic and triangle curves is somewhat involved and warrants a bit of notation.

notat-param

Notation 5.6. Denote by $P_{1^{\text{st}}}$ the $\mathbb{P}^9$-bundle over the Grassmannian $\text{Grass}(2, \mathbb{P}^b_k)$ parametrizing pairs $([\Lambda], [E])$ of a linear 2-plane $\Lambda \subset \mathbb{P}^b_k$, i.e., $[\Lambda] \in \text{Grass}(2, \mathbb{P}^b_k)$, together with a degree 3 Cartier divisor $E \subset \Lambda$. Denote by $\Delta_{1^{\text{st}}} \subset P_{1^{\text{st}}}$ the closed subscheme parametrizing pairs such that E is singular. Denote by $\Gamma_{1^{\text{st}}} \subset P_{1^{\text{st}}}$ the locally closed subscheme parametrizing pairs such that E is a union of 3 non-concurrent lines in Λ. Denote by $P_{2^{\text{nd}}} \subset P_{1^{\text{st}}}$ the maximal open subscheme parametrizing pairs such that

(i) $\overline{B} \times_{\mathbb{P}^b_k} E$ is contained in $\widetilde{B}$,
(ii) and the morphism $\overline{B} \times_{\mathbb{P}^b_k} E \to E$ is a finite, separable morphism of at-worst-nodal curves which is étale over a neighborhood of every node of E and maps every node of $\overline{B} \times_{\mathbb{P}^b_k} E$ to a node of E.

Denote $\Gamma_{2^{\text{nd}}} = \Gamma_{1^{\text{st}}} \cap P_{2^{\text{nd}}}$ and $\Delta_{2^{\text{nd}}} = \Delta_{1^{\text{st}}} \cap P_{2^{\text{nd}}}$. Denote by $P_{3^{\text{rd}}} \to P_{2^{\text{nd}}}$ the blowing up of $P_{2^{\text{nd}}}$ along $\Gamma_{2^{\text{nd}}}$. Denote the exceptional divisor by by $\Gamma_{3^{\text{rd}}}$ and denote the strict transform of $\Delta_{2^{\text{nd}}}$ by $\Delta_{3^{\text{rd}}}$.

Denote by P the open complement of $\Delta_{3^{\text{rd}}}$ in $P_{3^{\text{rd}}}$ and denote $\Gamma = \Gamma_{3^{\text{rd}}} \cap P$. The scheme P is an integral, smooth parameter space for projective cubic and triangle curves in $\widetilde{B}$ and the locus Γ parametrizing triangle curves is a integral, smooth Cartier divisor in P. Denote by $\mathcal{C} \subset \widetilde{B} \times_k P$ the associated family of projective cubic and triangle curves in V parametrized by P.

Finally, denote by

$$\Sigma \subset \mathrm{Hom}_P(\mathcal{C}, \mathcal{C} \times_{\widetilde{B}} \widetilde{Q})$$

the locally closed subscheme of the relative Hom scheme over P parametrizing $\mathcal{C}$-morphisms.

Because $\mathcal{C} \times_{\widetilde{B}} \widetilde{B}$ is a group scheme over $\mathcal{C}$, Σ is a group scheme over P. Because the restriction of $\widetilde{Q}$ over a general curve in $\widetilde{B}$ is strongly non-isotrivial, Lemma 4.9 implies that $\Sigma \to P$ is "locally generically finite", i.e., every quasi-compact open subset of Σ is finite over a dense, open subset of P.

Let Σ_i be an irreducible component of Σ and let

$$F : \mathcal{C} \times_P \Sigma_i \to \mathcal{C} \times_{\widetilde{B}} \widetilde{Q}$$

be the restriction of the universal $\mathcal{C}$-morphism. Denote by

$$y_\Sigma : \mathcal{C} \times_P \Sigma_i \to \widetilde{B}$$

the composition of F with projection to $\widetilde{B}$. Because F is a $\mathcal{C}$-morphism, $F = (\mathrm{Id}_{\mathcal{C}}, G)$ where

$$G : \mathcal{C} \times_P \Sigma_i \to \widetilde{Q}$$

is a $\widetilde{B}$-morphism, i.e., $\mathrm{pr}_{\widetilde{B}} \circ G = y_\Sigma$.

