MAT 310 L18 Tue Apr 8

<u>Recall</u>: Let $(V, \langle -,- \rangle)$ be an inner product space.

- The norm associated to $\langle -,- \rangle$ is defined as $\|x\| := \sqrt{\langle x,x \rangle}$.
  It measures the "length" of x.

- u and v are orthogonal ($u \perp v$) if $\langle u,v \rangle = 0$.

<u>Orthonormal bases</u> (6B)

<u>Def</u>: A list of vectors $(v_1, \dots, v_n)$ is called <u>orthonormal</u> if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

<u>Rem</u>: Equivalently, $(v_1, \dots, v_n)$ is orthonormal if

$$\|v_i\| = 1 \ \forall i \ \& \ v_i \perp v_j \text{ for } i \neq j.$$

Ex: (a) $(e_1, \dots, e_n)$ standard basis in $\mathbb{F}^n$ is orthonormal with respect to the dot product:

$$e_i = (0, \dots, 0, \underset{\uparrow\ i\text{-th comp}}{1}, 0, \dots, 0)$$

$$e_j = (0, \dots, 0, \underset{\uparrow\ j\text{-th comp}}{1}, 0, \dots, 0)$$

i-th comp

$$e_i \cdot e_i = 0 \cdot 0 + \cdots + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 + \cdots + 0 \cdot 0$$
$$= 1$$

and if $i \neq j$

$$e_i \cdot e_j = 0 \cdot 0 + \cdots + \underset{\uparrow\ i\text{-th term}}{0 \cdot 1} + \cdots + \underset{\uparrow\ j\text{-th term}}{1 \cdot 0} + \cdots + 0 \cdot 0$$
$$= 0$$

(b) $$\Big(\underbrace{\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)}_{v_1}, \underbrace{\left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)}_{v_2}\Big)$$ is

an orthonormal list in $\mathbb{F}^3$ (w/ dot product):

$$v_1 \cdot v_1 = \left(\tfrac{1}{\sqrt{3}}\right)^2 + \left(\tfrac{1}{\sqrt{3}}\right)^2 + \left(\tfrac{1}{\sqrt{3}}\right)^2 = \frac{3}{(\sqrt{3})^2} = \underline{1}$$

$$V_2 \cdot V_2 = \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$V_1 \cdot V_2 = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}}\right) \cdot 0 = 0$$

(c) In $C^0([-\pi,\pi])$ w/ $\langle f,g\rangle = \int_{-\pi}^{\pi} fg\,dx$ the list $(f,g) = \left(\frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}\right)$ is orthonormal.

$$\langle f,f\rangle = \int_{-\pi}^{\pi} \left(\frac{\cos x}{\sqrt{\pi}}\right)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,dx = 1$$

[Use $\cos^2 x = \frac{1+\cos 2x}{2}$ to compute.]

$$\langle f,g\rangle = \int_{-\pi}^{\pi} \frac{\cos x \cos 2x}{\pi}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \cos 2x\,dx = 0$$

[Use $\cos x \cos(2x) = \frac{\cos x + \cos(3x)}{2}$.]

In fact $\left(\frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \dots\right)$ is an infinite orthonormal list.

<u>Prop.</u> Let $(e_1,\dots,e_n)$ be an orthonormal list. Then $\|a_1e_1+\dots+a_ne_n\|^2 = |a_1|^2+\dots+|a_n|^2$.

Proof: Lets prove it for $n=2$ for simplicity. Since $e_1 \perp e_2$, Pythagorean theorem (from last time) gives

$$\|a_1e_1+a_2e_2\|^2 = \|a_1e_1\|^2 + \|a_2e_2\|^2$$
$$= |a_1|^2 \underbrace{\|e_1\|^2}_{=1} + |a_2|^2 \underbrace{\|e_2\|^2}_{=1} = |a_1|^2+|a_2|^2 \quad \square$$

Prop: Every orthonormal list of vectors is linearly independant.

Proof: If $a_1e_1+\cdots+a_ne_n=0$ & $(e_1,\ldots,e_n)$ is orthonormal, then

$$\|a_1e_1+\cdots+a_ne_n\|^2 = 0$$
$$\|$$
$$|a_1|^2+\cdots+|a_n|^2 \Rightarrow a_1=\cdots=a_n=0 \quad \square$$

Def: An orthonormal basis of V is an orthonormal list of vectors that's a basis.

Now we illustrate why orthonormal lists are useful:

Thm: Let V be fin dim. Every orthonormal list of $\dim V$ vectors is a basis.

