MAT131 Fall 2008 Midterm 2 Review Sheet

The topics tested on Midterm 2 will be among the following.

(i) Given a function, its first derivative and its second derivative, use the first derivative test to determine where the function is increasing and where decreasing, and use the second derivative test to determine where the function is concave up and where concave down. Combine this with even/odd, identification of vertical and horizontal asymptotes, and identification of discontinuities to give a rough sketch of the graph of the function.

(ii) Use the rules of differentiation: the sum rule, the product rule, the power rule and the derivatives of exponentials.

(iii) Given the limits $\lim_{h \to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{h \to 0} \frac{1 - \cos(x)}{x} = 0$, use trigonometric identities to find the formulas for the derivatives of $\sin(x)$ and $\cos(x)$ as limits of difference quotients.

(iv) Find the derivatives of other trigonometric functions – $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$ – using the derivatives for $\sin(x)$ and $\cos(x)$ and the rules for differentiation.

(v) Find derivatives using the chain rule.

(vi) Find the tangent slope to a parametric curve at a specified point.

(vii) Find derivatives using implicit differentiation, including derivatives of inverse functions.

(viii) Find derivatives using logarithmic differentiation.

(ix) Find the linear approximation to the value of a function, using a known nearby value and derivative or using differentials. Determine whether your approximation is an overestimate or an underestimate.

(x) Understanding differential notation and the geometric interpretation of differentials. Using differentials to approximate values of functions.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 1. In each of the following cases, a function $y = f(x)$ is given as well as the first derivative $f'(x)$ and the second derivative $f''(x)$. In each case do all of the following.

(i) Compute the first and second derivatives of $f(x)$ to verify the given formulas.

(ii) Determine whether $f(x)$ is even, odd or neither.

(iii) Find the equations of all vertical and horizontal asymptotes.

(iv) Determine where $y = f(x)$ is increasing and where $y = f(x)$ is decreasing. Pay careful attention to the change as x crosses a vertical asymptote.

(v) Determine where $y = f(x)$ is concave up and where $y = f(x)$ is concave down. Pay careful attention to the change as x crosses a vertical asymptote.

(vi) Give a rough sketch of the graph of $y = f(x)$ carefully labelling all of the above information on your sketch.

(a)

$$f(x) = x^3 - x,$$
$$f'(x) = 3x^2 - 1,$$
$$f''(x) = 6x.$$

(b)

$$f(x) = x^4 - 2x^2,$$

$$f'(x) = 4x^3 - 4x,$$
$$f''(x) = 12x^2 - 4.$$

(c)

$$f(x) = x^{-1}e^x = \frac{e^x}{x},$$
$$f'(x) = (x-1)x^{-2}e^x = \frac{(x-1)e^x}{x^2},$$
$$f''(x) = (x^2 - 2x + 2)x^{-3}e^x = \frac{(x^2 - 2x + 2)e^x}{x^3}.$$

(d)

$$f(x) = x^{-2}e^x = \frac{e^x}{x^2},$$
$$f'(x) = (x-2)x^{-3}e^x = \frac{(x-2)e^x}{x^3},$$
$$f''(x) = (x^2 - 4x + 6)x^{-4}e^x = \frac{(x^2 - 4x + 6)e^x}{x^4}.$$

(e)

$$f(x) = x^{-3}e^x = \frac{e^x}{x^3},$$
$$f'(x) = (x-3)x^{-4}e^x = \frac{(x-3)e^x}{x^4},$$
$$f''(x) = (x^2 - 6x + 12)x^{-5}e^x = \frac{(x^2 - 6x + 12)e^x}{x^5}.$$

(f)

$$f(x) = x^{-1}\ln(x^2) = \frac{\ln(x^2)}{x}$$
$$f'(x) = x^{-2}(2 - \ln(x^2)) = \frac{2 - \ln(x^2)}{x^2},$$
$$f''(x) = x^{-3}(2\ln(x^2) - 6) = \frac{2\ln(x^2) - 6}{x^3}.$$

