MAT131 Fall 2007 Midterm 1 Review Sheet

The topics tested on Midterm 1 will be among the following.

(i) Definition, basic properties and graphs of elementary functions: powers, exponentials, logarithms, and trigonometric.

(ii) The definition, basic properties and graphs of **even** and **odd** functions.

(iii) The definition and meaning of **increasing** and **decreasing** for functions and graphs.

(iv) Reflection, translation and scaling of graphs and the corresponding transformation of the functions.

(v) Definition, basic properties, and graphs of inverse functions. Computation of an inverse function.

(vi) Definition, basic laws, and techniques for computing limits, one-sided limits, limits using the squeeze theorem, limits equal to infinity, and limits at infinity.

(vii) Identifying all discontinuity points (both the location and type), the domain of a function, and all vertical and horizontal asymptotes. Application of these notions to curve-sketching.

(viii) The statement of the Intermediate Value Theorem and its use in finding zeroes of functions.

(ix) The definition of the derivative as the limit of a difference quotient, and methods for computing derivatives directly from the definition.

(x) Using the derivative to compute the equations of tangent lines.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 1. In each of the following cases, determine whether the limit exists as a finite number, and say its value if it is defined. If the limit does not exist as a finite number, determine whether the limit is positive or negative infinity. If the limit does not exist as a finite number or as positive/negative infinity, explain why.

(a)

$$\lim_{x\to 0} f(x), \text{ where } f(x) = \begin{cases} \sqrt{x}, & x > 0 \\ -\sqrt{-x}, & x \le 0 \end{cases}$$

Solution to (a)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = \boxed{0}.$$

(b)

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x}.$$

Solution to (b)

$$1 = \lim_{x\to\infty} \frac{x-1}{x} \le \lim_{x\to\infty} \frac{x+\sin(x)}{x} \le \lim_{x\to\infty} \frac{x+1}{x} = 1.$$

Thus, by the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x} = \boxed{1}.$$

(c)

$$\lim_{x\to 0} \frac{\sqrt{x^2}}{x}$$

Solution to (c)

$$\lim_{x\to 0^+} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0+} \frac{x}{x} = 1.$$

$$\lim_{x\to 0^-} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^-} \frac{(-x)}{x} = -1.$$

Since

$$\lim_{x\to 0^+} \frac{\sqrt{x^2}}{x} \neq \lim_{x\to 0^-} \frac{\sqrt{x^2}}{x},$$

thus

$$\lim_{x\to 0} \frac{\sqrt{x^2}}{x} \boxed{\text{ does not exist}}.$$

(d)

$$\lim_{x\to 2} \frac{x^3 - 2x^2 - 4x + 8}{x^2 - 4}$$

Solution to (d)

$$\lim_{x\to 2} \frac{(x-2)(x^2-4)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x^2-4}{x+2} = \frac{(2)^2-4}{2+2} = \frac{0}{4} = \boxed{0}.$$

(e)

$$\lim_{x\to\infty} \ln((5+e^{-x})^{1/x})$$

Solution to (e)

$$\lim_{x\to\infty} \ln((5+e^{-x})^{1/x}) = \lim_{x\to} \frac{\ln(5+e^{-x})}{x} = \lim_{x\to\infty} \frac{\ln(5)}{x} = \boxed{0}.$$

(f)

$$\lim_{x\to-\infty} (\sqrt{9x^2+6x} + 3x)$$

Solution to (f)

$$\lim_{x\to-\infty} (\sqrt{9x^2+6x} + 3x) =$$

$$\lim_{x\to-\infty} \frac{(9x^2+6x) - (3x)^2}{\sqrt{9x^2+6x} - 3x} =$$

$$\lim_{x\to-\infty} \frac{6x}{x(-\sqrt{9+(6/x)} - 3)} =$$

$$\lim_{x\to-\infty} \frac{6}{-\sqrt{9+(6/x)} - 3} = \frac{6}{-3-3} = \boxed{-1}.$$

(g)

$$\lim_{x\to 2}\frac{|x-2|}{x-2}$$

Solution to (g)

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2}=\lim_{x\to 2^+}\frac{x-2}{x-2}=1.$$

