I. DEFINITION. A quasi-coherent sheaf is locally finitely presented if there is a covering by opens on which the sheaf is a "tilde sheaf" of a finitely presented module.

Lemma. A quasi-coherent sheaf on an affine scheme is locally finitely presented if and only if the global sections module is finitely pres'd.

Lemma. The pullback of a locally finitely presented quasi-coherent sheaf is a locally finitely presented quasi-coherent sheaf.

Notation. A quasi-coherent sheaf on a Noetherian scheme is coherent if it is locally finitely presented (this is not quite correct on complex analytic spaces).

Devissage Lemma/Theorem (source "Stacks Project") Let $\mathcal{P}$ be a property of coherent sheaves on a Noetherian scheme X. Then $\mathcal{P}$ holds for all coherent sheaves if and only if both of the following hold

(1) for every short exact sequence where two of three have $\mathcal{P}$, so does the third, and

(2) for every integral closed subscheme Z, $\mathcal{P}$ holds for some $\mathcal{F}$ supported on Z that is generically isomorphic to $\mathcal{O}_Z$.

Application. For R a ring, for $S = \bigoplus_{n\geq 0} S_n$ a graded R-algebra generated by S_1 over $R = S_0$, recall the functor $\Gamma_* : QCoh_{\mathrm{Proj}\,S} \to \mathbb{Z}_{\geq 0}$-graded S-modules by $\Gamma_*(\mathcal{F}) := \bigoplus_{n\geq 0} \Gamma(\mathrm{Proj}\,S, \mathcal{F}(n))$. Together with $(\cdot)^\sim : \mathbb{Z}_{\geq 0}\text{-gr } S\text{-mod} \to QCoh_{\mathrm{Proj}\,S}$, this gives an equivalence of categories of $QCoh_{\mathrm{Proj}\,S}$ with the localization of $\mathbb{Z}_{\geq 0}$-gr S-mod at the system of morphisms that are "isomorphisms in high degree".

Theorem. If R is "excellent" (e.g. a finite type algebra over a field or over $\mathbb{Z}$), then Γ_* of each coherent sheaf is a finitely presented S-module.

Proof. Consider the class of closed subsets of Proj S such that for every coherent sheaf supported on that set, Γ_* is finitely presented. For a reducible closed subset $C = C' \cup C''$, the quotient by the subsheaf of sections supported on C' is a sheaf supported on C''. Thus by Noetherian induction it suffices to prove this class contains each irreducible closed subset C whose proper closed subsets, $D \subsetneq C$, are all in the class. Now let $\mathcal{P}$ be the property on coherent sheaves $\mathcal{F}$ supported on C that for every coherent subquotient, i.e., quotient $\mathcal{O}$-module of an $\mathcal{O}$-submodule of $\mathcal{F}$, Γ_* is finitely generated. For every short exact sequence, every subquotient of the first term, resp. the last term, is a subquotient of the middle term. Thus, if the middle term has $\mathcal{P}$, so do the other two terms. Conversely, every subquotient of the middle term has a subsheaf that is a subquotient of the first term and whose quotient is a subquotient of the last term. By left exactness of Γ_*, if the first and last terms have $\mathcal{P}$, so does the middle term. Thus $\mathcal{P}$ satisfies (1). By devissage, to show that $\mathcal{P}$ holds for all coherent sheaves supported on C, it suffices to show $\mathcal{P}$ holds for one coherent sheaf whose scheme-theoretic support equals C with its reduced structure, e.g., $\mathcal{F} = \mathcal{O}_C$. Every subsheaf is an ideal sheaf $\mathcal{J}$, and every quotient sheaf of $\mathcal{J}$ that is not an isomorphism is supported on a proper closed subset of C. By hypothesis, such a sheaf has finitely presented Γ_*. Thus, it suffices to prove that $\Gamma_*(\mathcal{J})$ is finitely presented. As an S-submodule of $\Gamma_*(\mathcal{O}_C)$, ~~thus~~ it suffices to prove that $\Gamma_*(\mathcal{O}_C)$ is finitely presented. Of course the kernel ideal of $S \to \Gamma_*(\mathcal{O}_C)$ is finitely presented and the quotient ring is a finitely presented S-module that is a $\mathbb{Z}_{\geq 0}$-graded integral domain. The S-module $\Gamma_*(\mathcal{O}_C)$ is "between" this quotient ring and its integral closure. Since R is excellent, the integral closure is finite over the quotient ring. $\square$

