MAT131 Fall 2007 Midterm 2 Review Sheet

The topics tested on Midterm 2 will be among the following.

(i) Using the rules of differentiation: the sum rule, the product rule, the power rule and the derivatives of exponentials.

(ii) Given the limits $\lim_{h\to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{h\to 0} \frac{1-\cos(x)}{x} = 0$, finding the formulas for the derivatives of $\sin(x)$ and $\cos(x)$.

(iii) Finding derivatives of other trigonometric functions – $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$ – using the derivatives for $\sin(x)$ and $\cos(x)$ and the rules for differentiation.

(iv) Finding derivatives using the chain rule.

(v) Finding the tangent slope to a parametric curve at a specified point.

(vi) Finding derivatives using implicit differentiation, including derivatives of inverse functions.

(vii) Finding derivatives using logarithmic differentiation.

(viii) Finding the linear approximation to the value of a function, using a known nearby value and derivative or using differentials.

(ix) Understanding differential notation and the geometric interpretation of differentials. Using differentials to approximate values of functions.

(x) Solving related rates problems.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 1. What is the differential of $y = x^3$? When $x = 2$, a small change $dx = 0.01$ in x produces what change dy in y?

Problem 2. The inverse function of

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

is called the *inverse hyperbolic cosine*, $\cosh^{-1}(x)$. For $y = \cosh^{-1}(x)$, find a formula for the derivative of y that is an expression only involving polynomials in x and square roots.

Hint. After you find an answer that involves y, consider what you get by squaring your formula for y' and squaring your formula for x. Can you relate these two formulas?

Problem 3. The double-angle formula for tangent is

$$\tan(2x) = \frac{2\tan(x)}{1 - (\tan(x))^2}.$$

Compute the derivative of each side of this equation. Which derivative is easier to compute?

Problem 4. Let a be a positive constant and consider the parametric curve

$$\begin{cases} x(t) &= 2at \\ y(t) &= \frac{2a}{1+t^2} \end{cases}$$

Compute the slope of the tangent line at the point where $t = 1/\sqrt{3}$.

Problem 5. Compute each of the following derivatives.

(a)

$$y = \ln(\ln(x)),\ x > 1$$

(b)

$$y = e^{e^x}$$

(c)

$$y = \frac{2x}{1+x^2}$$

(d)

$$y = \frac{x^3\sqrt{\sin(x)}}{\sqrt{\cos(x)}}$$

(e)

$$y = \ln(-x + \sqrt{x^2 - 1})$$

(Simplify your answer as much as possible.)

Problem 6. Using your known formulas for the derivatives of $\sin(x)$ and $\cos(x)$, find the limit

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h}$$

by relating it to the derivative of $\cos(x)$ for some value of x.

Problem 7. The value of $1/(1+\sqrt{10})$ is close to $1/4 = 0.25$. Using a linear approximation or differentials, estimate whether the true value is closer to 0.2 or closer to 0.3.

Problem 8. For the curve with implicit equation

$$y + \frac{1}{y} = 4x + 2x^2$$

find the slope of the tangent line to the curve at the point $(x, y) = (1/2, 2)$.

Problem 9. A calculus instructor of height 170 cm is moving at speed 5.2 km/hr away from a street lamp of height 3m. What is the speed of the shadow of the instructor's head?

Problem 10. A long, straight piece of wood of length 7 meters rests with its foot on the ground and its midpoint on the top of a 3 meter tall fence. The foot of the plank begins to slide straight away from the fence with the plank still touching the top of the fence. At the moment when the foot of the plank is 4 meters from the base of the fence, the distance between the top of the plank and the ground is decreasing at 0.3 meters per second. At what speed is the foot of the plank moving away from the base of the fence?

Problem 11. Find the derivative y' of the function y in each of the following cases.

(a)
$$y = \sin(\sqrt{x^2+1})$$

(b)
$$y = (\ln(x^2))^2$$

(c)
$$y = (\sqrt{x})^{\cos(x)}$$

(d)
$$y = \frac{x^2 e^x}{x^2+2x+1}$$

(e)
$$x^2 + 4y^2 = 5$$

(f)
$$y = \frac{(x^2+1)^3}{x^2-1}$$

Problem 12 In each of the following two cases, find the linearization of $f(x)$ near the point $x = a$.

(a)
$$f(x) = x^{2/f}, \quad a = 32.$$

(b)
$$f(x) = \frac{1}{\sqrt{1+x^2}}, \quad a = \sqrt{3}.$$

Problem 13 Using differentials or an appropriate linear approximation, approximate the following number.

$$\frac{1}{\sqrt{25.1}}$$

Problem 14 In a flat ceiling two hooks are fastened 21 cm apart. A length of 27 cm of inextensible wire is suspended between the hooks. A heavy weight is hung from the wire near the first hook and slides along the wire toward a point equidistant from both hooks, pulling the wire taut at each moment. At the moment when the weight is 10 cm from the first hook, it is moving away from the first hook at a speed of 1 cm/sec.

(a) With what speed is the weight moving towards the second hook at this moment?

(b) What is distance between the weight and the ceiling at this moment?

(c) With what speed is the weight moving away from the ceiling (i.e., what is the rate of change of the distance between the weight and the ceiling)?

Problem 15 A car approaches a large hotel at night, driving along a semi-circular driveway which becomes tangent to the front wall of the hotel at the entrance. The headlights of the car illuminate a spot on the front wall. If the radius of the driveway is 10 meters, and if the car is moving at a speed of 10 km/hour, with what speed is the spot moving when the angle between the entrance, the center of the circle and the car is $\pi/3$ radians, i.e., 60 degrees?

Problem 16 A cube of ice rests on a hot plate. It melts in such a way that its shape at every moment is a cube and the rate of decrease of the volume is a constant multiple of the area of the face resting on the hot plate. If after 5 minutes the volume of the cube is one eighth of its initial volume, how much longer is required before the cube melts entirely?