MAT 310 L12 Tue Mar 11

We will now work our way towards eigenvalues/eigenvectors & one of the most important results in linear algebra: Every linear map $V \to V$ on a complex vector space has an eigenvalue.

§ Invariant Subspaces (5A)

As usual $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and every vector space is over $\mathbb{F}$ unless mentioned otherwise.

Def: A linear map $V \to V$ is called a linear operator

Def: Let $T \in \mathcal{L}(V)$. A subspace $U \subset V$ is an invariant under T if $Tu \in U$ for all $u \in U$.

Ex: $D: \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$, $Dp = p'(x)$

Then any $\mathcal{P}_m(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$ for $m \geq 0$ is invariant under D because if $p \in \mathcal{P}_m(\mathbb{R})$ is of deg $1 \leq k \leq m$ then

$Dp = p'(x)$ is of degree $k-1 \leq m$ so $Dp \in \mathcal{P}_m(\mathbb{R})$.

<u>Ex</u>: If V is any V. sp. over $\mathbb{F}$ and $T \in \mathcal{L}(V)$, then we always have the following invariant subspaces:

(1) $\{0\}$ is invariant since $T(0) = 0$

(2) V is invariant since $Tv \in V$ $\forall v \in V$ by definition of T.

(3) null T is invariant. If $v \in$ null T Then $Tv = 0$, so $T(Tv) = T(0) = 0$ meaning $Tv \in$ null T.

(4) range T is invariant. If $v \in$ range T then $v = Tw$ for some $w \in V$.

So $Tv = T(Tw)$ and so $Tv \in$ range T.

If V is a vector space, and $v \in V$, then $\text{Span}(v) = \{\lambda v \mid \lambda \in \mathbb{F}\} \subset V$ is a 1-dimensional subspace. Invariant 1-dimensional subspaces are special and of fundamental importance.

Note that the condition that $\mathrm{span}(v)$ is an invariant subspace of $T \in \mathcal{L}(V)$ means $Tw \in \mathrm{span}(v)$ for all $0 \neq w \in \mathrm{span}\, v$, so for some $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq 0$:

$$w = \alpha v \quad \text{and} \quad Tw = \alpha Tv = \beta v$$

$$\Longleftrightarrow Tv = \frac{\beta}{\alpha} v \Longleftrightarrow Tv = \lambda v$$

for some $\lambda \in \mathbb{F}$.

This leads us to:

<u>Def.</u> Let $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an <u>eigenvalue</u> if there is some $v \neq 0$ such that $Tv = \lambda v$.

<u>Remark</u>: We require $v \neq 0$ because any $\lambda \in \mathbb{F}$ satisfies the equ $T0 = \lambda \cdot 0$.

Prop: Let $T \in \mathcal{L}(V)$. The v.sp. V has a 1-dim invariant subspace under T iff T has an eigenvalue.

Proof: $\Longrightarrow$: This is proven above (before the def of eigenvalue).

$\Longleftarrow$: If $Tv = \lambda v$, then let $0 \neq w \in \mathrm{span}(v)$ be arbitrary, so $w = \alpha v$ for some $\alpha \in \mathbb{F}$.

$$Tv = \lambda v \Leftrightarrow \alpha T v = \alpha \lambda v$$
$$\Leftrightarrow Tw = \alpha \lambda v \in \mathrm{span}(v),$$ so $\mathrm{span}(v) \subset V$ is invariant under T. $\square$

Ex: $$T: \mathbb{F}^3 \longrightarrow \mathbb{F}^3$$
$$(x,y,z) \longmapsto (7x+3z, 3x+6y+9z, -6y)$$

Then $T(3,1,-1) = (18,6,-6) = 6(3,1,-1)$

So $6 \in \mathbb{F}$ is an eigenvalue.

Prop: Let V be fin dim, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The following are equivalent:

(a) λ is an eigenvalue of T
(b) $T-\lambda I$ is not injective
(c) $T-\lambda I$ is not surjective
(d) $T-\lambda I$ is not invertible

<u>Proof</u>: (b), (c), (d) are equivalent since V is fin dim, by 3.65.

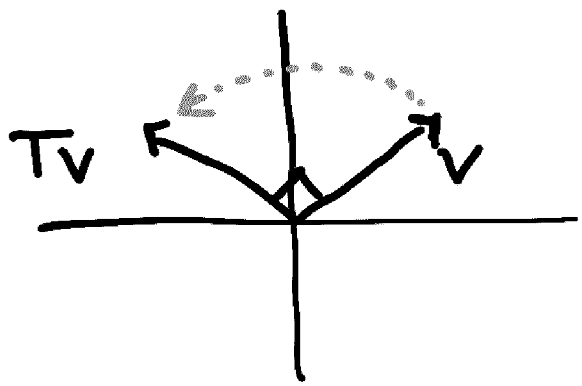
<u>(a) $\Rightarrow$ (b)</u>: If λ is an eigenvalue we have $\exists v \neq 0 : Tv = \lambda v \Leftrightarrow (T-\lambda I)v = 0$
so $\text{null}(T-\lambda I) \neq \{0\} \Leftrightarrow T-\lambda I$ not injective

<u>(b) $\Rightarrow$ (a)</u>: If $\text{null}(T-\lambda I) \neq \{0\}$ then $\exists v \neq 0 : (T-\lambda I)v = 0 \Leftrightarrow Tv = \lambda v$. $\square$

<u>Def</u>: Let $T \in \mathcal{L}(V)$ and let $\lambda \in \mathbb{F}$ be an eigenvalue for T. A vector v is called an <u>eigenvector</u> if $v \neq 0$ and $Tv = \lambda v$.