(g)

$$f(x) = x^{-2}\ln(x^2) = \frac{\ln(x^2)}{x^2},$$

$$f'(x) = x^{-3}(2 - 2\ln(x^2)) = \frac{2 - 2\ln(x^2)}{x^3},$$
$$f''(x) = x^{-4}(6\ln(x^2) - 10) = \frac{6\ln(x^2) - 10}{x^4}.$$

(h)

$$f(x) = (1 + x^{-2})^{1/2} = \frac{1}{\sqrt{1 + (1/x)^2}},$$
$$f'(x) = -x^{-3}(1 + x^{-2})^{-1/2} = \frac{-1}{x^3\sqrt{1 + (1/x)^2}},$$
$$f''(x) = (3x^2 + 2)x^{-6}(1 + x^{-2})^{-3/2} = \frac{3x^2 + 1}{x^6(\sqrt{1 + (1/x)^2})^3}.$$

Solution to Problem 1 There is a sheet with scanned solutions to this problem at http://www.math.sunysb.edu/ jstarr/M131f08/exams.html.

Problem 2. What is the differential of $y = x^3$? When $x = 2$, a small change $dx = 0.01$ in x produces what change dy in y?

Solution to Problem 2 Since $y' = 3x^2$,

$$\frac{dy}{dx} = y' = 3x^2, \quad \boxed{dy = 3x^2 dx.}$$

When $x = 2$, this gives $dy = 3(2)^2 dx = 12dx$. Thus a small change $dx = 0.01$ produces a change $dy = 12 \times (0.01)$ or $\boxed{0.12}$ in y. Since $y(2) = 8$, this gives $y(2.01) \approx 8.12$.

Problem 3. The inverse function of

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

is called the *inverse hyperbolic cosine*, $\cosh^{-1}(x)$. For $y = \cosh^{-1}(x)$, find a formula for the derivative of y that is an expression only involving polynomials in x and square roots.

Hint. After you find an answer that involves y, consider what you get by squaring your formula for y' and squaring your formula for x. Can you relate these two formulas?

Solution to Problem 3 Since $y = f^{-1}(x)$, this gives $f(y) = x$. This is the implicit equation

$$\frac{1}{2}(e^y + e^{-y}) = x.$$

Implicit differentiation gives

$$\frac{1}{2}(e^y - e^{-y})y' = 1, \quad y' = \frac{2}{e^y - e^{-y}}.$$

Squaring both sides gives

$$(y')^2 = \frac{4}{e^{2y} - 2 + e^{2y}}.$$

On the other hand, squaring both sides of the original implicit equation gives

$$\frac{e^{2y} + 2 + e^{-2y}}{4} = x^2.$$

Thus

$$\frac{e^{2y} - 2 + e^{-2y}}{4} = \frac{e^{2y} + 2 + e^{-2y}}{4} - 1 = x^2 - 1.$$

Substituting this into the equation for $(y')^2$ gives

$$(y')^2 = \frac{1}{x^2 - 1}.$$

On the interval $[1, \infty)$ where $\cosh^{-1}(x)$ is usually defined, y' is given by the positive square root,

$$\boxed{y' = 1/\sqrt{x^2 - 1}.}$$

Problem 4. The double-angle formula for tangent is

$$\tan(2x) = \frac{2\tan(x)}{1 - (\tan(x))^2}.$$

Compute the derivative of each side of this equation. Which derivative is easier to compute?

