Recall: • $(v_1, \dots, v_n)$ is an orthonormal list if $\langle v_i, v_j \rangle = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$.

• If $(e_1, \dots, e_n)$ is an orthonormal basis for V & $v \in V$, then

$$v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n.$$

• Gram-Schmidt procedure: Turns any linearly indep list to an orthonormal one.

If $(v_1, \dots, v_n)$ is lin indep, then Let $f_1 = v_1$ and for $k = 2, \dots, n$ def

$$f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \dots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}$$

Then $(f_1/\|f_1\|, \dots, f_n/\|f_n\|)$ is an orthonormal list.

Ex: Let's consider $P_2(\mathbb{F})$ & the standard basis $(1, x, x^2)$

Consider the inner product $\langle f,g\rangle = \int_{-1}^{1} fg\,dx$. The basis is not orthonormal:

$$\langle 1,x\rangle = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\langle 1,x^2\rangle = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(\frac{-1}{3}\right) = \frac{2}{3} \neq 0$$

Now we use Gram-Schmidt:

$f_1 = e_1 = 1$. Then

$$f_2 = e_2 - \frac{\langle e_2, f_1\rangle}{\|f_1\|^2} f_1 = x - \frac{\overbrace{\langle x,1\rangle}^{=0}}{\|1\|^2} 1 = x$$

$$f_3 = e_3 - \frac{\langle e_3, f_1\rangle}{\|f_1\|^2} f_1 - \frac{\langle e_3, f_2\rangle}{\|f_2\|^2} f_2$$

$$= x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2} 1 - \frac{\langle x^2, x\rangle}{\|x\|^2} x$$

$$\langle x^2, 1\rangle = \frac{2}{3}$$

$$\|1\|^2 = \int_{-1}^{1} 1^2\,dx = [x]_{-1}^{1} = 1 - (-1) = 2$$

$$\langle x^2, x\rangle = \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

So $f_3 = x^2 - \frac{2/3}{2} 1 = x^2 - \frac{1}{3}$.

Now $\|f_1\| = \|1\| = \sqrt{2}$

$\|f_2\|^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$

$\|f_3\|^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 \, dx = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9} \, dx$

$= \left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9} \right]_{-1}^{1} = \frac{8}{45}$

$\left(\frac{f_1}{\|f_1\|}, \frac{f_2}{\|f_2\|}, \frac{f_3}{\|f_3\|} \right) = \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \sqrt{\frac{45}{8}} (x^2 - \frac{1}{3}) \right)$

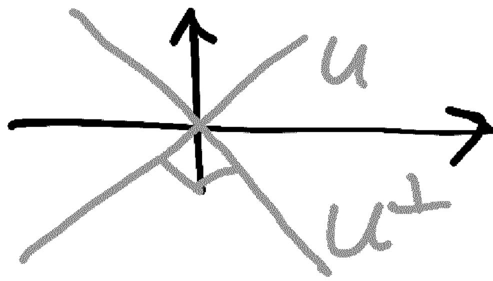
is an orthonormal basis for $P_2(\mathbb{F})$.

Orthogonal Complements (6C)

Def: Let $U \subset V$ be a subspace of an inner product space V. The orthogonal complement to U is

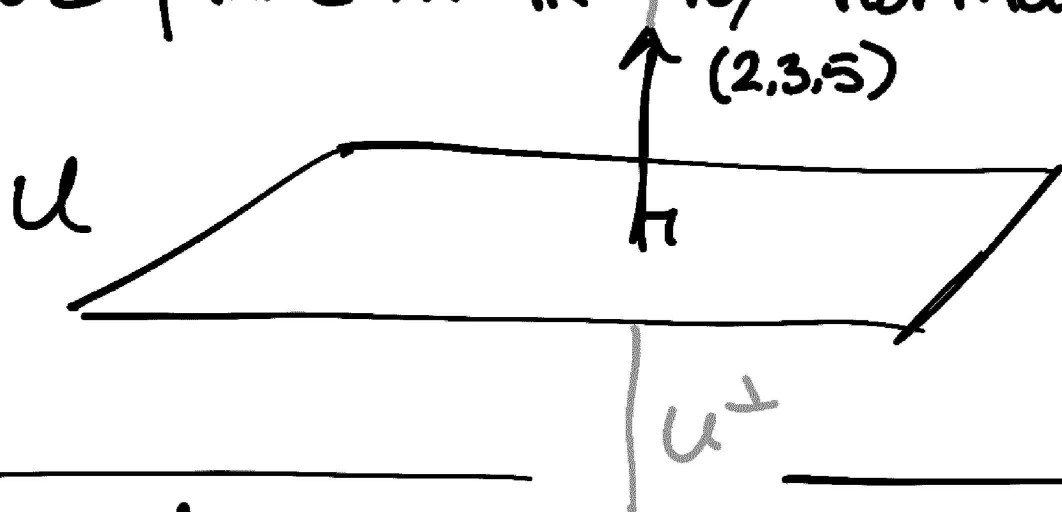
$U^\perp := \{ v \in V \mid \langle u, v \rangle = 0 \ \forall u \in U \}$.