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2}=\lim_{x\to 2^-}\frac{-(x-2)}{x-2}=-1.$$

Since

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2}\neq\lim_{x\to 2^-}\frac{|x-2|}{x-2},$$

thus

$$\lim_{x\to 2}\frac{|x-2|}{x-2}\boxed{\text{ does not exist }}.$$

(i)

$$\lim_{x\to 1}\frac{x^2-4x+3}{x^2-1}$$

Solution to (i)

$$\lim_{x\to 1}\frac{x^2-4x+3}{x^2-1}=\lim_{x\to 1}(x-1)(x-3)(x-1)(x+1)=\lim_{x\to 1}\frac{x-3}{x+1}=\frac{-2}{2}=\boxed{-1}.$$

(j)

$$\lim_{x\to 0}\frac{\cos x}{x}$$

Solution to (j)

$$\lim_{x\to 0^+}\frac{\cos(x)}{x}=\lim_{x\to 0^+}\frac{\cos(0)}{x}=\lim_{x\to 0^+}\frac{1}{x}=+\infty$$

$$\lim_{x\to 0^-}\frac{\cos(x)}{x}=\lim_{x\to 0^-}\frac{\cos(0)}{x}=\lim_{x\to 0^-}\frac{1}{x}=-\infty.$$

Each one-sided limit is defined as $+\infty$ or $-\infty$, but the two one-sided limits are not the same. Thus

$$\lim_{x\to 0}\frac{\cos(x)}{x}\boxed{\text{ does not exist }}.$$

(k)

$$\lim_{x\to 0} x\cos(x)$$

Solution to (k)

$$\lim_{x\to 0} x\cos(x) = 0\cos(0) = 0\times 1 = \boxed{0}.$$

(l)

$$\lim_{x\to \infty} x\cos(x)$$

Solution to (l)

$$\lim_{x\to \infty} x\cos(x) \boxed{\text{ does not exist}}$$

since the values oscillate between very positive numbers (when x is between $(2N-1/2)\pi$ and $(2N+1/2)\pi$ for positive integers N), and very negative numbers (when x is between $(2N+1/2)\pi$ and $(2N+3/2)\pi$ for positive integers N).

(m)

$$\lim_{x\to \infty} \frac{3x^3+2x-1}{4x^3+1}$$

Solution to (m)

$$\lim_{x\to \infty} \frac{3x^3+2x-1}{4x^3+1} = \lim_{x\to \infty} \frac{x^3(3+(2/x^2)-(1/x^3))}{x^3(4+(1/x^3))} = \lim_{x\to \infty} \frac{3+(2/x^2)-(1/x^3)}{4+(1/x^3)} = \boxed{\tfrac{3}{4}}.$$

(n)

$$\lim_{x\to 0} \frac{\sin(x)}{\sin(2x)}$$

(Hint: Use the angle addition formulas or the double angle formula for sine.)

Solution to (n) By the double angle formula, $\sin(2x) = 2\sin(x)\cos(x)$. Thus

$$\lim_{x\to 0} \frac{\sin(x)}{\sin(2x)} = \lim_{x\to 0} \frac{\sin(x)}{2\sin(x)\cos(x)} = \lim_{x\to 0} \frac{1}{2\cos(x)} = \frac{1}{2\cos(0)} = \boxed{\tfrac{1}{2}}.$$

(o)

$$\lim_{x\to \pi/2} \left(\frac{\cos(x)}{\sin(2x)}\right)$$

Solution to (o) Again by the double angle formula, $\sin(2x) = 2\sin(x)\cos(x)$. Thus

$$\lim_{x\to\pi/2}\left(\frac{\cos(x)}{\sin(2x)}\right) = \lim_{x\to\pi/2}\left(\frac{\cos(x)}{2\sin(x)\cos(x)}\right) = \lim_{x\to\pi/2}\frac{1}{2\sin(x)} = \frac{1}{2\sin(\pi/2)} = \boxed{\tfrac{1}{2}}.$$