lem-deform

Lemma 5.7. *If char$(k) = 0$ and if Σ_i dominates P, then the image of Σ_i in P intersects Γ in a dense subset of Γ.*

Proof. Let $\Sigma_i \hookrightarrow \overline{\Sigma}_i$ be a dense, open immersion of P-schemes with $\overline{\Sigma}_i$ a normal, projective P-scheme. The $\mathcal{B}$-morphism G gives a rational transformation of $\mathcal{B}$-schemes

$$G : \mathcal{C} \times_P \overline{\Sigma}_i \dashrightarrow \widetilde{Q}.$$

By the universal property of Σ, $\Sigma_i \subset \overline{\Sigma}_i$ is the maximal open subscheme over which G is regular. Thus to prove the lemma, it is equivalent to prove that G is regular over an open subscheme of $\overline{\Sigma}_i$ whose image in P intersects Γ in a dense subset.

Denote by $\widetilde{\Sigma}_i$ the smooth locus of $\overline{\Sigma}_i$.

claim-nonisoA

Claim 5.8. *There exists an irreducible component $D \subset \widetilde{\Sigma}_i \times_P \Gamma$ such that $D \to \Gamma$ is dominant and generically finite. And D is a Cartier divisor in $\widetilde{\Sigma}_i$.*

Because $\overline{\Sigma}_i \to P$ is dominant and projective, it is surjective. Thus also $\overline{\Sigma}_i \times_P \Gamma \to \Gamma$ is surjective. Thus some generic point of $\overline{\Sigma}_i \times_P \Gamma$ dominates the generic point of Γ. Because Γ is a Cartier divisor in P, $\overline{\Sigma}_i \times_P \Gamma$ is a Cartier divisor in $\overline{\Sigma}_i$. So each of its generic points is codimension 1 in $\overline{\Sigma}_i$. Because $\overline{\Sigma}_i$ is normal, $\widetilde{\Sigma}_i$ contains every codimension 1 point, thus it contains every generic point of $\overline{\Sigma}_i \times_P \Gamma$. In particular, it contains a generic point dominating the generic point of Γ. Denote the closure of this generic point in $\widetilde{\Sigma}_i$ by D and give it the reduced, induced scheme structure. Then D dominates Γ. Because $\widetilde{\Sigma}_i \to P$ is generically finite and $\widetilde{\Sigma}_i$ is integral, the morphism is quasi-finite at every codimension 1 point of $\widetilde{\Sigma}_i$. Thus $D \to \Gamma$ is generically finite. Finally D is a Cartier divisor in $\widetilde{\Sigma}_i$ since $\widetilde{\Sigma}_i$ is smooth and D is an integral, codimension 1, closed subvariety. This proves Claim 5.8.

Denote by

$$y_\Gamma : \mathcal{C} \times_P \Gamma \xrightarrow{\mathrm{pr}_{\mathcal{C}}} \mathcal{C} \xrightarrow{\mathrm{pr}_{\widetilde{B}}} \widetilde{B}$$

the natural morphism.

claim-nonisoB

Claim 5.9. *The morphism y_Γ is smooth.*

By chasing diagrams, y_Γ is the composition of an open immersion and the base change by $\widetilde{B} \to \mathbb{P}^b_k$ of the same morphism y_Γ in the special case that $\widetilde{B} = \mathbb{P}^b_k$. Thus it suffices to prove Claim 5.9 in the special case that $\widetilde{B} = \mathbb{P}^b_k$. In this case y_Γ is the fiber bundle over $\mathbb{P}^b_k$ whose fiber over $p \in \mathbb{P}^b_k$ is the variety parametrizing all triangles T of non-concurrent lines containing p as a smooth point, which is clearly smooth. This proves Claim 5.9.