<u>Remark</u>. Later we will have a different proof that does not require that R is an excellent Noetherian ring.

Corollary. For every (weakly) projective morphism to an (excellent) Noetherian scheme, the pushforward of every coherent sheaf is coherent.

Corollary. The same holds for every proper morphism.

Proof. The same method as above. However, to prove the pushforward of some $\mathcal{F}$ is coherent where $\mathcal{F}$ is generically isomorphic to $\mathcal{O}_C$, use Chow's Lemma to find a birational, strongly projective morphism, $\nu : \widetilde{C} \to C$, whose composition with the proper morphism is also projective. Let $\mathcal{F}$ be $\nu_* \mathcal{O}_{\widetilde{C}}$. $\square$

The corollary is also equivalent to the theorem via a diagram chase that is important in its own right. For $X = \operatorname{Proj} S$ with its morphism $\pi : X \to \operatorname{Spec} R$ and the invertible quotient $q : \pi^* \widetilde{S}_1 \twoheadrightarrow \mathcal{O}(1)$, form $\varepsilon : \mathbb{E} \to X$ and its universal morphism of quasi-coherent sheaves, $r : \varepsilon^* \mathcal{O}(1) \to \mathcal{O}_{\mathbb{E}}$, where $\mathbb{E} = \operatorname{Spec}_X(\operatorname{Sym}^{\bullet}_{\mathcal{O}_X} \mathcal{O}(1))$. The morphism $r \circ \varepsilon^* q : (\pi \circ \varepsilon)^* \widetilde{S}_1 \to \mathcal{O}_{\mathbb{E}}$ defines a morphism from $\mathbb{E}$ to $Y := \operatorname{Spec} S$, say $\lambda : \mathbb{E} \to \operatorname{Spec} S$. Chasing universal properties, the morphism λ with its surjective morphism of quasi-coherent sheaves, $\varepsilon^* q : \lambda^* \widetilde{S}_1 \twoheadrightarrow \mathcal{O}(1)$, is the relative Proj over Y of the <u>Rees algebra</u> of the ideal sheaf $\mathcal{I}$ of the vertex $\mathfrak{z} : \operatorname{Spec} R \overset{\text{closed}}{\hookrightarrow} Y$, i.e., $\mathcal{I} = (\bigoplus_{n \geq 1} S_n)^{\sim}$ on $\operatorname{Spec} S$. The Rees algebra of an ideal sheaf $\mathcal{I}$ on a scheme Y is the $\mathbb{Z}_{\geq 0}$-graded quasi-coherent $\mathcal{O}_Y$-algebra, $R_Y(\mathcal{I}) := \bigoplus_{n \geq 0} \mathcal{I}^n \cdot \text{"}t^n\text{"}$, with its obvious multiplication. The relative Proj of $R_Y(\mathcal{I})$ is the <u>blowing up</u> (nowadays called the <u>blowup</u>) of the ideal sheaf $\mathcal{I}$ in Y. Since λ is proper, the corollary implies that for every coherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S$, the pushforward by λ of $\varepsilon^* \mathcal{F}$ is coherent, and thus the global sections module is a finitely presented S-module. Of course this module is $\Gamma_*(\mathcal{F})$.