Ex: • If $U \subset \mathbb{F}^2$ is a line, then $U^\perp$ is the line perpendicular to U.



• If $U = \{(x,y,z) \in \mathbb{R}^3 \mid 2x+3y+5z=0\}$ then $U^\perp = \text{span}((2,3,5))$

U is the plane in $\mathbb{R}^3$ w/ normal $(2,3,5)$



Prop: Let V be an inner product sp and $U \subset V$ a subspace.

(a) $U^\perp \subset V$ is a subspace.

(b) $\{0\}^\perp = V$

(c) $V^\perp = \{0\}$

(d) $U \cap U^\perp = \{0\}$

(e) If $U_1 \subset U_2$ then $U_2^\perp \subset U_1^\perp$.

<u>Proof</u>: (a) Need to check that $U^\perp$ is closed under addition & scalar mult.

$v, w \in U^\perp$ means $\langle v, u\rangle = \langle w, u\rangle = 0$ $\forall u \in U$. Then $\langle v+w, u\rangle = \langle v, u\rangle + \langle w, u\rangle = 0$ $\forall u \in U$

so $v+w \in U^\perp$. Also $\langle \lambda v, u\rangle = \lambda\langle v, u\rangle = 0$ $\forall u \in U$

so $\lambda v \in U^\perp$

(b) $\{0\}^\perp \subset V$ from (a), so suffices to prove $V \subset \{0\}^\perp$. If $v \in V$ then $\langle v, 0\rangle = 0$ so $v \in \{0\}^\perp$.

(c) Again suffices to show $\{0\} \subset V^\perp$.

$\langle 0, v\rangle = 0 \ \forall v \in V$ so $0 \in V^\perp$.

(d) If $v \in U \cap U^\perp$ then $\langle v, u\rangle = 0$ $\forall u \in U$. Since $v \in U$ in particular $\langle v, v\rangle = 0 \Rightarrow v = 0$ so $U \cap U^\perp \subset \{0\}$.

(e) If $v \in U_2^\perp$ & $U_1 \subset U_2$ then $\langle v, u\rangle = 0$ $\forall u \in U_2$ so in particular $\langle v, u\rangle = 0$ $\forall u \in U_1$, i.e. $v \in U_1^\perp$. $\square$

<u>Prop</u>: If $U \subset V$ is fin dim, then $V = U \oplus U^\perp$.

<u>Proof</u>: Since $U \cap U^\perp = \{0\}$ it suffices to show $V = U + U^\perp$ & in particular that any vector $v \in V$ can be written as $v = u + w$ $u \in U$, $w \in U^\perp$.

Let $(e_1, .., e_m)$ be an orthonormal basis for U. Then

$$v = \underbrace{\langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m}_{=u}$$

$$+ \underbrace{v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m}_{w} = u + w$$

Now $u \in U$ since $(e_1, \ldots, e_m)$ spans U. By Gram-Schmidt, w is orthogonal to each e_i so $w \in U^\perp$. $\square$

<u>Prop</u>: If V is fin dim and $U \subset V$, then $\dim U^\perp = \dim V - \dim U$.

