MAT 310 L15 Thu Mar 27

Recall: Let V be fin dim, $T \in \mathcal{L}(V)$.
Pick a basis $(v_1, \dots, v_n)$ for V.

- $\mathcal{M}(T)$ is upper triangular if

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

In this case eigenvalues of T are $\lambda_1, \dots, \lambda_n$ & minimal polynomial for T is $(z-\lambda_1)\cdots(z-\lambda_n)$.

- v_1 is always an eigenvector & v_k is an eigenvector if k-th column in $\mathcal{M}(T)$ is $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Ex: If $T: \mathbb{F}^3 \to \mathbb{F}^3$, $T(x,y,z) = (8x, 5y, 5z)$ then the matrix of T wrt the standard basis is

$$\mathcal{M}(T) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

From last lecture we therefore learn that the eigenvalues for T are $8, 5, 5$. We now also have that $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are eigenvectors. Therefore it seems like diagonal matrices are special among upper triangular ones.

<u>Def</u>: A square matrix is <u>diagonal</u> if its of the form $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

<u>Def</u>: Let V be fin dim, and $T \in \mathcal{L}(V)$. We say that T is <u>diagonalizable</u> if there's a basis for V such that $\mathcal{M}(T)$ is diagonal.

<u>Ex</u>: $T: \mathbb{F}^2 \longrightarrow \mathbb{F}^2$, $T(x,y) = (6x-y, 2x+3y)$

Then wrt standard basis $e_1 = (1,0)$
$e_2 = (0,1)$

the matrix is $\mathcal{M}(T) = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$

not even upper triangular. Instead pick basis $v_1 = (1,1)$, $v_2 = (1,2)$. Then

$$T(1,1) = (5,5) = 5 \cdot (1,1) + 0 \cdot (1,2)$$

$$T(1,2) = (4,8) = 0 \cdot (1,1) + 4 \cdot (1,2)$$

so $\mathcal{M}(T) = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$ diagonal

wrt the basis (v_1, v_2), so T is diagonalizable.

<u>Def</u>: Let $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. The eigenspace of T corr. to λ is

$$\begin{aligned} E(\lambda, T) &= \operatorname{null}(T - \lambda I) \\ &= \{v \in V \mid Tv = \lambda v\}. \\ &= \{\text{all eigenvectors corresponding to } \lambda \text{ \& the zero vector}\} \end{aligned}$$

<u>Remark</u>: By def $E(\lambda, T) \neq \{0\}$
$\iff \lambda$ is an eigenvalue for T

Ex: For T as in the ex above

$E(5,T) = \operatorname{span}((1,1))$

$E(4,T) = \operatorname{span}((1,2))$

$E(\lambda,T) = \{0\}$ for $\lambda \neq 4,5$.

Prop: Let $T \in \mathcal{L}(V)$ & suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues for T. Then

$$E(\lambda_1,T) + \dots + E(\lambda_m,T) \subset V$$

is a direct sum.

Furthermore if V is fin dim

$$\dim E(\lambda_1,T) + \dots + \dim E(\lambda_m,T) \leq \dim V$$

Proof: To prove the first claim, we need to show if

$v_i \in E(\lambda_i,T)$ $\forall i \in \{1,\dots,m\}$ such that

(*) $v_1 + \dots + v_m = 0$ then $v_1 = \dots = v_m = 0$.

If $v_i \in E(\lambda_i,T)$ it means $Tv_i = \lambda_i v_i$.

So each v_i is an eigenvector. We previously showed (5.11) that eigenvectors corresp. to different eigenvalues are lin. indep. Therefore (*) must imply $v_1 = \cdots = v_m = 0$.

Now if V is fin dim

$$\dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T)$$
$$= \dim(E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T)) \leq \dim V$$

since $E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T)$ is a subspace of V. $\square$

Let's now return to diagonalizability:

<u>Prop</u>: Let V be fin dim & $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. The following are equivalent:

1. T is diagonalizable
2. V has a basis consisting of eigenvectors
3. $V = E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T)$
4. $\dim V = \dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T)$

Proof: We already know ③ $\Leftrightarrow$ ④ by 3.94.

