MAT 310 L13 Thu Mar 13

Recall: Let $T \in \mathcal{L}(V)$.

- $U \subset V$ invariant under T if $Tu \in U$ $\forall u \in U$.

- $\lambda \in \mathbb{F}$ eigenvalue if $\exists v \neq 0$ s.th. $Tv = \lambda v$
- $v \in V$ eigenvector if $v \neq 0$ & $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$.

§ The minimal polynomial (5B)

Thm. Every linear operator $T \in \mathcal{L}(V)$ on a fin dim non-zero complex vector space has an eigenvalue.

Proof: Let V be a n-dim cpx v.sp. and let $T \in \mathcal{L}(V)$. Then $(v, Tv, \dots, T^n v)$ is a list of $n+1$ vectors, so it is lin dep. Therefore we can find some minimal k

Sth $(v, Tv, \dots, T^k v)$ is lin dep,
so $\exists a_0, \dots, a_k \in \mathbb{F}$ with $a_k \neq 0$

$$a_0 v + a_1 Tv + \cdots + a_k T^k v = 0$$

$$\Leftrightarrow (a_0 I + a_1 T + \cdots + a_k T^k) v = 0$$

$$\Leftrightarrow p(T) v = 0 \quad (*)$$

where $p(z) = a_0 + a_1 z + \cdots + a_k z^k$.
By the fundamental thm of algebra this polynomial has a zero $\lambda \in \mathbb{C}$, $p(\lambda) = 0$. So $\exists q \in \mathcal{P}(\mathbb{C})$ such that $p(z) = (z - \lambda) q(z)$. Therefore

$$(*) \quad 0 = p(T) v = (T - \lambda I)(q(T) v)$$

Since $\deg q = \deg p - 1$ we must have $q(T) v \neq 0$ by minimality of the degree of p, so $(T - \lambda I) v = 0$. $\square$

<u>Ex</u>: The assumption that the vector space in the above thm is fin dim

is important! Consider $T \in \mathcal{L}(\mathcal{P}(\mathbb{C}))$ $(Tp)(z) = zp(z)$. Then if T had an eigenvalue $\lambda \in \mathbb{C}$ we would need to have some $p \in \mathcal{P}(\mathbb{C})$ with

$$Tp = \lambda p \iff zp(z) = \lambda p(z)$$

$$\iff (z-\lambda)p(z) = 0 \quad \forall z$$

Which can only be true if $p=0$ so there are no eigenvalues.

<u>Def</u>: A monic polynomial is a polynomial whose highest degree term is 1.

<u>Ex</u>: $2+9z^2+z^3$ is a monic poly. of degree 3.

<u>Thm</u>: Let V be a fin dim v.sp. over $\mathbb{F}$, and $T \in \mathcal{L}(V)$. Then there exists a monic polynomial of smallest degree such that $p(T) = 0$. Furthermore $\deg p \leq \dim V$.

<u>Proof</u>: We will prove this by induction on dim V.

<u>Base Case</u>: If dim $V = 0 \iff V = \{0\}$ we have T being the identity operator $T0 = 0$, so we can take $p = 1$.

Assume the theorem is true for all vector spaces & all lin operators of dim $\leq p$ for some $p \geq 0$.

Now assume dim $V = p+1$

Let $v \in V$. The list $(v, Tv, \dots, T^{p+1}v)$ Consists of $p+2$ vectors so its lin dep. Hence there's a minimal $k \leq p+1$ s.th $(v, Tv, \dots, T^k v)$ is lin dep.

So $\exists c_0, \dots, c_k \in \mathbb{F} : c_k \neq 0$ s.th.

$$c_0 v + c_1 Tv + \dots + c_k T^k v = 0$$

$$\iff \frac{c_0}{c_k} v + \frac{c_1}{c_k} Tv + \dots + \frac{c_{k-1}}{c_k} T^{k-1} v + T^k v = 0$$

$\iff P(T)v = 0$ where p is the

monic polynomial

$$p(z) = \frac{c_0}{c_k} + \frac{c_1}{c_k} z + \cdots + \frac{c_{k-1}}{c_k} z^{k-1} + z^k$$

of minimal degree s.th. $p(T)v = 0$.

By minimality of k, $(v, Tv, \ldots, T^{k-1}v)$ is lin indep.

For any $l \geq 0$ we have

$$p(T)(T^l v) = (p(T)T^l)v = T^l(p(T)v) = T^l 0 = 0.$$

So $\dim \operatorname{null} p(T) \geq k$ by lin indep of $(v, Tv, \ldots, T^{k-1}v)$. Fund thm for lin maps therefore implies

$$\dim \operatorname{range} p(T) = \dim V - \dim \operatorname{null} p(T) \leq \dim V - k.$$

Now, $\operatorname{range} p(T) \subset V$ is invariant under T, so the restriction of T is a linear map

$$T|_{\operatorname{range} p(T)} : \operatorname{range} p(T) \longrightarrow \operatorname{range} p(T).$$

& by our induction hypothesis there's a unique monic $s \in \mathcal{P}(\mathbb{F})$ w/