Solution to Problem 4 The derivative of $\tan(u)$ is given by

$$d\tan(u) = \sec^2(u)du.$$

Thus, by the chain rule,

$$d\tan(2x) = \sec^2(2x)d(2x) = \sec^2(2x)2dx.$$

This gives

$$\boxed{\frac{d\tan(2x)}{dx} = 2\sec^2(2x).}$$

And the quotient rule gives

$$\frac{d}{dx}\left(\frac{2\tan(x)}{1-(\tan(x))^2}\right) = \frac{(1-\tan^2(x))(2\sec^2(x)) - (2\tan(x))(-2\tan(x)\sec^2(x))}{(1-\tan^2(x))^2}.$$

After simplification this gives

$$\frac{2\sec^2(x)(1+\tan^2(x))}{(1-\tan^2(x))^2}.$$

Using the identity that

$$1+\tan^2(x) = \sec^2(x),$$

this becomes

$$\frac{d}{dx}\left(\frac{2\tan(x)}{1-(\tan(x))^2}\right) = \boxed{2\sec^4(x)/(1-\tan^2(x))^2.}$$

Problem 5. Let a be a positive constant and consider the parametric curve

$$\begin{cases} x(t) &= 2at \\ y(t) &= \frac{2a}{1+t^2} \end{cases}$$

Compute the slope of the tangent line at the point where $t = 1/\sqrt{3}$.

Solution to Problem 5 The derivatives are

$$\begin{cases} dx/dt &= 2a \\ dy/dt &= \frac{-4at}{(1+t^2)^2} \end{cases}$$

Thus,

$$dx = 2adt, \quad dy = \frac{-4at}{(1+t^2)^2}dt.$$

Since dx is nonzero, this gives

$$\frac{dy}{dx} = \frac{-4at}{(1+t^2)^2}\frac{1}{2a} = \frac{-2t}{(1+t^2)^2}$$

at the point $(x(t), y(t))$. In particular, when $t = 1/\sqrt{3}$,

$$\frac{dy}{dx} = \frac{-2}{\sqrt{3}(1+(1/\sqrt{3})^2)^2} = \boxed{-3\sqrt{3}/8.}$$

Problem 6. Compute each of the following derivatives.

(a)

$$y = \ln(\ln(x)),\ x > 1$$

Solution to (a)

$$y' = \frac{1}{\ln(x)}\frac{1}{x} = \boxed{1/(x\ln(x)).}$$

(b)

$$y = e^{e^x}$$

Solution to (b)

$$y' = e^{e^x}e^x = \boxed{e^{x+e^x}.}$$

(c)

$$y = \frac{2x}{1+x^2}$$

Solution to (c)

$$y' = \frac{(1+x^2)(2)-(2x)(2x)}{(1+x^2)^2} = \boxed{2(1-x^2)/(1+x^2)^2.}$$

(d)

$$y = \frac{x^3\sqrt{\sin(x)}}{\sqrt{\cos(x)}}$$

Solution to (d) Denote by u the logarithm

$$u = \ln(y) = 3\ln(x) + \frac{1}{2}\ln(\sin(x)) - \frac{1}{2}\ln(\cos(x)).$$

Then

$$\frac{du}{dx} = \frac{3}{x} + \frac{1}{2}\frac{1}{\sin(x)}\cos(x) - \frac{1}{2}\frac{1}{\cos(x)}(-\sin(x)).$$

Simplifying, this becomes

$$\frac{du}{dx} = \frac{6\sin(x)\cos(x) + 1}{2x\sin(x)\cos(x)}.$$

And since also

$$du = \frac{1}{y}dy, \quad dy = ydu$$

this gives

$$\frac{dy}{dx} = y\frac{du}{dx} = \boxed{x^2(6\sin(x)\cos(x) + 1)/(2\sqrt{\sin(x)\cos^3(x)}).}$$

(e)

$$y = \ln(-x + \sqrt{x^2 - 1})$$

(Simplify your answer as much as possible.)

Solution to (e)

$$y' = \frac{1}{-x + \sqrt{x^2 - 1}}\frac{d}{dx}(-x+\sqrt{x^2 - 1}) = \frac{1}{-x + \sqrt{x^2 - 1}}\left(-1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{1}{-x + \sqrt{x^2 - 1}}\frac{x - \sqrt{x}}{\sqrt{x^2}}$$

Simplifying gives

$$y' = \boxed{1/\sqrt{x^2 - 1}.}$$

Problem 7. Using your known formulas for the derivatives of $\sin(x)$ and $\cos(x)$, find the limit

$$\lim_{h\to 0}\frac{\cos(h) - 1}{h}$$

by relating it to the derivative of $\cos(x)$ for some value of x.