Proof: By the previous prop an orthonormal list is lin indep, and since it's of length $\dim V$, it's automatically a basis (2.38) $\square$

Ex: $((1,1),(-1,1))$ in $\mathbb{F}^2$ (w/ dot prod) is not orthonormal:

$$\sqrt{v_1 \cdot v_1} = \sqrt{1^2+1^2} = \sqrt{2} \neq 1$$
$$\sqrt{v_2 \cdot v_2} = \sqrt{(-1)^2+1^2} = \sqrt{2} \neq 1$$
$$v_1 \cdot v_2 = 1\cdot(-1) + 1\cdot 1 = 0$$

so $v_1 \perp v_2$, but the vectors are too long; their norms are not 1. This can easily be fixed by rescaling:

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\tilde{v}_2 = \frac{\tilde{v}_2}{\|v_2\|} = \frac{(-1,1)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$(\tilde{v}_1, \tilde{v}_2)$ is then orthonormal.

<u>Prop</u>: Let $(e_1, \dots, e_n)$ be an orthonormal basis of V, and $u, v \in V$.

(a) $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$

(b) $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$

<u>Proof</u>: (a) Since $(e_1, \dots, e_n)$ is a basis, write $v = a_1 e_1 + \cdots + a_n e_n$.

Therefore

$$\langle v, e_k \rangle = \langle a_1 e_1 + \cdots + a_n e_n, e_k \rangle$$

$$= a_1 \langle e_1, e_k \rangle + \cdots + a_n \langle e_n, e_k \rangle = a_k$$

since $\langle e_k, e_k \rangle = 1$ while $\langle e_i, e_k \rangle = 0$ for $i \neq k$

(b) $\|v\|^2 = \|a_1 e_1 + \cdots + a_n e_n\|^2$

$$= \|\langle v, e_1\rangle e_1 + \cdots + \langle v, a_n\rangle e_n\|^2$$

$$= |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 \quad \square$$

<u>Ex</u>: From previous example, $(v_1, v_2) = \left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right)$ is an orthonormal basis for $\mathbb{F}^2$ w/ dot product.

Now, how to express $v = (-8, 2)$ as a linear combination of v_1, v_2?

From previous prop,

$$v = (-8,2) = (\underline{v} \cdot v_1) v_1 + (v \cdot v_2) v_2$$

$$v \cdot v_1 = (-8,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (-8)\frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{2}}$$

$$= -4\sqrt{2} + \sqrt{2} = -3\sqrt{2}$$

$$v \cdot v_2 = (-8,2) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (-8)\left(-\frac{1}{\sqrt{2}}\right) + 2\frac{1}{\sqrt{2}}$$

$$= 4\sqrt{2} + \sqrt{2} = 5\sqrt{2}$$

$$\boxed{(-8,2) = -3\sqrt{2}\, v_1 + 5\sqrt{2}\, v_2}$$

We saw earlier that given two orthogonal vectors $v_1 \perp v_2$, we can make them orthonormal by rescaling them:

$(v_1/\|v_1\|, v_2/\|v_2\|)$ is orthonormal.

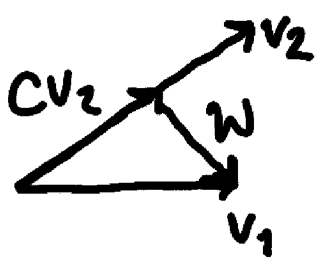
<u>Ex</u>: Take $(v_1, v_2) = ((1,0), (1,1))$ in $\mathbb{F}^2$ w/ dot product.

Then $\|v_1\| = \sqrt{1^2+0^2} = 1$

$\|v_2\| = \sqrt{1^2+1^2} = \sqrt{2}$

$v_1 \bullet v_2 = 1 \cdot 1 + 0 \cdot 1 = 1.$

We should first make them orthogonal.



From last time $w = v_1 - \frac{v_1 \bullet v_2}{\|v_2\|^2} v_2$ is orthogonal to v_2.

$$w = (1,0) - \frac{(1,0)\bullet(1,1)}{(\sqrt{2})^2}(1,1) = (1,0) - \frac{1}{2}(1,1)$$

$$= \left(\frac{1}{2}, -1\right)$$

(w, v_2) is sth $w \perp v_2$. Now rescale to get an orthonormal basis:

$\|w\| = \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2} = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2}$

$\|v_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\left(\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right)$ orthonormal.

<u>Thm</u> (Gram-Schmidt procedure)

Let $(v_1, \dots, v_m)$ be linearly indep. Let $f_1 = v_1$. For $k = 2, \dots, m$ def f_k inductively by

$$f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \dots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}$$

For each $k = 1, \dots, m$ let $e_k = f_k / \|f_k\|$. Then $(e_1, \dots, e_m)$ is an orthonormal list of vectors sth $\mathrm{span}(v_1, \dots, v_k) = \mathrm{span}(e_1, \dots, e_k)$ $\forall k$.