(p)

$$\lim_{x\to 0}\frac{\sin(x)}{\tan(x)}$$

Solution to (p) Since $\tan(x)$ equals $\sin(x)/\cos(x)$,

$$\lim_{x\to 0}\frac{\sin(x)}{\tan x} = \lim_{x\to 0}\cos(x) = \cos(0) = \boxed{1}.$$

(q)

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}$$

Solution to (q) By factoring, $x^2-5x+6 = (x-2)(x-3)$ and $x^2-4 = (x-2)(x+2)$. Thus,

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4} = \lim_{x\to 2}\frac{x-3}{x+2} = \frac{2-3}{2+2} = \boxed{-1/4}.$$

(r)

$$\lim_{x\to 3}\frac{x^2-6x+9}{2x-6}$$

Solution to (r) By factoring, $x^2-6x+9 = (x-3)^2$ and $2x-6 = 2(x-3)$. Thus,

$$\lim_{x\to 3}\frac{x^2-6x+9}{2x-6} = \lim_{x\to 3}\frac{x-3}{2} = \frac{3-3}{2} = \boxed{0}.$$

(s)

$$\lim_{x\to\pi/2}\left(\frac{\cos(x)}{\sin(x+\pi/2)}\right)$$

Solution to (s) By the angle addition formulas,

$$\sin(x+\pi/2) = \sin(x)\cos(\pi/2)+\cos(x)\sin(\pi/2) = \sin(x)\cdot 0+\cos(x)\cdot 1 = \cos(x).$$

Thus,

$$\lim_{x\to\pi/2}\left(\frac{\cos(x)}{\sin(x+\pi/2)}\right)=\lim_{x\to\pi/2}\frac{\cos(x)}{\cos(x)}=\boxed{1}.$$

(t)

$$\lim_{x\to 0}\left(\frac{x+1}{x}+1+\frac{x-1}{x}\right)$$

Solution to (t) Clearing denominators gives,

$$\lim_{x\to 0}\left(\frac{x+1}{x}+1+\frac{x-1}{x}\right)=\lim_{x\to 0}\left(\frac{x+1}{x}+\frac{x}{x}+\frac{x-1}{x}\right)=\lim_{x\to 0}\frac{(x+1)+x+(x-1)}{x}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{x+1}{x}+1+\frac{x-1}{x}\right)=\lim_{x\to 0}\frac{3x}{x}=\boxed{3}.$$

(u)

$$\lim_{x\to 0}\left(\frac{1}{\frac{1}{x}-\frac{x^2+1}{x^3}}\right)$$

Solution to (u) Multiplying numerator and denominator by the common factor of x^3 gives,

$$\frac{1}{\frac{1}{x}-\frac{x^2+1}{x^3}}=\frac{x^3}{x^2-(x^2+1)}=-x^3$$

for all $x\neq 0$. Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\frac{1}{x}-\frac{x^2+1}{x^3}}\right)=\lim_{x\to 0}-x^3=\boxed{0}.$$

(v)

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

$$\lim_{x\to-\infty}\left(\sqrt{x^2+1}-x\right)$$

Solution to (v) Using difference of squares,

$$(u-v)(u+v)=u^2-v^2$$

with the substitutions $u = \sqrt{x^2+1}$, $v = x$ yields,

$$\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right) = (x^2+1) - x^2 = 1.$$

Dividing gives,

$$\left(\sqrt{x^2+1} - x\right) = \frac{1}{\sqrt{x^2+1}+x}.$$

Thus

$$\lim_{x\to\infty}\left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = \boxed{0}.$$

On the other hand, clearly,

$$\lim_{x\to-\infty}\left(\sqrt{x^2+1} - x\right) = \infty + \infty = \boxed{\infty}.$$

(w)

$$\lim_{x\to\infty}\left(\sqrt{x^2+1} + x\right)$$

$$\lim_{x\to-\infty}\left(\sqrt{x^2+1} + x\right)$$

Solution to (w) By the same method as above,

$$\lim_{x\to-\infty}\left(\sqrt{x^2+1} + x\right) = \lim_{x\to-\infty}\frac{1}{\sqrt{x^2+1}-x} = \boxed{0}.$$