<u>Proof</u>: Follows from $V = U \oplus U^\perp$. $\square$

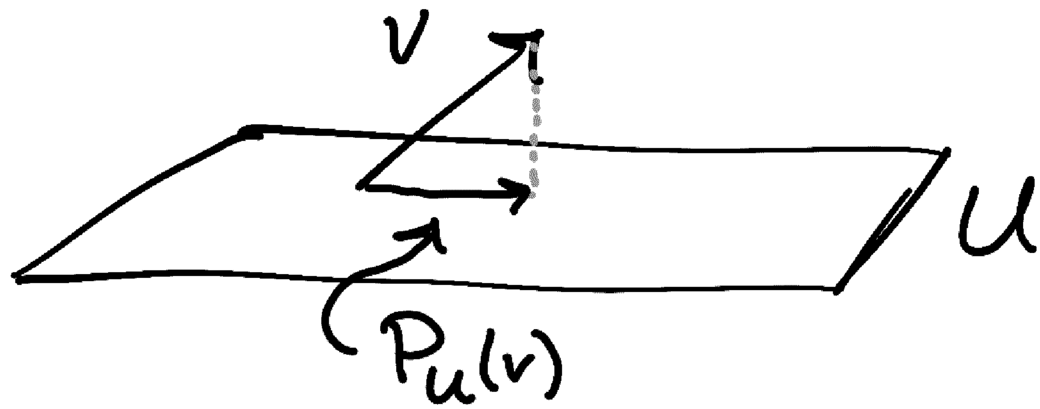
<u>Prop</u>: If $U \subset V$ is fin dim, then
$$U = (U^\perp)^\perp.$$

<u>Def</u>: Let $U \subset V$ be fin dim. The orthogonal projection onto U is the linear map $P_U \in \mathcal{L}(V)$ def

as follows: Any $v \in V$ can be written $v = u + w$ uniquely since $V = U \oplus U^\perp$.

$$\boxed{P_U(v) = u.}$$

Ex:

① 

② If $U = \text{span}(u) \subset V$ is 1-dim, then orthogonal projection onto U has a nice formula:

If $v \in V$, then

$$v = \underbrace{\frac{\langle v,u\rangle}{\|u\|^2} u}_{\text{in } U} + \underbrace{\left(v - \frac{\langle v,u\rangle}{\|u\|^2} u\right)}_{\text{in } U^\perp}$$

so $P_U(v) = \dfrac{\langle v,u\rangle}{\|u\|^2} u.$

Prop: Let $\mathcal{U} \subset V$ be fin dim.

(a) $P_{\mathcal{U}} \in \mathcal{L}(V)$

(b) $P_{\mathcal{U}}(u) = u \quad \forall u \in \mathcal{U}$

(c) $P_{\mathcal{U}}(w) = 0 \quad \forall w \in \mathcal{U}^{\perp}$

(d) range $P_{\mathcal{U}} = \mathcal{U}$

(e) null $P_{\mathcal{U}} = \mathcal{U}^{\perp}$

(f) $v - P_{\mathcal{U}}(v) \in \mathcal{U}^{\perp} \quad \forall v \in V$

(g) $P_{\mathcal{U}}^2 = P_{\mathcal{U}}$

(h) $\|P_{\mathcal{U}} v\| \leq \|v\| \quad \forall v \in V$

(i) If $(e_1, \dots, e_m)$ is an orthonormal basis of $\mathcal{U}$ and $v \in V$, then

$$P_{\mathcal{U}} v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m.$$

Proof: (a) Exercise

(b) If $u \in \mathcal{U}$ then $u = \underset{\text{in } \mathcal{U}}{u} + \underset{\text{in } \mathcal{U}^{\perp}}{0}$

So $P_{\mathcal{U}} u = u$

(c) If $w \in U^\perp$, $w = \underset{\text{in } U}{0} + \underset{\text{in } U^\perp}{w}$

So $P_U w = 0$.

(d) range $P_U = \{v \in V \mid v = P_U w$ for some $w \in V\}$

Note $U \subset$ range P_U since if $u \in U$ then $u = P_U u$.

range $P_U \subset U$ is by def.

(e) null $P_U = \{v \in V \mid P_U v = 0\}$.

By (c) $U^\perp \subset$ null P_U, and if $P_U v = 0$ it means $v = 0 + v$ was the direct sum decomposition of v so $v \in U^\perp$.

(f) The direct sum decomp of v is $v = \underset{\text{in } U}{P_U v} + \underset{\text{in } U^\perp}{(v - P_U v)}$

(g) By def $P_U v \in U$ & $P_U u = u$ for $u \in U$ so $P_U^2 v = P_U(P_U v) = P_U v$ $\forall v \in V$.

(h) $\|v\|^2 = \|P_U v + (v - P_U v)\|^2$

$$\underset{\substack{\uparrow \\ \text{Pythagorean thm}}}{=} \|P_U v\|^2 + \|v - P_U v\|^2 \geq \|P_U v\|^2 + 0 \quad \square$$

(i) This is a consequence of the Gram-Schmidt procedure. $\square$
