Recall: • If $U \subset V$ is a subspace of an inner product sp V, the orthogonal complement of U is

$$U^\perp := \{v \in V \mid \langle u, v\rangle = 0 \ \forall u \in U\}.$$

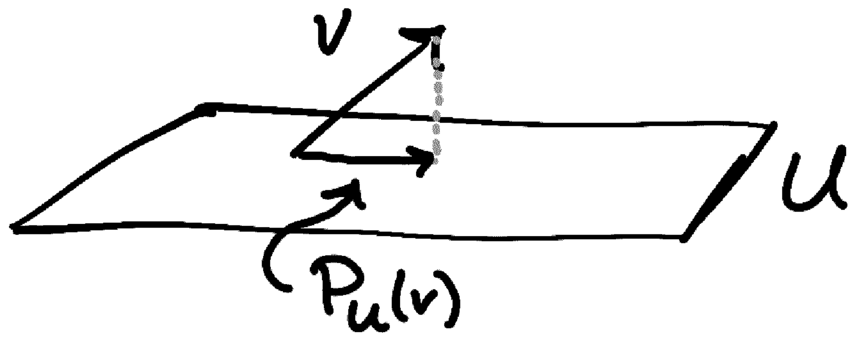
• If U is fin dim,

$$V = U \oplus U^\perp.$$

Def: Let $U \subset V$ be fin dim. The orthogonal projection onto U is the linear map $P_U \in \mathcal{L}(V)$ def as follows: Any $v \in V$ can be written $v = u + w$ uniquely since $V = U \oplus U^\perp$.

$$\boxed{P_U(v) = u.}$$

Ex: (1)



② If $\mathcal{U} = \operatorname{span}(u) \subset V$ is 1-dim, then orthogonal projection onto $\mathcal{U}$ has a nice formula:

If $v \in V$, then

$$v = \underbrace{\frac{\langle v,u\rangle}{\|u\|^2} u}_{\text{in } \mathcal{U}} + \underbrace{\left(v - \frac{\langle v,u\rangle}{\|u\|^2} u\right)}_{\text{in } \mathcal{U}^\perp}$$

so $P_{\mathcal{U}}(v) = \frac{\langle v,u\rangle}{\|u\|^2} u.$

<u>Prop</u>: Let $\mathcal{U} \subset V$ be fin dim.

(a) $P_{\mathcal{U}} \in \mathcal{L}(V)$

(b) $P_{\mathcal{U}}(u) = u \quad \forall u \in \mathcal{U}$

(c) $P_{\mathcal{U}}(w) = 0 \quad \forall w \in \mathcal{U}^\perp$

(d) range $P_{\mathcal{U}} = \mathcal{U}$

(e) null $P_{\mathcal{U}} = \mathcal{U}^\perp$

(f) $v - P_{\mathcal{U}}(v) \in \mathcal{U}^\perp \quad \forall v \in V$

(g) $P_{\mathcal{U}}^2 = P_{\mathcal{U}}$

(h) $\|P_{\mathcal{U}} v\| \leq \|v\| \quad \forall v \in V$

(i) If $(e_1, \ldots, e_m)$ is an orthonormal basis of U and $v \in V$, then
$$P_U v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m.$$

<u>Proof</u>: (a) Exercise

(b) If $u \in U$ then $u = \underset{\text{in } U}{u} + \underset{\text{in } U^\perp}{0}$

So $P_U u = u$.

(c) If $w \in U^\perp$, $w = \underset{\text{in } U}{0} + \underset{\text{in } U^\perp}{w}$

So $P_U w = 0$.

(d) $\operatorname{range} P_U = \{v \in V \mid v = P_U w \text{ for some } w \in V\}$

Note $U \subset \operatorname{range} P_U$ since if $u \in U$ then $u = P_U u$.

$\operatorname{range} P_U \subset U$ is by def.

(e) $\operatorname{null} P_U = \{v \in V \mid P_U v = 0\}$.

By (c) $U^\perp \subset \operatorname{null} P_U$, and if

$P_U v = 0$ it means $v = 0 + v$ was the direct sum decomposition of v so $v \in U^\perp$.

(f) The direct sum decomp of v is $v = \underbrace{P_U v}_{\text{in } U} + (\underbrace{v - P_U v}_{\text{in } U^\perp})$

(g) By def $P_U v \in U$ & $P_U u = u$ for $u \in U$ so $P_U^2 v = P_U(P_U v) = P_U v$ $\forall v \in V$.

(h) $\|v\|^2 = \|P_U v + (v - P_U v)\|^2$

$\underset{\text{Pythagorean thm}}{=} \|P_U v\|^2 + \|v - P_U v\|^2 \geq \|P_U v\|^2 + 0$

(i) This is a consequence of the Gram-Schmidt procedure. $\square$

Recall dual spaces:

$V' = \mathcal{L}(V, \mathbb{F})$ is called the dual of V. It's the vector sp of linear functionals on V.

<u>Thm</u> (Riesz representation thm)
Let V be fin dim, and let $\varphi \in \mathcal{L}(V, \mathbb{F})$ be a linear functional.
Then there's a unique vector $v \in V$ such that

$$\boxed{\varphi(u) = \langle u, v \rangle \quad \forall u \in V.}$$

<u>Proof</u>: <u>Existence</u>: Pick an ON basis $(e_1, \ldots, e_n)$ for V. Then any $u \in V$ can be written as

$$u = \langle u, e_1 \rangle e_1 + \cdots + \langle u, e_n \rangle e_n.$$

So:

$$\varphi(u) = \varphi\big(\langle u, e_1 \rangle e_1 + \cdots + \langle u, e_n \rangle e_n\big)$$

$$= \langle u, e_1 \rangle \varphi(e_1) + \cdots + \langle u, e_n \rangle \varphi(e_n)$$

Scalars

$= \langle u, \overline{\varphi(e_1)}e_1\rangle + \cdots + \langle u, \overline{\varphi(e_n)}e_n\rangle$

$= \langle u, \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n\rangle$

So the vector v is

$$v = \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n.$$

<u>Uniqueness</u>: Assume that $v_1, v_2 \in V$ are such that

$$\varphi(u) = \langle u, v_1\rangle = \langle u, v_2\rangle \quad \forall u \in V.$$

Then $\langle u, v_1\rangle = \langle u, v_2\rangle$

$$\Leftrightarrow \langle u, v_1 - v_2\rangle = 0 \quad \forall u \in V$$

In particular for $u = v_1 - v_2$ we get $\langle v_1 - v_2, v_1 - v_2\rangle = 0 \Rightarrow v_1 - v_2 = 0$

$\Leftrightarrow v_1 = v_2.$ $\square$

Review: Midterm 2.

- Sections 4–6A (inclusive).
- 4 problems like first midterm.
- Some important definitions:
    - Invariant subspace
    - Eigenvalue, eigenvector
    - Minimal polynomial
    - Upper triangular and diagonal matrices
    - Inner product & norm
    - Cauchy–Schwarz inequality & triangle inequality.

Make sure you know the definitions.
Repeat examples from the lectures.