<u>Ex</u> (1) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(z,w) = (-w,z)$

This linear operator is a $90°$ counter-clockwise rotation in the plane:



This linear operator has no eigenvalues (and therefore no eigenvectors).

(2) $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $T(z,w) = (-w, z)$
now the geometric meaning is less obvious. But there are now eigenvalues. How to find them?

Must solve $T(z,w) = \lambda(z,w)$
& find $\lambda \in \mathbb{F}$ s.th. there are non-zero solutions.

$$T(z,w) = (-w,z) = \lambda(z,w)$$

$$\Leftrightarrow \begin{cases} w = -\lambda z \\ z = \lambda w \end{cases} \begin{matrix} \Rightarrow w = -\lambda z = -\lambda^2 w \\ \Leftrightarrow (\lambda^2+1) w = 0. \end{matrix}$$

w can't be 0 because then $z=0$.
So we must have $\lambda^2+1=0 \Leftrightarrow \lambda = \pm i$

so T has two eigenvalues: i & $-i$.

<u>Thm</u>: Let $T \in \mathcal{L}(V)$. If $\lambda_1, \dots, \lambda_m$ are eigenvalues that are all different, and $(v_1, \dots, v_m)$ is a list of eigenvectors of the corresponding eigenvalues, then $(v_1, \dots, v_m)$ is lin indep.

<u>Proof</u>: By assumption $Tv_i = \lambda_i v_i$ for all $i \in \{1, \dots, m\}$. Assume for a contradiction that it's false, i.e, that $(v_1, \dots, v_m)$ is lin dep.

Then there is some minimal $k \leq m$ such that $(v_1, \dots, v_k)$ is lin dep, so can find $a_1, \dots, a_k \in \mathbb{F}$ (all $\neq 0$) so that
$$a_1 v_1 + \dots + a_k v_k = 0.$$

Apply $T - \lambda_k I$ to both sides:

$(T - \lambda_k I)(a_1 v_1 + \dots + a_k v_k)$

$= (a_1\lambda_1 v_1 - a_1\lambda_k v_1) + \cdots + (a_k\lambda_k v_k - a_k\lambda_k v_k)$

$= a_1(\lambda_1 - \lambda_k)v_1 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0.$

Since $\lambda_1 - \lambda_k, \ldots, \lambda_{k-1} - \lambda_k$ are all $\neq 0$ it shows $(v_1, \ldots, v_{k-1})$ lin dep, which is a contradiction. $\square$

<u>Prop</u>: Let V be fin dim. Any $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvalues.

<u>Proof</u>: If $\lambda_1, \ldots, \lambda_m$ are all distinct eigenvalues of T. Then $(v_1, \ldots, v_m)$ is lin indep, so $m \leq \dim V$. $\square$

<u>Def</u>. Let $T \in \mathcal{L}(V)$. We use the notation

- $T^m := \underbrace{T \cdots T}_{m \text{ factors}} = \underbrace{T \circ \cdots \circ T}_{m \text{ times}}$ for any $m \geq 1$.
- $T^0 := I$

• $T^{-m} := (T^{-1})^m$ (assuming that T is invertible).

Exercise: Verify $T^m T^n = T^{m+n}$ and $(T^m)^n = T^{mn}$.

Def: Let $T \in \mathcal{L}(V)$ & $p \in \mathcal{P}(\mathbb{F})$ be a polynomial

$$p(z) = a_0 + a_1 z + \dots + a_m z^m.$$

Define

$$p(T) := a_0 I + a_1 T + \dots + a_m T^m.$$

Ex: $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$, $Dp = p'(x)$.

If $q(x) = 7 - 3x + 5x^2$, then

$q(D) = 7I - 3D + 5D^2$ and

$$(q(D))p = (7I - 3D + 5D^2)p$$

$$= 7p(x) - 3Dp + 5D(Dp)$$

$$= 7p(x) - 3p'(x) + 5p''(x).$$

<u>Prop</u>: If $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$, then null $p(T)$ and range $p(T)$ are invariant under T.

<u>Proof</u>: Let $v \in$ null $p(T)$. Then $p(T)(v) = 0$. So we need to prove $Tv \in$ null $p(T)$. We have

$$p(T)(Tv) = (p(T)T)(v) = (Tp(T))(v)$$
$$= T(p(T)v) = T(0) = 0$$

So $Tv \in$ null $p(T)$.

Similarly let $v \in$ range $p(T)$, i.e., $\exists w \in V$: $p(T)w = v$. Apply T to both sides:

$$Tv = T(p(T)w) = p(T)(Tw)$$

$\Leftrightarrow$ $Tv \in$ range $p(T)$. $\square$