And also

$$\lim_{x\to\infty}\left(\sqrt{x^2+1} + x\right) = \boxed{\infty}.$$

(x)

$$\lim_{x\to\infty}\frac{e^{x+1}-e^{x-1}}{e^{x+1}+e^{x-1}}$$

Solution to (x) Using the exponent laws,

$$e^{x+1} = e^x \cdot e, \quad e^{x-1} = e^x \cdot e^{-1}.$$

Thus,

$$\frac{e^{x+1}-e^{x-1}}{e^{x+1}+e^{x-1}} = \frac{e^x\cdot e - e^x \cdot e^{-1}}{e^x\cdot e + e^x\cdot e^{-1}} = \frac{e^x(e-e^{-1})}{e^x(e+e^{-1})} = \frac{e-e^{-1}}{e+e^{-1}}.$$

Therefore,

$$\lim_{x\to\infty} \frac{e^{x+1}-e^{x-1}}{e^{x+1}+e^{x-1}} = \lim_{x\to infty} \frac{e-e^{-1}}{e+e^{-1}} = \boxed{(e-e^{-1})/(e+e^{-1})}.$$

(y)

$$\lim_{x\to 0} \frac{1}{\sin(x)}$$

$$\lim_{x\to 0} \frac{1}{\mid \sin(x) \mid}$$

Solution to (y) Of course,

$$\lim_{x\to 0^+} \frac{1}{\sin(x)} = +\infty$$

and

$$\lim_{x\to 0^-} \frac{1}{\sin(x)} = -\infty.$$

Therefore $\lim\limits_{x\to 0}(1/\sin(x))$ is $\boxed{\text{undefined both as a finite number and as } +\infty \text{ or } -\infty}$.

On the other hand, both

$$\lim_{x\to 0^+} \frac{1}{\mid \sin(x) \mid} = +\infty$$

and

$$\lim_{x\to 0^-} \frac{1}{\mid \sin(x) \mid} = +\infty.$$

Therefore $\lim\limits_{x\to 0}(1/\mid \sin(x) \mid)$ equals $\boxed{\infty}$.

(z)

$$\lim_{x\to 0} \ln(x^2)$$

$$\lim_{x\to 0^+} [\ln(x)]^2.$$

Solution to (z) Since $\lim_{x\to 0}(x^2)$ equals 0^+,

$$\lim_{x\to 0} \ln(x^2) = \lim_{y\to 0^+} \ln(y) = \boxed{-\infty}.$$

On the other hand, since

$$\lim_{z \to -\infty} z^2 = \infty,$$

also

$$\lim_{x \to 0^+} [\ln(x)]^2 = \boxed{\infty}.$$

Problem 2 For the following function, state the domain, whether the function is even, odd or neither, and the location and type of any and all discontinuities.

$$f(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x}.$$

Solution to Problem 2 The domain is the union of $[-1/2, 0)$ and $(0, 1/2]$. The function is odd. Note that

$$\lim_{x \to 0} \frac{1 - \sqrt{1 - 4x^2}}{2x} = \lim_{x \to 0} \frac{1^2 - (1 - 4x^2)}{2x(1 + \sqrt{1 - 4x^2})} = \lim_{x \to 0} \frac{4x^2}{2x(1 + \sqrt{1 - 4x^2})} =$$

$$\lim_{x \to 0} \frac{2x}{1 + \sqrt{1 - 4x^2}} = \frac{2 \cdot 0}{1 + \sqrt{1}} = \frac{0}{2} = 0.$$

Thus $f(x)$ has a removable discontinuity at $x = 0$.

Problem 3 For each of the following functions, state the domain of the function, and the location and type of any and all discontinuities.

(a)

$$y = \frac{x}{1 + \cos(x)}$$

Solution to (a) The function is defined except when $1 + \cos(x) = 0$, i.e., when $x = (2n + 1)\pi$ for n an arbitrary integer. Each of these discontinuities is an infinite discontinuity.