Solution to Problem 7 By definition, the derivative of $\cos(x)$ at $x = 0$ is

$$\lim_{h\to 0}\frac{\cos(h) - \cos(0)}{h}.$$

Since $\cos(0) = 1$, this gives

$$\frac{d\cos(x)}{dx}|_{x=0} = \lim_{h\to 0}\frac{\cos(h) - 1}{h}.$$

On the other hand, the formula for the derivative is

$$\frac{d\cos(x)}{dx} = -\sin(x).$$

Since $\sin(0) = 0$, this gives

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \boxed{0.}$$

Problem 8. The value of $1/(1 + \sqrt{10})$ is close to $1/4 = 0.25$. Using a linear approximation or differentials, estimate whether the true value is closer to 0.2 or closer to 0.3.

Solution to Problem 8 Observe 10 is close to 9 and $\sqrt{9} = 3$. Thus let

$$y = \frac{1}{1 + \sqrt{x}}.$$

The differential of y is

$$dy = \frac{-1}{2\sqrt{x}(1 + \sqrt{x})^2} dx.$$

Thus at $x = 9$ the differential is

$$dy = \frac{-1}{2(3)(1 + 3)^2} dx = \frac{-1}{96} dx.$$

So the change $dx = 10 - 9 = 1$ gives a change $dy = (-1/96)(1)$ for x close to 9. This gives a linear approximation

$$\frac{1}{1 + \sqrt{10}} \approx \frac{1}{4} - \frac{1}{96} = \boxed{23/96}$$

which is closer to $\boxed{0.2}$ than to 0.3.

Problem 9. For the curve with implicit equation

$$y + \frac{1}{y} = 4x + 2x^2$$

find the slope of the tangent line to the curve at the point $(x, y) = (1/2, 2)$.

Solution to Problem 9 Differentiating both sides gives

$$\frac{y^2-1}{y^2}y' = 4+4x.$$

When y is not 0 or ± 1, we can divide and get

$$y' = \frac{4(x+1)y^2}{y^2-1}.$$

Plugging in $(x, y) = (1/2, 2)$ gives

$$y' = \boxed{8.}$$

Problem 10. Find the derivative y' of the function y in each of the following cases.

(a)

$$y = \sin(\sqrt{x^2+1})$$

Solution to (a)

$$\frac{dy}{dx} = \cos(\sqrt{x^2+1})\left(\frac{1}{2}(x^2+1)^{-1/2}\right)(2x) = \boxed{x\cos(\sqrt{x^2+1})/\sqrt{x^2+1}.}$$

(b)

$$y = (\ln(x^2))^2$$

Solution to (b) First of all,

$$y = (2\ln(x))^2 = 4(\ln(x))^2.$$

Thus,

$$\frac{dy}{dx} = 4(2\ln(x))\left(\frac{1}{x}\right) = \boxed{8\ln(x)/x.}$$

(c)

$$y = (\sqrt{x})^{\cos(x)}$$

Solution to (c) Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln(\sqrt{x}^{\cos(x)}) = \cos(x)\ln(\sqrt{x}) = \frac{1}{2}\cos(x)\ln(x).$$

The derivative of this is

$$\frac{du}{dx} = \frac{1}{2}(-\sin(x))\ln(x) + \frac{1}{2}\cos(x)\left(\frac{1}{x}\right) = \frac{\cos(x) - x\sin(x)\ln(x)}{2x}.$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = (\sqrt{x})^{\cos(x)}\frac{\cos(x) - x\sin(x)\ln(x)}{2x} = \boxed{(\cos(x) - x\sin(x)\ln(x))(\sqrt{x})^{\cos(x)}/2x.}$$