Denote by

$$y_{\widetilde{\Sigma}} : \mathcal{C} \times_P \widetilde{\Sigma}_i \to \widetilde{B}$$

the composition

$$\mathcal{C} \times_P \widetilde{\Sigma}_i \xrightarrow{\mathrm{pr}_{\mathcal{C}}} \mathcal{C} \xrightarrow{\mathrm{pr}_{\widetilde{B}}} \widetilde{B}.$$

For every open subset $U \subset \widetilde{\Sigma}$, denote by

$$y_U : \mathcal{C} \times_P U \to \widetilde{B}$$

the restriction of $y_{\widetilde{\Sigma}}$ over U.

claim-nonisoC

Claim 5.10. *There exists an open subset $U \subset \widetilde{\Sigma}$ intersecting D such that y_U is smooth.*

The smooth locus of $y_{\widetilde{\Sigma}}$ is an open subscheme of $\mathcal{C} \times_P \widetilde{\Sigma}$. Its complement is closed. Because $\mathcal{C} \to P$ is proper, the image of the closed complement in $\widetilde{\Sigma}$ is closed. The open complement $U \subset \widetilde{\Sigma}$ is the maximal open subscheme such that y_U is smooth. The claim is precisely that U intersects D.

Since $\mathrm{char}(k) = 0$, the morphism $D \to \Gamma$ is smooth on a dense open subset $U_D \subset D$. Thus the product

$$\mathcal{C} \times_P U_D \to \mathcal{C} \times_P \Gamma$$

is smooth. Denote by

$$y_{U_D} : \mathcal{C} \times_P U_D \to \widetilde{B}$$

the composition of this product with y_Γ. By Claim 5.9, y_Γ is smooth. Thus the composition y_{U_D} is smooth. Chasing diagrams, y_{U_D} equals the restriction of $y_{\widetilde{\Sigma}}$ to $\mathcal{C} \times_P U_D$. Since $\mathcal{C} \times_P D$ is a Cartier divisor in $\mathcal{C} \times_P \widetilde{\Sigma}_i$, $y_{\widetilde{\Sigma}}$ is smooth at every point of $\mathcal{C} \times_P U_D$. Since the singular locus of $y_{\widetilde{\Sigma}}$ does not intersect the inverse image of U_D, U_D is contained in U. Therefore U intersects D. This proves Claim 5.10.

claim-nonisoD

Claim 5.11. *There exists an open subset $V \subset \widetilde{B}$ containing B and all codimension 1 points such that G is regular on the open subset $y_U^{-1}(V) \subset \mathcal{C} \times_P U$.*

Since y_U is smooth,

$$y_U : y_U^{-1}(B) \to B$$

is also smooth. Since $Q \to B$ is an Abelian scheme, the rational transformation of smooth B-schemes

$$G : y_U^{-1}(B) \dashrightarrow Q$$

is a regular morphism by Weil's extension theorem, cf. [BLR90, §4.4], [Kol96, Theorem VI.1.9.3]. Because $\widetilde{Q} \to \widetilde{B}$ is a Néron model, by the Neron extension property, there exists an open subset V of $\widetilde{B}$ containing B and all codimension 1 points such that

$$G : y_U^{-1}(V) \dashrightarrow \widetilde{Q}$$

is a regular morphism. This proves Claim 5.11.

Let $P_V \subset P$ denote the open subscheme parametrizing curves in $\widetilde{B}$ entirely contained in V. Because V contains all codimension 1 points of $\widetilde{B}$, a general triangle curve is contained in P_V. Since Γ is irreducible, $P_V \cap \Gamma$ is dense. Since $D \to \Gamma$ is dominant, $D \times_P P_V$ is also dense in D. Since $U \cap D$ is also dense in D, the two open subsets of D intersect. In other words, $U \times_P P_V \subset \widetilde{\Sigma}_i$ is an open subset intersecting D. Since D is integral, this subset is dense in D. Since $D \to \Gamma$ is dominant, the image of $U \times_P P_V$ in P intersects Γ in a dense subset. By definition of P_V,

$$\mathcal{C} \times_P (U \times_P P_V) \subset y_U^{-1}(V).$$

Therefore G is regular over $U \times_P P_V$ by Claim 5.11. Thus $U \times_P P_V \subset \overline{\Sigma}_i$ is an open subset satisfying the conditions at the beginning of the proof. Therefore the image of Σ_i in P intersects Γ in a dense subset. $\square$

lem-noniso2

Lemma 5.12. *Assume char$(k) = 0$. Let B be an integral, smooth, quasi-projective k-scheme of dimension $b \geq 2$. Let Q be an Abelian scheme over B. Assume that Q is strongly non-isotrivial, cf. Definition 4.7. For a very general cubic curve C in B, the map*

$$H^0_{\acute{e}t}(B, Q) \to H^0_{\acute{e}t}(C, C \times_B Q)$$

is a bijection.