(b)

$$y = \frac{x + 2}{x^3 + x^2 - 2x}$$

Solution to (b) Factoring gives $x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x - 1)(x + 2)$. Thus the denominator is 0 for $x = -2$, $x = 0$ and $x = 1$. So $f(x)$ is undefined when $x=-2$, $x=0$ and $x=1$. The discontinuities $x=0$ and $x=1$ are each infinite discontinuities. But since

$$\lim_{x\to -2}\frac{x+2}{x(x-1)(x+2)}=\lim_{x\to -2}\frac{1}{x(x-1)}=\frac{1}{(-2)(-3)}=\frac{1}{6},$$

$x=-2$ is a removable discontinuity.

Problem 4 Find the equations of all tangent lines to the graph of $y=x^2$ which contain the point $(3,5)$. Please note this point is *not* on the graph. You may compute the derivative by any (correct) method you know.
Note. If this review problem is discussed in lecture, we will draw a picture. For a nice Java applet illustrating this problem, scan down to the "Archimedes triangle" section of this webpage on the parabola.

Solution to 4. The derivative of $y=x^2$ at $x=a$ is

$$y'(a)=2a.$$

Thus the equation of the tangent line to $y=x^2$ at (a,a^2) is

$$y-a^2=2a(x-a), \quad y=2ax-a^2.$$

Substituting in $(x,y)=(3,5)$, the point $(3,5)$ lies on the tangent line at (a,a^2) if and only if

$$5=2a(3)-a^2.$$

Rewriting gives

$$a^2-6a+5=0.$$

This factors as $(a-5)(a-1)=0$. Thus $a=1$ or $a=5$. The equations of the corresponding tangent lines are

$$\boxed{y=2x-1} \text{ and } \boxed{y=10x-25}.$$

Problem 5 In each of the following cases, use the definition of the derivative as a limit of a difference quotient to compute the derivative of $y=f(x)$ at the point $x=a$. Then find the equation of the tangent line to the graph of $y=f(x)$ at the point $(a,f(a))$.

(a) $y=\sqrt{x+1}$ at $x=3$

Solution to (a) The difference quotient is

$$\frac{1}{h}(y(3+h)-y(3))=\frac{1}{h}(\sqrt{(3+h)+1}-\sqrt{3+1})=\frac{1}{h}(\sqrt{4+h}-\sqrt{4})=$$

$$\frac{1}{h}\frac{(4+h)-(4)}{\sqrt{4+h}+\sqrt{4}} = \frac{1}{\sqrt{4+h}+\sqrt{4}}$$

for $h \neq 0$. Therefore

$$y'(3) = \lim_{h\to 0}\frac{1}{h}(y(3+h)-y(3)) = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+\sqrt{4}} = \frac{1}{\sqrt{4}+\sqrt{4}} = \frac{1}{2\sqrt{4}} = \boxed{1/4}.$$

So the equation of the tangent line is

$$y-\sqrt{4} = (1/4)(x-3), \quad \boxed{y = (1/4)x + (5/4)}.$$

(b) $y = x + \frac{1}{x}$ at $x = -1$

Solution to (b) The difference quotient is

$$\frac{1}{h}(y(-1+h)-y(-1)) = \frac{1}{h}((-1+h)+\frac{1}{-1+h}-(-2)) = \frac{1}{h}(1+h+\frac{1}{-1+h}).$$

Clearing denominators gives,

$$\frac{1}{h}(1+h+\frac{1}{-1+h}) = \frac{1}{h}\frac{h^2-1}{-1+h}+\frac{1}{-1+h} = \frac{1}{h}\frac{(h^2-1)+1}{-1+h} = \frac{1}{h}\frac{h^2}{-1+h} = \frac{h}{-1+h}.$$