(d)

$$y = \frac{x^2 e^x}{x^2 + 2x + 1}$$

Solution to (d) Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln(x^2e^x/(x^2+2x+1)) = 2\ln(x)+x-\ln(x^2+2x+1) = 2\ln(x)+x-2\ln(x+1)$$

The derivative of this is

$$\frac{du}{dx} = \frac{2}{x} + 1 - 2\frac{1}{x+1} = \frac{x^2 + x + 2}{x(x+1)}.$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = \frac{x^2e^x}{(x+1)^2}\frac{x^2 + x + 2}{x(x+1)} = \boxed{x(x^2 + x + 2)e^x/(x+1)^3.}$$

(e)

$$x^2 + 4y^2 = 5$$

Solution to (e) Implicitly differentiating both sides with respect to x gives

$$2x + 8y\frac{dy}{dx} = 0.$$

Solving for the derivative gives

$$\frac{dy}{dx} = \boxed{-x/4y.}$$

Solving for y as well gives an answer depending only on x,

$$\frac{dy}{dx} = \boxed{-x/(2\sqrt{5 - x^2}).}$$

(f)

$$y = \frac{(x^2+1)^3}{x^2-1}$$

Solution to (f) Rewrite this as

$$y = \frac{(x^2+1)^3}{(x+1)(x-1)}.$$

Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln\left(\frac{(x^2+1)^3}{(x+1)(x-1)}\right) = 3\ln(x^2+1) - \ln(x+1) - \ln(x-1).$$

The derivative of this is

$$\frac{du}{dx} = 3\frac{1}{x^2+1}(2x) - \frac{1}{x+1} - \frac{1}{x-1} = \frac{4x(x^2-2)}{(x^2+1)(x+1)(x-1)}$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = \frac{(x^2+1)^3}{(x+1)(x-1)}\frac{4x(x^2-2)}{(x^2+1)(x+1)(x-1)} = \boxed{4x(x^2-2)(x^2+1)^2/((x+1)(x-1)).}$$

Problem 11 In each of the following two cases, find the linearization of $f(x)$ near the point $x = a$.

(a)

$$f(x) = x^{2/5}, \quad a = 32.$$

Solution to (a) The derivative is

$$f'(x) = \frac{2}{5}x^{-3/5}.$$

Plugging in $x = 32$ so that $x^{1/5} = 2$, this gives

$$f'(32) = \frac{2}{5}2^{-3} = \frac{1}{20}.$$

So the linearization is

$$f(x) \approx f(a) + f'(a)(x-a) = \boxed{2 + (1/20)(x-32).}$$

(b)

$$f(x) = \frac{1}{\sqrt{1+x^2}}, \quad a = \sqrt{3}.$$

Solution to (b) The derivative is

$$f'(x) = \frac{-1}{2}(1+x^2)^{-3/2}(2x) = \frac{-x}{(1+x^2)^{3/2}}.$$

Plugging in $a = \sqrt{3}$ so that $f(\sqrt{3}) = 1/\sqrt{1+3} = 1/2$, the derivative is

$$f'(\sqrt{3}) = \frac{-\sqrt{3}}{(1/2)^3} = -8\sqrt{3}.$$

So the linearization is

$$f(x) \approx f(a) + f'(a)(x-a) = \boxed{(1/2) - 8\sqrt{3}(x-\sqrt{3}).}$$

Problem 12 Using differentials or an appropriate linear approximation, approximate the following number.

$$\frac{1}{\sqrt{25.1}}$$

Solution to Problem 12 Let $f(x) = x^{-1/2}$ and let $a = 25$ so that $f(a) = 1/5$. The differential is

$$dy = df(x) = \frac{-1}{2}x^{-3/2}dx.$$

When $x = 25$ this gives

$$dy = \frac{-1}{2}5^{-3}dx = -\frac{1}{250}dx.$$

Thus the linear approximation is

$$f(25.1) \approx 1/5 - \frac{1}{250}(0.1) = \boxed{0.1996} = \boxed{499/2500.}$$