Recall: Let $T \in \mathcal{L}(V)$.

- $\lambda \in \mathbb{F}$ eigenvalue if $\exists v \neq 0$ s.th. $Tv = \lambda v$
- $v \in V$ eigenvector if $v \neq 0$ & $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$.
- Minimal polynomial for T is the monic polynomial p of minimal degree s.th. $p(T) = 0$. Its zeros are the eigenvalues of T.

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§ Upper-triangular matrices (5C)

Let V be a fin dim vector sp. over $\mathbb{F}$. Let $T \in \mathcal{L}(V)$. Recall that after choosing a basis for V, we may def a matrix $\mathcal{M}(T)$ associated to T.

Namely let $(v_1, \dots, v_n)$ be a basis for V. Then

$$\mathcal{M}(T) = \begin{pmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{pmatrix}$$

Where each column is $\begin{pmatrix} | \\ T(v_k) \\ | \end{pmatrix} = \begin{pmatrix} a_{1,k} \\ \vdots \\ a_{n,k} \end{pmatrix}$

where $a_{1,k}, \dots, a_{n,k} \in \mathbb{F}$ are the scalars s.th

$$T(v_k) = a_{1,k}v_1 + \cdots + a_{n,k}v_n.$$

Note.

① The matrix $\mathcal{M}(T)$ depends on the choice of the basis $(v_1, \dots, v_n)$.

② Since $T: V \to V$ is a linear operator its matrix has size $n \times n$, so its a square matrix (#rows = #columns).

Ex: $T: \mathbb{F}^3 \to \mathbb{F}^3$. $T(x,y,z) = (2x+y, 5y+3z, 8z)$

Pick standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ $e_3 = (0,0,1)$. Then

$T(1,0,0) = (2,0,0)$

$T(0,1,0) = (1,5,0)$

$T(0,0,1) = (0,3,8)$

$$\mathcal{M}(T) = \begin{pmatrix} | & | & | \\ T(1,0,0) & T(0,1,0) & T(0,0,1) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}.$$

Def. The entries $a_{11}, a_{22}, \ldots, a_{nn}$ in an $(n \times n)$-matrix is called the diagonal.

Ex: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$ diagonal is 2, 5, 8

Def. A square matrix is called upper triangular if all entries below the diagonal are $=0$.

Such a matrix is of the form: $\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Ex: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$ these entries are $=0$ so this matrix is upper triangular.

$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$ is not upper triangular since this element is $\neq 0$.

<u>Prop</u>: Let $(v_1, \dots, v_n)$ be a basis for V, and let $T \in \mathcal{L}(V)$. The following are equivalent:

① $\mathcal{M}(T)$ is upper triangular

② $\text{span}(v_1, \dots, v_k)$ is invariant under T $\forall k \in \{1, \dots, n\}$

③ $Tv_k \in \text{Span}(v_1, \dots, v_k)$ $\forall k \in \{1, \dots, n\}$

<u>Proof</u>: We will show ① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ①.

<u>① $\Rightarrow$ ②</u>: Assume $\mathcal{M}(T)$ is upper triangular.

We need to show for each $k \in \{1, \dots, n\}$ if $w = a_1 v_1 + \dots + a_k v_k$ then $Tw \in \text{Span}(v_1, \dots, v_k)$.

Upper triangular means

$$Tv_j = a_{j,1} v_1 + \dots + a_{j,j} v_j \quad \forall j \text{, so}$$

$$Tw = T(a_1 v_1 + \dots + a_k v_k) = a_1 Tv_1 + \dots + a_k Tv_k$$

$$= a_1 (a_{1,1} v_1) + a_2 (a_{2,1} v_1 + a_{2,2} v_2)$$

$+\dots+a_k(a_{k,1}v_1+\dots+a_{k,k}v_k)$

$=(a_1a_{1,1}+a_2a_{2,1}+\dots+a_ka_{k,1})v_1$

$+\dots+(a_{k-1}a_{k-1,k-1}+a_ka_{k,k-1})v_{k-1}$

$+a_ka_{k,k}v_k$

$= c_1v_1+\dots+c_kv_k \in \operatorname{span}(v_1,\dots,v_k)$.

2 ⇒ 3: This is immediate.

$v_k \in \operatorname{span}(v_1,\dots,v_k)$ so

$Tv_k \in \operatorname{span}(v_1,\dots,v_k)$ since $\operatorname{span}(v_1,\dots,v_k)$ is invariant under T by assumption.