$\dim s \leq \dim \operatorname{range} p(T) \leq \dim V - k$

$s(T|_{\operatorname{range} p(T)}) = 0.$

So $\forall v \in V$ we have

$$(sp)(T)v = s(T)(p(T)v) = 0$$

since $p(T)v \in \operatorname{range} p(T)$

Therefore sp is a monic polynomial w/

$$\deg sp = \deg s + \deg p \leq \dim V - k + k = \dim V$$

$$sp(T) = 0$$

This shows existence. To show uniqueness, suppose $r \in \mathcal{P}(\mathbb{F})$ is another monic polynomial of the same deg as p & $r(T) = 0$. Then

$(p-r)(T) = 0$ and $\deg(p-r) < \deg p$

(because the top degree term in

both p & r is z^m). So if $p-r \neq 0$ we would have

$$p-r = a_0 + a_1 z + \cdots + a_\ell z^\ell \quad a_\ell \neq 0$$

$$\Leftrightarrow \frac{p-r}{a_\ell} = \frac{a_0}{a_\ell} + \frac{a_1}{a_\ell} z + \cdots + \frac{a_{\ell-1}}{a_\ell} z^{\ell-1} + z^\ell$$

monic of deg $< m$ such that

$\frac{1}{a_\ell}(p-r)(T) = 0$, which is a contradiction, so we must have $p=r$. $\square$

The previous theorem justifies the following def.

<u>Def</u>: If V is a fin dim v.sp. and $T \in \mathcal{L}(V)$, the <u>minimal polynomial</u> is the unique monic polynomial p of smallest degree s.th. $p(T) = 0$.

To compute the minimal polynomial we need to find the smallest positive integer $m \leq \dim V$ s.th. the equ

$$c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} = -T^m$$

has a solution $c_0, c_1, \ldots, c_{m-1} \in \mathbb{F}$. Then $p(x) = c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} + x^m$ is the minimal polynomial.

Ex. Let $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$T(z,w) = (-w,z)$. Then we pick any $(z,w) \in \mathbb{C}^2$. Let $c_0, c_1 \in \mathbb{C}$.

$$\begin{aligned}(c_0 I + c_1 T)(z,w) = -T^2(z,w) &= -T(-w,z) \\ &= -(-z,-w) \\ &= (z,w)\end{aligned}$$

$\iff c_0(z,w) + c_1(-w,z) = (z,w)$

$\iff (c_0 z - c_1 w, c_0 w + c_1 z) = (z,w)$

$\iff \begin{cases} z = c_0 z - c_1 w \\ w = c_0 w + c_1 z \end{cases}$

This has the Sol $c_0 = 1$, $c_1 = 0$

The minimal polynomial is therefore

$p(x) = 1 + x^2.$

<u>Prop</u>: Let V be fin dim, $T \in \mathcal{L}(V)$.
(a) The zeros of the minimal polynomial of T are the eigenvalues of T.
(b) If V is a cpx vector space, then the minimal polynomial is

$$(z-\lambda_1)\cdots(z-\lambda_m)$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of T, possibly with repetition.

<u>Proof</u>: Let p be the minimal poly.

(a) Assume $\lambda \in \mathbb{F}$ is a zero of p. Then $p(z) = (z-\lambda)q(z)$ for some polynomial $q(z)$ of $\deg < \deg p$.

So $p(T) = (T - \lambda I)q(T)$ &
$q(T) \neq 0$ by minimality of the degree of p. So $\forall v \in V$

$$0 = p(T)v = (T-\lambda I)q(T)v \quad (*)$$

and we can find $v \neq 0$ s.t.
$q(T)v \neq 0$, which implies $(T-\lambda I)v = 0$
by (*).

Next we show that <u>every</u> eigenvalue of T is a zero of p. Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of T, i.e., $Tv = \lambda v$ for some $v \neq 0$. This also implies

$$T^k v = \lambda^k v \quad \forall k \geq 1$$

Therefore $0 = p(T)v = p(\lambda)v \Rightarrow p(\lambda) = 0$.

(b) Follows from part (a) + the fund thm of algebra. $\square$

<u>Prop</u>: Let V be fin dim, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbb{F})$. If $q(T) = 0$ then

$q = ps$ where p is the minimal poly for T.

<u>Prop</u>: Let V be fin dim and $T \in \mathcal{L}(V)$. If $U \subset V$ is invariant under T, then the minimal poly p of T is

$p = qs$ where q = minimal poly for $T|_U$.

<u>Prop</u>: Let V be fin dim & $T \in \mathcal{L}(V)$. Denote the minimal poly of T by p.
T is invertible $\Leftrightarrow$ constant term in p is $\neq 0$.

<u>Proof</u>: Recall $T - \lambda I$ not invertible $\Leftrightarrow$ λ eigenvalue for T. Therefore

T not invertible $\Leftrightarrow$ 0 is an eigenval
$\Leftrightarrow$ 0 is a zero for p
$\Leftrightarrow$ constant term for $p = 0$. $\square$

<u>Thm</u>: Every linear operator on an odd-dim real vector space has an eigenvalue.

<u>Remark</u>. Recall that every lin op on any <u>cpx</u> vector space has an eigenvalue & how it's a reflection of the fundamental thm of algebra

(any cpx polynomial has a zero).

The above thm is in the same vein a reflection of the fact that any real polynomial of odd degree has a zero.