① $\Leftrightarrow$ ②: Diagonalizable $\Leftrightarrow$ $\exists$ basis $(v_1, \ldots, v_n)$ s.th. $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\Leftrightarrow$ $\exists$ basis $(v_1, \ldots, v_n)$: $Tv_k = \lambda_k v_k$ $\forall k$

② $\Rightarrow$ ③: If $(v_1, \ldots, v_n)$ basis of eigenvectors then $E(\lambda_1, T) + \cdots + E(\lambda_m, T) = V$. By previous prop (5.54), LHS is a direct sum, so

$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

④ $\Rightarrow$ ②:

Suppose $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Choose basis for each $E(\lambda_k, T)$. Put these vectors together to form the list $(v_1, \ldots, v_n)$, where $n = \dim V$. Since it's a list of the correct length, it suffices to show $(v_1, \ldots, v_n)$ is lin. independent.

Assume $a_1v_1 + \cdots + a_nv_n = 0$ (*)
for some $a_1, \ldots, a_n \in \mathbb{F}$. Let
u_k = sum of terms a_jv_j such that
$u_k \in E(\lambda_k, T)$ $\forall k \in \{1, \ldots, m\}$.

Then (*) turns into $u_1 + \cdots + u_m = 0$
& each u_k is an eigenvector corr to λ_k, they are lin indep, so $u_k = 0$ $\forall k$. Therefore

u_k = sum of a_jv_j s.th. $v_j \in E(\lambda_k, T)$ are basis vectors. Lin indep of these (inside $E(\lambda_k, T)$) implies $a_j = 0$ for these. So we must have $a_1 = \cdots = a_n = 0$ after doing this for all $k \in \{1, \ldots, m\}$ $\square$

Ex: Let $T: \mathbb{F}^3 \to \mathbb{F}^3$, $T(x,y,z) = (y,z,0)$.
Wrt standard basis for $\mathbb{F}^3$, the matrix is $\mathcal{M}(T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Let's find eigenvalues for T:

Assume $\exists v \neq 0$ s.th. $Tv = \lambda v$

$\iff T(x,y,z) = \lambda(x,y,z)$

$\iff (y,z,0) = (\lambda x, \lambda y, \lambda z)$

Since $v = (x,y,z) \neq (0,0,0)$ it means $\lambda z = 0 \Rightarrow \lambda = 0$.

So $(y,z,0) = (0,0,0)$ implies $y = z = 0$. E.g. $(1,0,0)$ is an eigenvector corr. to this eigenval.

So there is only one non-zero eigenspace:

$$E(0,T) = \{v \in \mathbb{F}^3 \mid Tv = 0\}$$
$$= \{(x,0,0) \mid x \in \mathbb{F}\}$$
$$= \operatorname{Span}((1,0,0))$$

So $\dim \mathbb{F}^3 = 3$

$\dim E(0,T) = 1 \neq 3$

So T is <u>not</u> diagonalizable.

The are too few lin indep eigenvectors for T to be able to build a basis for V with them.

Prop: If V is n-dim, and $T \in \mathcal{L}(V)$ has n distinct eigenvalues, then T is diagonalizable.

Proof: If λ is an eigenvalue, then $E(\lambda, T) \neq \{0\}$, so $\dim E(\lambda, T) \geq 1$.
If $\lambda_1, \dots, \lambda_n$ are the n distinct eigenvals we therefore have

$$\dim E(\lambda_1, T) + \dots + \dim E(\lambda_n, T) \geq n$$

so 5.54 implies

$$\dim E(\lambda_1, T) + \dots + \dim E(\lambda_n, T) = n$$

& the result follows from 5.55. $\square$

Prop: Let V be fin dim & let $T \in \mathcal{L}(V)$.

Then T is diagonalizable $\Leftrightarrow$ the minimal polynomial of T is

$(z-\lambda_1)\cdots(z-\lambda_m)$ for some distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.

Remark:

T upper triangular

$\iff p_T(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ (repeats allowed).

The above prop says that T is diagonalizable iff there are no repeats among these $\lambda_1, \ldots, \lambda_m$.

Prop: If $T \in \mathcal{L}(V)$ is diagonalizable, and $U \subset V$ is a subspace that is invariant under T, then $T|_U$ is diagonalizable.