Therefore,

$$y'(-1) = \lim_{h\to 0}\frac{h}{-1+h} = \boxed{0}.$$

So the equation of the tangent line is

$$\boxed{y = -2.}$$

(c) $y = x^3 + x^2$ at $x = -2$

Solution to (c) The difference quotient is

$$\frac{1}{h}(y(2+h)-y(2)) = \frac{1}{h}((2+h)^3+(2+h)^2-12) = \frac{1}{h}((8+12h+6h^2+h^3)+(4+4h+h^2)-12) =$$

$$\frac{1}{h}(16h+7h^2+h^3) = 16+7h+h^2.$$

Therefore,

$$y'(2) = \lim_{h\to 0}(16+7h+h^2) = \boxed{16}.$$

So the equation of the tangent line is

$$y - 12 = 16(x-2), \quad \boxed{y = 16x - 20}.$$

(d) $y = \frac{x+1}{x-1}$ at $x = 0$

Solution to (d) The difference quotient is

$$\frac{1}{h}(y(h)-y(0)) = \frac{1}{h}\left(\frac{h+1}{h-1} - (-1)\right) = \frac{1}{h}\left(\frac{h+1}{h-1} + \frac{h-1}{h-1}\right) = \frac{1}{h}\frac{(h+1)+(h-1)}{h-1} =$$

$$\frac{1}{h}\frac{2h}{h-1} = \frac{2}{h-1}.$$

Therefore,

$$y'(0) = \lim_{h\to 0} \frac{2}{h-1} = \boxed{-2}.$$

So the equation of the tangent line is

$$y + 1 = -2x, \quad \boxed{y = -2x - 1}.$$

Problem 6 Use the definition of the derivative as a limit of a difference quotient to compute the derivative of $y = x^2 + \ln(1)\sin(x)$ at the point $x = 7$.

Solution to Problem 6 Since $\ln(1)$ equals 0, this is the same as $y = x^2$. The difference quotient is

$$\frac{y(7+h)-y(7)}{h} = \frac{(7+h)^2 - 7^2}{h} = \frac{(49+14h+h^2)-49}{h} = \frac{14h+h^2}{h} = 14+h$$

for $h \neq 0$. Thus

$$y'(7) = \lim_{h\to 0} \frac{y(7+h)-y(7)}{h} = \lim_{h\to 0}(14+h) = \boxed{14}.$$

Problem 7 Use the definition of the derivative as a limit of a difference quotient to compute the derivative at $x = 0$ for the following function

$$y = \begin{cases} x^2, & x > 0 \\ 0, & x = 0 \\ -x^2, & x < 0 \end{cases}$$

Note. The derivative *is* defined at this point.

Solution to Problem 7 For $h > 0$, the difference quotient is

$$\frac{y(h) - y(0)}{h} = \frac{h^2 - 0}{h} = h.$$

And for $h < 0$, the difference quotient is

$$\frac{y(h) - y(0)}{h} = \frac{-h^2 - 0}{h} = -h.$$

Thus

$$\lim_{h \to 0^+} \frac{y(h) - y(0)}{h} = \lim_{h \to 0^+} h = 0,$$

and

$$\lim_{h \to 0^-} \frac{y(h) - y(0)}{h} = \lim_{h \to 0^+} (-h) = 0.$$

Since

$$\lim_{h \to 0^+} \frac{y(h) - y(0)}{h} = \lim_{h \to 0^-} \frac{y(h) - y(0)}{h} = 0,$$

also

$$y'(0) = \lim_{h \to 0} \frac{y(h) - y(0)}{h} = \boxed{0}.$$

Problem 8 Determine whether or not the following function is continuous at $x = 0$.

$$y = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Also determine whether or not the derivative of $y = f(x)$ is defined at $x = 0$. If it is defined, compute it. If it is not defined, explain why not.

Solution to Problem 8 The function is squeezed between $+x^2$ and $-x^2$ since $\sin(1/x)$ is trapped between $+1$ and -1. Since

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} (-x^2) = 0,$$

by the Squeeze Theorem,

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$

Since $y(0) = 0$ also, y is continuous at $x = 0$.

Moreover, $(y(h) - y(0))/h = h\sin(1/h)$ for $h \neq 0$. Since this is squeezed between $|h|$ and $-|h|$, and since

$$\lim_{h\to 0} |h| = \lim_{h\to 0}(-|h|) = 0,$$

also the derivative

$$y'(0) = \lim_{h\to 0} h\sin(1/h) = \boxed{0}$$

by the Squeeze Theorem.

Problem 9 In each of the following cases, use the definition of the derivative as a limit of a difference quotient to compute the *derivative function*.

