

Homotopy rigidity of nearby Lagrangian cocores

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joint work with Yash Deshmukh and Alex Pieloch. (arxiv:2511.09548)

Setup
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Main result
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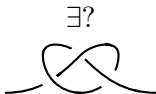
Sketch of proof
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Setup

Motivation

Problem (Arnol'd)

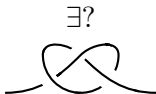
Is there a Lagrangian embedding $\mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ (standard at infinity) that is not smoothly isotopic (rel boundary) to the standard embedding?



Motivation

Problem (Arnol'd)

Is there a Lagrangian embedding $\mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ (standard at infinity) that is not smoothly isotopic (rel boundary) to the standard embedding?



Theorem (Eliashberg–Polterovich 1996)

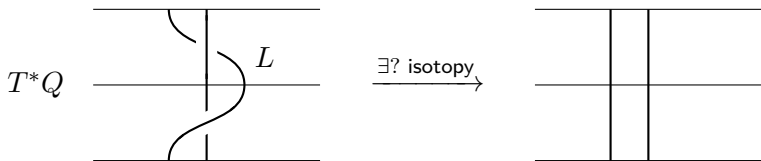
Any Lagrangian embedding $\mathbb{R}^2 \rightarrow \mathbb{R}^4$ (standard at infinity) is Hamiltonian isotopic to the standard embedding.

(The answer to the problem is no for $n \geq 4$ by an h -principle, and it is open for $n = 3$.)

Cotangent bundle version

Problem

Let $L \subset T^*Q$ be an exact Lagrangian embedding that coincides with a cotangent fiber T_q^*Q at infinity and such that its image is disjoint from some other cotangent fiber. Is L smoothly isotopic (rel boundary) to a cotangent fiber in the complement of the missed fiber?

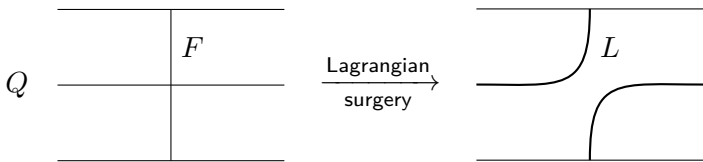


We call such L a “nearby Lagrangian cocore.”

Cotangent bundle version

Claim

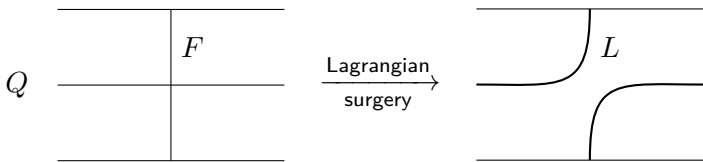
Without the assumption that L is disjoint from some cotangent fiber, the answer is no.



Cotangent bundle version

Claim

Without the assumption that L is disjoint from some cotangent fiber, the answer is no.



Proof.

If L is smoothly isotopic to a cotangent fiber, the composition $L \rightarrow T^*Q \xrightarrow{\pi} Q$ is null-homotopic. However, the degree of the projection $\pi|_L: L \rightarrow Q$ is equal to 1. □

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Sketch of proof
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Main result

Main result

Theorem A (A.–Deshmukh–Pieloch)

Let $k \geq 1$, and let $n \geq 2k + 2$. Let Q^n be one of the following smooth manifolds:

- $\mathbb{C}P^2$, $S^5 \times S^5 \times \mathbb{R}^k$, $S^5 \times S^5 \times S^1$, or
- $M^{n-k} \times N^k$ where N^k is a k -dimensional $K(\pi, 1)$ manifold, and M is an $(n - k)$ -dimensional manifold such that $\pi_2(M) = 0$ and either
 1. $\pi_n(M) = 0$, or
 2. $\widetilde{M} \simeq \bigvee_i S^{n-k_i}$ where $k_i \leq k$.

Then any nearby Lagrangian cocore in T^*Q is smoothly isotopic (rel boundary) to a cotangent fiber in the complement of the missed fiber.

Main result

Examples of manifolds M^{n-k} allowed in the theorem include:

- \mathbb{R}^{n-k} , S^{n-k} , $\mathbb{R}P^{n-k}$, $\mathbb{O}P^2$
- connected sums of $S^{n-k-1} \times S^1$
- high dimensional lens spaces

Main result

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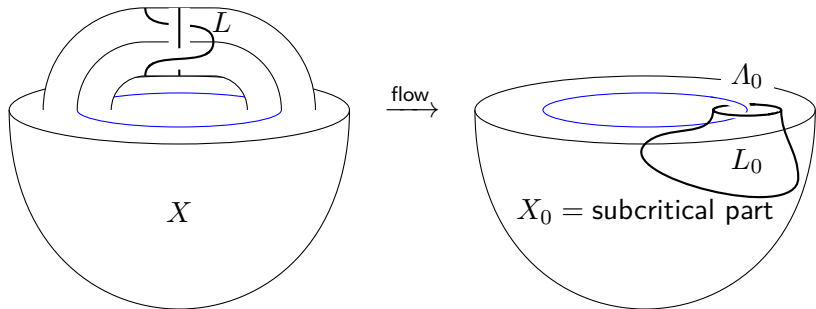
Remark