Proof. The group $H^0_{\text{ét}}(B, Q)$ is finitely generated by the Lang-Néron theorem. Applying the Néron extension property to each of the finitely many generators, there exists a single open subset of $\widetilde{B}$ containing B and all codimension 1 points such that every section of Q over B extends to a section of $\widetilde{Q}$ over this open subset. Replacing $\widetilde{B}$ by this open subset, we may assume that every section of Q over B extends to a section of $\widetilde{Q}$ over $\widetilde{B}$. Then restriction to curves in $\widetilde{B}$ determines a homomorphism of group schemes over P

$$\text{restr} : H^0_{\text{ét}}(B, Q) \times P \to \Sigma,$$

where the first group scheme is the finite, flat group scheme over P whose geometric fibers are all canonically identified with $H^0_{\text{ét}}(B, Q)$. The lemma is equivalent to the statement that the induced homomorphism of geometric generic fibers over P is a bijection.

Denote by $\eta \in P$ the generic point and denote by $\gamma \in P$ the generic point of Γ. Denote $R = \mathcal{O}_{P,\eta_\Gamma}$. Because P is smooth and $\Gamma \subset P$ is a Cartier divisor, this is a

DVR. Consider the base change of restr over Spec R. The geometric generic fiber of $\Sigma \times_P \operatorname{Spec} R$ over Spec R is

$$\Sigma_{\overline{\eta}} = \Sigma \otimes_{\mathcal{O}_P} \overline{\kappa(\eta)},$$

and the geometric closed fiber is

$$\Sigma_{\overline{\gamma}} = \Sigma \otimes_{\mathcal{O}_P} \overline{\kappa(\gamma)}.$$

By Lemma 4.9, $\Sigma \times_P \operatorname{Spec} R$ is a group scheme over Spec R which is "locally quasi-finite", i.e., every quasi-compact component $\Sigma_i \times_P \operatorname{Spec} R$ is quasi-finite over Spec R. By Zariski's Main Theorem, the morphism to Spec R is the composition of a dense open immersion and a finite morphism. By Lemma 5.12, points of $\Sigma_{\overline{\eta}}$ specialize to points of $\Sigma_{\overline{\gamma}}$. Thus $\Sigma_i \times_P \operatorname{Spec} R \to \operatorname{Spec} R$ is finite. Since the characteristic is zero, all group schemes are reduced. Thus $\Sigma \times_P \operatorname{Spec} R$ is unramified over Spec R. In particular, the specialization homomorphism

$$\text{spec} : \Sigma_{\overline{\eta}} \to \Sigma_{\overline{\gamma}}$$

is injective. But this is compatible with the restriction homomorphism, i.e.,

$$\begin{array}{ccc} H^0_{\text{ét}}(B, Q) & \xrightarrow{=} & H^0_{\text{ét}}(B, Q) \\ {\scriptstyle \text{restr}_{\overline{\eta}}}\downarrow & & \downarrow{\scriptstyle \text{restr}_{\overline{\gamma}}} \\ \Sigma_{\overline{\eta}} & \xrightarrow{\text{spec}} & \Sigma_{\overline{\gamma}} \end{array}$$

is a commutative diagram of group homomorphisms. By Lemma 5.3, the right vertical homomorphism is a bijection. Since spec is injective, also the right vertical homomorphism is a bijection, proving the lemma. ☐

Proof of Theorem 1.2. **(ii)** First we prove (ii), i.e., we prove that $\mathcal{P}(B, f, A)$ holds, cf. Definition 4.1. By Corollary 4.11, there exists a dense, open subscheme $U \subset B$, a finite, étale, Galois morphism $U' \to U$, and an isogeny of Abelian schemes over U',

$$u = u_0 \oplus u_Q : (U' \times_k A_0) \times_{U'} Q \to U' \times_B A$$

where A_0 is an Abelian k-variety and Q is a strongly non-isotrivial Abelian scheme over U'. By the elementary reductions Lemma 4.2 and Lemma 4.3, it suffices to prove $\mathcal{P}(U', f', U' \times_B A)$ holds. By the isogeny reduction, Lemma 4.12, it suffices to prove the two cases $\mathcal{P}(U', f', U' \times_k A_0)$ and $\mathcal{P}(U', f', Q)$ hold. The first case follows from Lemma 5.1 for triangle curves, resp. Lemma 5.2 for cubic curves. The second case follows from Lemma 5.3 for triangle curves, resp. Lemma 5.12 for cubic curves in characteristic 0. This proves (ii).