(a)

$$f(x) = \frac{1}{x+3}, \text{ for } x \neq 3, \; f'(x) =?$$

Solution to (a) By definition,

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)+3} - \frac{1}{x+3}\right) = \lim_{h\to 0} \frac{1}{h}\frac{(x+3)-(x+h+3)}{(x+h+3)(x+3)} =$$

$$\lim_{h\to 0} \frac{1}{h}\frac{-h}{(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} = \boxed{-1/(x+3)^2}.$$

(b)

$$g(x) = 2x^2 - 4, \quad g'(x) =?$$

Solution to (b) By definition,

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{(2(x+h)^2-4)-(2x^2-4)}{h} =$$

$$\lim_{h\to 0} \frac{(2(x^2+2xh+h^2)-4)-2x^2+4}{h} = \lim_{h\to 0} \frac{4xh+2h^2}{h} = \lim_{h\to 0}(4x+2h) = \boxed{4x}.$$

(c)

$$f(x) = \sqrt{2x-7}, \quad f'(x) =?$$

Solution to (c) By definition,

$$f'(x) = \lim_{h\to 0} \frac{1}{h}(\sqrt{2x+2h-7} - \sqrt{2x-7}) = \lim_{h\to 0} \frac{1}{h}\frac{(2x+2h-7)-(2x-7)}{\sqrt{2x+2h-7}+\sqrt{2x-7}} =$$

$$\lim_{h\to 0}\frac{2}{\sqrt{2x+2h-7}+\sqrt{2x-7}} = \frac{2}{2\sqrt{2x-7}} = \boxed{1/\sqrt{2x-7}}.$$

(d)

$$i(x) = \frac{1}{x+1} - \frac{1}{x-1}, \quad i'(x) = ?$$

Solution to (d) By definition,

$$i'(x) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h+1} - \frac{1}{x+h-1} - \frac{1}{x+1} + \frac{1}{x-1}\right) =$$

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{x+1}{(x+1)(x+h+1)} - \frac{x-1}{(x+h-1)(x-1)} - \frac{x+h+1}{(x+1)(x+h+1)} + \frac{x+h-1}{(x-1)(x+h-1)}\right) =$$

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{(x+1)-(x+h+1)}{(x+1)(x+h+1)} - \frac{(x-1)-(x+h-1)}{(x-1)(x+h-1)}\right) =$$

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{(x+1)(x+h+1)} - \frac{-h}{(x-1)(x+h-1)}\right) =$$

$$\lim_{h\to 0}\left(\frac{1}{(x-1)(x+h-1)} - \frac{1}{(x+1)(x+h+1)}\right) = \boxed{\tfrac{1}{(x-1)^2} - \tfrac{1}{(x+1)^2}.}$$

Problem 10 Sketch the graph of a function $f(x)$ satisfying all of the following properties.

1. $\lim_{x\to 1^+} f(x) = 1$
2. $\lim_{x\to 1^-} f(x) = 0$
3. $f(1) = 1$
4. $\lim_{x\to -\infty} f(x) = 2$
5. $f(-2) = 4$
6. $\lim_{x\to -1^-} f(x) = -\infty$
7. $\lim_{x\to -1^+} f(x) = \infty$
8. $\lim_{x\to \infty} f(x) = -1$

Problem 11 In each of the following cases, say whether the statement is true or false for an everywhere continuous function $f(x)$ satisfying the stated hypothesis. If the statement is false, sketch a graph demonstrating it is false.

1. If $y = f(x)$ is increasing, then $y = -f(x)$ is increasing. FALSE
2. If $y = f(x)$ is increasing, then $y = -f(x)$ is decreasing. TRUE
3. If $y = f(x)$ is increasing, then $y = f(-x)$ is increasing. FALSE
4. If $y = f(x)$ is increasing, then $y = f(-x)$ is decreasing. TRUE
5. If $y = f(x)$ is even, it cannot be everywhere decreasing. TRUE
6. If $y = f(x)$ is odd, it cannot be everywhere decreasing. FALSE
7. An inverse function $y = f^{-1}(x)$ defined on an interval $[a, b]$ cannot be both increasing on (a, c) and decreasing on (c, b). TRUE