MAT 310 L24 Thu May 1

Isometries (7D)

Def: A linear map $T \in \mathcal{L}(V,W)$ is called an isometry if

$$\|Tv\| = \|v\| \quad \forall v \in V.$$

Prop: Any isometry is injective.

Proof: Let $T \in \mathcal{L}(V,W)$ and let $v \in \operatorname{null} T$. I.e. $Tv = 0$. Then

$$\|v\| = \|Tv\| = 0 \Rightarrow v = 0,$$

so $\operatorname{null} T = \{0\}$. $\square$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (y,x)$ is an isometry. Namely:

$$\|T(x,y)\| = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2} = \|(x,y)\|.$$

Isometries are those linear maps that preserve the norm. It preserves length of vectors.

<u>Thm</u>: Let $T \in \mathcal{L}(V,W)$. Let $(e_1, \ldots, e_n)$ be an ON basis for V. The following are equiv:

(a) T is an isometry

(b) $T^*T = I$

(c) $\langle Tu, Tv \rangle = \langle u, v \rangle$ for all $u, v \in V$

(d) $(Te_1, \ldots, Te_n)$ is an ON list in W.

<u>Proof</u>: We will prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

<u>(a) $\Rightarrow$ (b)</u>: We will show $(I - T^*T)v = 0$

In fact by (7.16) its enough to prove that $I - T^*T$ is self-adjoint, and

$\langle (I-T^*T)v, v\rangle = 0$ for all $v \in V$. First

$$\langle (I-T^*T)v, w\rangle = \langle v, w\rangle - \langle T^*Tv, w\rangle$$

$$= \langle v, w\rangle - \langle Tv, Tw\rangle = \langle v, w\rangle - \langle v, T^*Tw\rangle$$

$= \langle v, (I-T^*Tw)\rangle$ So $I-T^*T$ is self-adjoint. Next

$$\langle (I-T^*T)v, v\rangle = \langle v, v\rangle - \langle T^*Tv, v\rangle$$

$$= \|v\|^2 - \langle Tv, Tv\rangle = \|v\|^2 - \|Tv\|^2 = 0.$$

<u>(b) ⇒ (c)</u>: For any $u, v \in V$ we have

$$\langle Tu, Tv\rangle = \langle u, \underbrace{T^*T}_{=I}v\rangle = \langle u, v\rangle.$$

<u>(c) ⇒ (d)</u>: By assumption $(e_1, \ldots, e_n)$ is ON. So

$$\langle Te_i, Te_j\rangle = \langle e_i, e_j\rangle = \begin{cases} 1 & i=j \\ 0 & \text{else.} \end{cases}$$

meaning $(Te_1, \ldots, Te_n)$ is ON.

<u>(d) ⇒ (a)</u>: Since $(e_1, \ldots, e_n)$ is an ON basis for V, if $v \in V$, then

$$v = \langle v,e_1\rangle e_1 + \cdots + \langle v,e_n\rangle e_n$$

$$Tv = \langle v,e_1\rangle Te_1 + \cdots + \langle v,e_n\rangle Te_n.$$

and $\|v\|^2 = |\langle v,e_1\rangle|^2 + \cdots + |\langle v,e_n\rangle|^2$.
Since $(Te_1,\ldots,Te_n)$ is ON we also get $\|Tv\|^2 = |\langle v,e_1\rangle|^2 + \cdots + |\langle v,e_n\rangle|^2$. $\square$

<u>Def.</u> An operator $T \in \mathcal{L}(V)$ is called <u>unitary</u> if T is an invertible isometry.

We already saw above that isometries are injective. So an isometry is invertible exactly when it's surjective.
In fact, if V is fin dim, any isometry is invertible! So isometries that are not invertible can only

occur in infinite dimensional inner product spaces.

<u>Ex</u>: Let $\theta \in \mathbb{R}$ and def $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_\theta(x,y) = ((\cos\theta)x - (\sin\theta)y, (\sin\theta)x + (\cos\theta)y)$$

The matrix wrt standard basis in $\mathbb{R}^2$ is

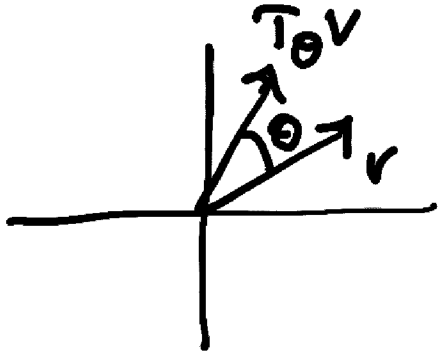
$$\mathcal{M}(T_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Wrt dot product, we can calculate

$$\|T_\theta(x,y)\| = \sqrt{(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2}$$

$$= \sqrt{x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{x^2+y^2} = \|(x,y)\|$$ so T_θ is an isometry. The lin op T_θ is rotation CCW by angle θ in the plane.



<u>Thm</u>: Let $T \in \mathcal{L}(V)$. If $(e_1, \dots, e_n)$ is an ON basis for V, the following are equivalent.

(a) T is unitary
(b) $T^*T = TT^* = I$
(c) T is invertible and $T^{-1} = T^*$
(d) $(Te_1, \dots, Te_n)$ is an ON basis of V
(e) T^* is unitary.

<u>Proof</u>: <u>(a) $\Rightarrow$ (b)</u>:

T is unitary, so its an isometry & we know $T^*T = I$. Since T is invertible we have $T^* = T^{-1}$, so

$TT^{-1} = T^{-1}T = I \Leftrightarrow T^*T = TT^* = I$.

<u>(b) $\Rightarrow$ (c)</u>: Clear from definitions

(c) $\Rightarrow$ (d) We have $T^*T = I$, so

$$\langle Te_i, Te_j\rangle = \langle e_i, T^*Te_j\rangle = \langle e_i, e_j\rangle = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

so $(Te_1, \dots, Te_n)$ is an ON list. Its a basis since it has length $\dim V = n$.

(d) $\Rightarrow$ (e): Since $(Te_1, \dots, Te_n)$ is an ON basis for V, we can write

$$v = \langle v, Te_1\rangle Te_1 + \dots + \langle v, Te_n\rangle Te_n.$$

$$T^*v = \langle T^*v, e_1\rangle e_1 + \dots + \langle T^*v, e_n\rangle e_n$$

so $\|v\|^2 = |\langle v, Te_1\rangle|^2 + \dots + |\langle v, Te_n\rangle|^2$

$= |\langle T^*v, e_1\rangle|^2 + \dots + |\langle T^*v, e_n\rangle|^2$

$= \|T^*v\|^2 \Rightarrow T^*$ isometry, and since $\dim V < \infty$, it's automatically invertible.

(e) $\Rightarrow$ (a): If T^* is unitary, then by the chain of implications

(a) $\Rightarrow \cdots \Rightarrow$ (e), $(T^*)^* = T$ is unitary. $\square$

Prop: If $T \in \mathcal{L}(V)$ is unitary, and $\lambda \in \mathbb{F}$ is an eigenvalue, then $|\lambda| = 1$.

Proof: T is in particular isometry, so if $Tv = \lambda v$ for $0 \neq v \in V$,

$$\|Tv\| = \|\lambda v\| = |\lambda| \, \|v\| = \|v\|$$

$$\Leftrightarrow |\lambda| = 1. \qquad \square$$