(i) Next we prove (i), by reducing it to (ii). Let T be an A-torsor over B. Because this is a torsor for the étale topology, there exists a dense open subset $U \subset B$ and a finite, étale, Galois morphism $U' \to U$ with Galois group G such that $U' \times_B T$ is a trivial $U' \times_B A$-torsor over U'. By (i), for a very general triangle curve C in B,

$$H^0_{\text{ét}}(U', U' \times_B A) \to H^0_{\text{ét}}(U' \times_B C, U' \times_B C \times_B A)$$

is a bijection. This is an isomorphism of Galois modules. Thus the induced morphism of Galois invariants is an isomorphism. By étale descent, the morphism of Galois invariants is the restriction map

$$\operatorname{Hom}_{U-\text{sch}}(U, U \times_B T) \to \operatorname{Hom}_{U \times_B C-\text{sch}}(U \times_B C, U \times_B C \times_B T).$$

In particular, if $C \times_B T$ has a section over C, then $U \times_B T$ has a section over U. This section extends over all of B by [Kol96, Theorem VI.1.9.3]. Thus, if $C \times_B T$ is a trivial torsor, then T is a trivial torsor.

Also, since sections of T over very general cubic curves specialize to sections of T over very general triangle curves, if $C \times_B T$ has a section over C for a very general cubic curve, then is also has a section over a very general triangle curve. Therefore, by the previous paragraph, T is a trivial torsor.

Finally, consider the statement with planar surfaces in place of curves. Let C be a very general triangle curve in B satisfying (ii). Since $C = B \times_{\mathbb{P}^b_k} E$, where E is a plane cubic curve in a linear 2-plane $\Pi \subset \mathbb{P}^b_k$, C is contained in the planar surface $S = B \times_{\mathbb{P}^b_k} \Pi$. If C is very general, then S is very general. Moreover, since (ii) holds for B for one triangle curve in S, it holds for every very general triangle curve in S. But for a very general triangle curve in S, (ii) also applies with S in place of B. Therefore both

$$H^0_{\text{ét}}(B, A) \to H^0_{\text{ét}}(C, C \times_B A)$$

and

$$H^0_{\text{ét}}(S, S \times_B A) \to H^0_{\text{ét}}(C, C \times_B A)$$

are bijections. Since the first map factors through the second map by restriction

$$H^0_{\text{ét}}(B, A) \to H^0_{\text{ét}}(S, S \times_B A),$$

this restriction is also a bijection. And (i) follows for planar surfaces in a similar way. □

sec-ex

6. Examples and the proof of Proposition 1.5

In this section we give some examples demonstrating limits to further generalization of Theorem 1.2.

Example (i). First of all, the restriction map

$$H^1_{\text{ét}}(B, A) \to H^1_{\text{ét}}(C, C \times_B A)$$

can fail to be surjective for all triangle curves, resp. all cubic curves. For an example, let $B = \mathbb{P}^2$, let f be the identity map, and let $A = B \times_k A_0$, where A_0 is a simple Abelian k-variety of dimension $g \geq 2$. Let l be an integer not divisible by $\text{char}(k)$. Associated to the multiplication map

$$0 \longrightarrow B \times_k A_0[l] \longrightarrow B \times_k A_0 \xrightarrow{\text{mult}_l} B \times_k A_0 \longrightarrow 0$$

there is a long exact sequence of cohomology groups part of which is

$$0 \longrightarrow H^0_{\text{ét}}(B, B \times_k A_0) \otimes \mathbb{Z}/l\mathbb{Z} \longrightarrow H^0_{\text{ét}}(B, B \times_k A_0[l]) \longrightarrow H^0_{\text{ét}}(B, B \times_k A_0)[l] \longrightarrow 0.$$