3 ⇒ 1:

Since column k in $\mathcal{M}(T)$ is Tv_k and $Tv_k = a_{k,1}v_1+\dots+a_{k,k}v_k$ by assump we have $a_{l,k}=0$ for $l>k$ which are precisely the elements below the diagonal of $\mathcal{M}(T)$. $\square$

Computing eigenvalues of linear ops

w/ upper triangular matrices is easy.

Prop: Suppose $T \in \mathcal{L}(V)$ and V has a basis $(v_1, \ldots, v_n)$ s.t. $\mathcal{M}(T)$ is upper triangular w/ diagonal entries $\lambda_1, \ldots, \lambda_n$. Then

$$(T-\lambda_1 I)\cdots(T-\lambda_n I)=0$$

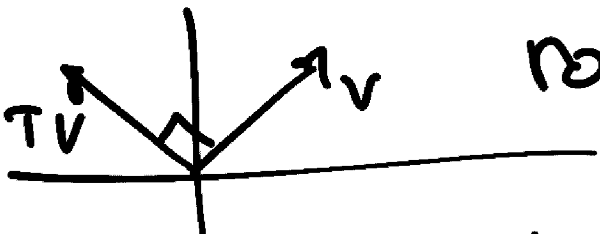
Prop: Suppose $T \in \mathcal{L}(V)$ and V has a basis $(v_1, \ldots, v_n)$ s.t. $\mathcal{M}(T)$ is upper triangular. Then the eigenvalues of T are precisely the diagonal entries.

These two results imply that if we can find a basis for V such that $\mathcal{M}(T)$ is upper triangular w/ diagonal entries $\lambda_1, \ldots, \lambda_n$ then

$p(z)=(z-\lambda_1)\cdots(z-\lambda_n)$ is the minimal polynomial for T.

Sometimes it's not possible to find a basis such that $\mathcal{M}(T)$ is upper triangular!

<u>Ex</u>: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (-y, x)$



rotation CCW $90°$ in the plane.

Since this lin op has <u>no</u> 1-dim invariant subspaces, there can not be a basis for $\mathbb{R}^2$ s.th. $\mathcal{M}(T)$ is upper triangular.

<u>Ex</u>: $T: \mathbb{F}^3 \to \mathbb{F}^3$. $T(x,y,z) = (2x+y, 5y+3z, 8z)$

Matrix wrt standard basis is

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix},$$ which is

upper triangular, so the minimal polynomial is $p(z) = (z-2)(z-5)(z-8)$ & eigenvalues are $2, 5, 8$.

<u>Ex</u>: Let $T: \mathbb{F}^4 \to \mathbb{F}^4$

$T(x,y,z,w) = (-y, x, 2x+3z, z+3w)$

Wrt the standard basis for $\mathbb{F}^4$, the matrix is

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

not upper triangular wrt this basis.

Can we find another basis so that it becomes upper triangular?

Turns out the answer depends on whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

In both cases, one can compute the minimal polynomial to be

$$p(z) = (z^2+1)(z-3)^2$$

If $\mathbb{F} = \mathbb{C}$ this can be further factored:

$$p(z) = (z-i)(z+i)(z-3)^2$$

One can verify that wrt the basis

$V_1 = (4-3i, -3-4i, -3+i, 1)$

$V_2 = (4+3i, -3+4i, -3-i, 1)$

$V_3 = (0,0,1,0)$

$V_4 = (0,0,0,1)$

the matrix is $\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (upper triangular)

If $\mathbb{F} = \mathbb{R}$, it turns out that we can <u>not</u> find such a basis.

<u>Prop</u>: Assume V is fin dim, and $T \in \mathcal{L}(V)$. Then T has an upper triangular matrix wrt some basis of V if and only if the minimal polynomial is $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{F}$.

<u>Prop</u>: Suppose V is a fin dim <u>complex</u> vsp, and $T \in \mathcal{L}(V)$. Then V admits a basis s.th. $\mathcal{M}(T)$ is upper triangular.

If we can find a basis for V $(v_1, \dots, v_n)$ such that

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ then v_1 will be an eigenvector. The vectors $v_2, \dots, v_n$ do <u>not</u> have to be eigenvectors.

In fact v_k is an eigenvector for T if column k in $\mathcal{M}(T)$ is

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$