<u>Ample invertible sheaves</u>. Let B be a scheme, and let $\pi: X \to B$ be a quasi-compact, quasi-separated morphism. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and let $\varepsilon: \mathbb{E} \to X$ and $r: \varepsilon^*\mathcal{L} \to \mathcal{O}_{\mathbb{E}}$ be a universal morphism, i.e., $\mathbb{E} = \mathrm{Spec}_X(\mathrm{Sym}^\bullet_{\mathcal{O}_X} \mathcal{L})$. Denote by $\mathbb{E}^* \subseteq \mathbb{E}$ the maximal open on which r is an isomorphism, i.e., the complement of the "vertex" $\mathfrak{z}: X \to \mathbb{E}$ where $\mathfrak{z}^* r$ equals the zero morphism.

<u>Definition</u>. The invertible $\mathcal{O}_X$-module $\mathcal{L}$ is ~~the~~ <u>π-relatively ample</u> if for every locally finitely presented, quasi-coherent $\mathcal{O}_X$-module, for the ε-pullback $\mathcal{F}$ of that $\mathcal{O}_X$-module, the natural morphism, $(\pi\circ\varepsilon)^*(\pi\circ\varepsilon)_*\mathcal{F} \to \mathcal{F}$, is surjective on $\mathbb{E}^*$.

<u>Equivalent formulation</u>. For every ~~quasi-compact~~ affine open $B^\circ \subseteq B$, for every loc. f. pres'd, q-coh $\mathcal{O}_X$-module $\mathcal{E}$ on $X^\circ := \pi^{-1}(B^\circ)$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{L}^{\otimes n})$ is globally generated.

<u>Main Example</u>. The previous argument proves that the Serre twisting sheaf $\mathcal{O}(1)$ on $X = \mathrm{Proj}\, S$ is π-relatively ample. Since the restriction of a relatively ample invertible sheaf on a locally closed subscheme (quasi-compact, of course) is also relatively ample, also $\mathcal{O}(1)$ is relatively ample on each quasi-projective open subscheme of $\mathrm{Proj}\, S$.

<u>Cartier Divisors</u>. Recall that an <u>effective Cartier divisor</u> on a scheme X is an injective $\mathcal{O}_X$-module homomorphism to an invertible $\mathcal{O}_X$-module $\mathcal{L}$ from $\mathcal{O}_X$, $s: \mathcal{O}_X \to \mathcal{L}$, up to the equivalence, $(\mathcal{O}_X \xrightarrow{s} \mathcal{L}) \sim (\mathcal{O}_X \xrightarrow{s'} \mathcal{L}')$ if there exists an $\mathcal{O}_X$-mod isomorphism $i: \mathcal{L} \to \mathcal{L}'$ with $i \circ s = s'$. The set of effective Cartier divisors has a monoid structure via $((\mathcal{O}_X \xrightarrow{s_1} \mathcal{L}_1), (\mathcal{O}_X \xrightarrow{s_2} \mathcal{L}_2)) \mapsto (\mathcal{O}_X \xrightarrow{s_1 \otimes s_2} \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2)$. The group associated to this monoid has a natural description in terms of the sheaf $\mathcal{K}_X$ of "total rings of fractions" on X,

i.e., the sheaf associated to the presheaf of $\mathcal{O}_X$-algebras associating to $\mathcal{U}$ the fraction ring of $\mathcal{O}_X(\mathcal{U})$ at the multiplicatively closed subset of ~~[illegible]~~ injective $\mathcal{O}_{\mathcal{U}}$-Module homomorphisms $\mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}$. In particular, a morphism between invertible $\mathcal{O}_X$-modules is injective if and only if it becomes an isomorphism after applying the functor $\mathcal{K}_X \otimes_{\mathcal{O}_X} -$.

<u>Definition.</u> A <u>Cartier divisor</u> on X is an invertible $\mathcal{O}_X$-submodule of $\mathcal{K}_X$. It is effective if it contains the invertible $\mathcal{O}_X$-submodule $\mathcal{O}_X \hookrightarrow \mathcal{K}_X$ (morphism of $\mathcal{O}_X$-algebras).