Because $\text{Alb}(P) = 0$,

$$H^0_{\text{ét}}(B, B \times_k A_0) = A_0(k).$$

Because A_0 is divisible, the exact sequence above gives

$$H^1_{\text{ét}}(B, B \times_k A_0)[l] \cong H^1_{\text{ét}}(B, B \times_k A_0[l]) \cong H^1_{\text{ét}}(B, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g}.$$

A similar argument applies to plane cubic curves and triangle curves in B. Of course the Albanese variety of a plane curve is not zero. But because A_0 is a simple

Abelian variety of dimension $g \geq 2$, every morphism from C to A_0 is a constant morphism. Thus

$$H^1_{\text{ét}}(C, C \times_k A_0)[l] \cong H^1_{\text{ét}}(C, C \times_k A_0[l]) \cong H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g}.$$

Thus the restriction homomorphism on l-torsion is equivalent to

$$H^1_{\text{ét}}(\mathbb{P}^2_k, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g} \to H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z})^{\oplus 2g}.$$

Now

$$H^1_{\text{ét}}(\mathbb{P}^2_k, \mathbb{Z}/l\mathbb{Z}) = 0$$

whereas

$$H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z}) = (\mathbb{Z}/l\mathbb{Z})^{\oplus 2}$$

for a smooth plane cubic and

$$H^1_{\text{ét}}(C, \mathbb{Z}/l\mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}$$

for a triangle curve. Thus the restriction map is not surjective.

Example (ii). This example is similar. Let $B = \mathbb{A}^2_k$ and let $f : \mathbb{A}^2_k \hookrightarrow \mathbb{P}^2_k$ be the usual inclusion. Let A_0 be an Abelian k-variety having a p-torsion k-point,

$$a \in A_0(k)[p],$$

e.g., A_0 is an ordinary elliptic curve over k. Let $A_0 \to A_1$ be the étale isogeny of Abelian varieties whose kernel is generate by a. There is an exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{a} A_0 \longrightarrow A_1 \longrightarrow 0.$$

This gives rise to a long exact sequence. For the same reason as in Example (i), the induced map

$$H^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{Z}/p\mathbb{Z}) \to H^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{A}^2_k \times_k A_0)[p]$$

is an injection. Thus to produce an A_0-torsor over $\mathbb{A}^2_k$ of order p whose restriction to C is trivial, it suffices to produce a $\mathbb{Z}/p\mathbb{Z}$-torsor T over $\mathbb{A}^2_k$ whose restriction to C is trivial. Let $f \in k[x, y]$ be a polynomial function on $\mathbb{A}^2_k$ vanishing on C and whose degree d is not divisible by p.

Let $T \to \mathbb{A}^2_k$ be the Artin-Schreier cover determined by the ring homomorphism

$$k[x, y] \to k[x, y, t]/\langle t^p - t - f\rangle.$$

Because d is not divisible by p, there is no element $g \in k[x, y]$ such that $g^p - g - f = 0$. Since $k[x, y]$ is normal and $t^p - t - f$ is a monic polynomial in t, also there is no element $g \in k(x, y)$ such that $g^p - g - f = 0$. Thus T is integral and so $T \to \mathbb{A}^2_k$ is a nontrivial torsor. On the other hand, since f is zero on C,

$$C \times_{\mathbb{A}^2_k} T \cong C \times \text{Spec } k[t]/\langle t^p - t\rangle,$$

which is the trivial $\mathbb{Z}/p\mathbb{Z}$-torsor over C. Thus

$$H^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{A}^2_k \times_k A_0)[p] \to H^1_{\text{ét}}(C, C \times_k A_0)[p]$$

is not injective.

Example (iii). Let $B = \mathbb{P}^2_k$ and let f be the identity map. Let A_0 be an ordinary elliptic curve over k and let $A = \mathbb{P}^2_k \times_k A_0$. Then

$$H^0_{\text{ét}}(\mathbb{P}^2_k, \mathbb{P}^2_k \times_k A_0) = A_0(k).$$

But every open subset of the set of plane cubic curves in $\mathbb{P}^2_k$ contains curves C isogenous to A_0. For these, the map

$$A_0(k) \to H^0_{\text{ét}}(C, C \times_k A_0)$$

is not surjective. Thus, it is necessary to use very general cubic curves in Theorem 1.2 – general cubic curves do not suffice.

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