There is a version of the problem for general Weinstein sectors, and our theorem also answers the problem in the affirmative for any plumbing of copies of $T^*((S^{n-k} \times \mathbb{R}^k)^{\#i} \# (\Sigma^n)^{\#j})$ where Σ^n is a homotopy n -sphere.

Previous results

- Ekholm–Smith (2018) proved this result in the case of $Q = \mathbb{R}^n$ for $n \geq 4$.
- Côté–Dimitroglou Rizell (2022) proved that a nearby Lagrangian cocore in $T^*(\text{Riemann surface})$ is *Hamiltonian isotopic* to a cotangent fiber.
- The cases of (exact) Lagrangian embeddings $\mathbb{R}^3 \rightarrow \mathbb{R}^6$ and nearby Lagrangian cocores in $T^*\mathbb{R}^3$ are still open.

Unknot fillings



- $\Lambda_0 \subset \partial_\infty X_0$ is a Legendrian unknot with exact Lagrangian filling $L_0 \subset X_0$.
- Main result (for those X it applies to) \Rightarrow Unique exact Lagrangian filling up to smooth isotopy.

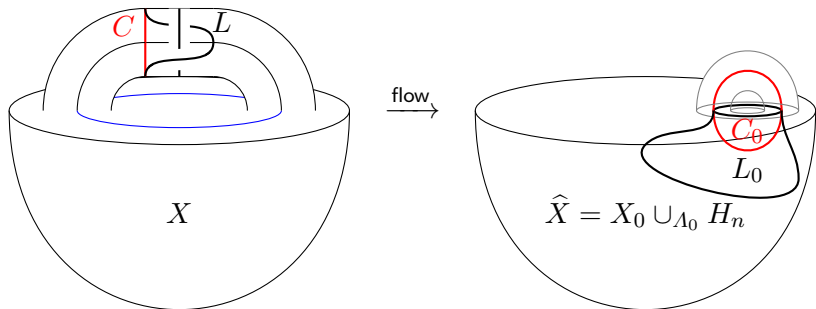
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Sketch of proof
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Sketch of proof

Compare fillings



Attaching a Weinstein n -handle to Λ_0 gives two exact Lagrangian spheres $\hat{C}, \hat{L} \subset \hat{X}$.

Strategy

1. Prove $\widehat{C} \cong \widehat{L}$ in $\mathcal{W}(\widehat{X}; MO\langle k+1 \rangle)$.
2. Show that it implies for the various Q in the main theorem that the projection

$$\pi|_L: L/\partial_\infty L \longrightarrow Q \setminus \{\text{pt}\}$$

is null-homotopic.

3. This implies that the maps $L \rightarrow T^*Q \setminus F$ and $C \rightarrow T^*Q \setminus F$ are homotopic maps (rel boundary).
4. An h -principle due to Haefliger (1961) implies that they are smoothly isotopic (rel boundary).

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Remark

Generally the composition

$$L/\partial_\infty L \xrightarrow{\pi|_L} \text{Core } X_0 \xrightarrow{\text{quotient}} \text{Core } X_0 / (\text{Core } X_0)_{n-k-1}$$

is null-homotopic.

Step 2 in $T^*\mathbb{C}P^2$

1. We show $\widehat{C} \cong \widehat{L}$ in $\mathcal{W}(\widehat{X}; MSpin)$.
2. The projection is $\pi: S^4 \simeq L/\partial_\infty L \rightarrow \mathbb{C}P^2 \setminus \{\text{pt}\} \simeq S^2$
3. Spectral open-closed map implies $\pi_*[L] = \pi_*[C] = 0$ in $\widetilde{H}_4(S^2; MSpin) \cong \pi_2(MSpin) \cong \pi_2^{\text{st}}$.
4. Early stabilization phenomenon: $\pi_2^{\text{st}} \cong \pi_4(S^2)$, and the class $\pi_*[L]$ corresponds to the homotopy class of π .
5. Hence π is null-homotopic.

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The general step 2 relies on the fact that there are isomorphisms

$$\widetilde{H}_n(S^{n-k}; MO\langle k+1 \rangle) \cong \pi_k(MO\langle k+1 \rangle) \cong \pi_k^{\text{st}} \cong \pi_n(S^{n-k})$$

for $k \geq 1$ and $n \geq 2k + 2$.

Thank you!