Since the multiplication map, $\mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{K}_X \to \mathcal{K}_X$, is an isomorphism of $\mathcal{O}_X$-modules, the monoid multiplication on effective Cartier divisors extends to an (Abelian) group multiplication on the set of all Cartier divisors. By definition, there is a group homomorphism from the group of Cartier divisors to the Picard group of X (group of isomorphism classes of invertible sheaves). The kernel equals the group of global sections of the sheaf $\mathcal{K}_X^*$ of multiplicatively invertible sections of $\mathcal{K}_X$. The image equals the subgroup of $\mathrm{Pic}(X)$ of those invertible sheaves $\mathcal{L}$ that have an invertible rational section, $\Gamma(\mathcal{L}^*)$. This is all of $\mathrm{Pic}(X)$ if either X is quasi-projective over a field or X is normal ~~locally factorial (e.g., regular)~~. The group of Cartier divisors is $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, the kernel is $\Gamma(X, \mathcal{K}_X^*)$, and the image is the kernel of $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{K}_X^*)$.

<u>Hypothesis.</u> Let X be a quasi-compact, separated scheme whose local rings are integrally closed integral domains (i.e., X is <u>normal</u>).

<u>Definition.</u> A <u>prime Weil divisor</u> on X is an integral closed subscheme whose prime ideal has height one (in each stalk of $\mathcal{O}_X$ at a point of the closed subsch.

Under the hypotheses, $\mathcal{K}_X$ is a flasque quasi-coherent sheaf, so that $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to \mathrm{Pic}(X)$ is surjective. For each prime Weil divisor $D \subseteq X$ with generic point $\iota_D : \mathrm{Spec}\, \kappa_D \to \{\eta_D\} \subseteq D$, the local ring $\mathcal{O}_{X,\eta_D}$ is a DVR (with residue field $\kappa_D := \mathcal{O}_{X,\eta_D}/\mathfrak{m}_{X,\eta_D}$). There is a discrete valuation, $v_D : \mathcal{K}_X^*/\mathcal{O}_X^* \to \iota_{D*}(\mathbb{Z}) \cdot \text{"}\underline{D}\text{"}$, sending each generator of $\mathfrak{m}_{X,\eta_D}$ to $1 \in \mathbb{Z}$.

Definition. A Weil divisor on X is a global section of the flasque sheaf of Abelian groups $\bigoplus_{\text{prime } D} \iota_{D*}(\mathbb{Z}) \cdot \text{"}\underline{D}\text{"}$. The associated Weil divisor of a Cartier divisor is the image of the global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ under the morphism of sheaves of Abelian groups, $(v_D)_D : \mathcal{K}_X^*/\mathcal{O}_X^* \to \bigoplus_{\text{prime } D} \iota_{D*}(\mathbb{Z}) \cdot \text{"}\underline{D}\text{"}$.

The hypotheses imply that $\mathcal{K}_X^*/\mathcal{O}_X^* \to \bigoplus_D \iota_{D*}(\mathbb{Z}) \cdot \text{"}\underline{D}\text{"}$ is injective. It is an isomorphism if and only if every stalk of $\mathcal{O}_X$ is a unique factorization ring. It is a theorem that a regular local ring is a unique factorization ring.

Definition. The Weil class group equals the quotient of the group of Weil divisors, $H^0(X, \bigoplus_D \iota_{D*}(\mathbb{Z}))$, by the image of the group of principal divisors, $H^0(X, \mathcal{K}_X^*)/H^0(X, \mathcal{O}_X^*)$.

The Picard group is contravariant for all morphisms. The group of Cartier divisors is contravariant for dominant morphisms between integral schemes. The group of Weil divisors is (covariant and) contravariant for finite surjective morphisms between normal, integral schemes. The pushforward of the pullback, in this case, is multiplication by the degree of the morphism. If both schemes are locally factorial, this defines pushforward on Picard groups (the "determinant of pushforward").