<u>Recall</u>: • $U \subset V$ subspace, then
$V/U = \{v+U \mid v \in V\} = \{\text{translates of } U\}$
is a vector space w/ addition and scalar mult

- $(v+U)+(w+U) = (v+w)+U$
- $\lambda(v+U) = (\lambda v)+U$.

<u>Def</u>: Let $U \subset V$ be a subspace. The <u>quotient map</u> $\pi: V \to V/U$ is the linear map def by

$$\pi(v) = v+U.$$

<u>Prop</u>. Let $U \subset V$ be a subspace, and assume V is fin dim. Then

$$\dim V/U = \dim V - \dim U.$$

<u>Proof</u>: For the quotient map $\pi: V \to V/U$, we see that it's obviously surjective, so range $\pi = V/U$

Its null space is

$$\text{null } \pi = \{v \in V \mid \pi(v) = 0 + U\}$$
$$= \{v \in V \mid v + U = 0 + U$$
$$= \{v \in V \mid v - 0 \in U\} = U.$$

Fund thm for linear maps gives

$$\dim V = \dim \text{null } \pi + \dim \text{range } \pi$$
$$= \dim U + \dim V/U. \qquad \square$$

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§ Duality (3F)

Def. Let V be a vector space over $\mathbb{F}$. A linear functional is a linear map $V \longrightarrow \mathbb{F}$.

Ex • $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\varphi(x, y, z) = 4x - 5y + 2z$

• $\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}$, $\varphi(p) = \int_0^1 p(x)\,dx$.

Def. The dual space of V is defined as $V' := \mathcal{L}(V, \mathbb{F})$.

Prop: $\dim V' = \dim V$

Proof:
$$\dim V' = \dim \mathcal{L}(V,\mathbb{F}) = (\dim V)(\dim \mathbb{F}) = \dim V. \qquad \square$$

Def: If $v_1,\dots,v_n$ is a basis for V, the dual basis is the list $(\varphi_1,\dots,\varphi_n)$ of linear functionals s.th.
$$\varphi_i : V \to \mathbb{F}, \quad \varphi_i(v_k) = \begin{cases} 1 & \text{if } k=i \\ 0 & \text{if } k\neq i \end{cases}$$
for any $i=1,\dots,n$.

Ex: Consider $\mathbb{F}^n$ & its standard basis $e_1,\dots,e_n$ where $e_i = (0,\dots 0,\underset{\text{i-th pos.}}{1},0,\dots,0)$.
Let's describe its dual basis $\varphi_1,\dots,\varphi_n$. The linear map $\varphi_i : \mathbb{F}^n \to \mathbb{F}$ is defined as $\varphi_i(x_1,\dots,x_n) = x_i$.

<u>Prop</u>: If $v_1,\dots,v_n$ is a basis for V, the dual basis $\varphi_1,\dots,\varphi_n$ satisfies

1. The dual basis is a basis for V'
2. For any $v\in V$,

$$v=\varphi_1(v)v_1+\cdots+\varphi_n(v)v_n.$$

$\square$

<u>Def</u>: Suppose $T\in\mathcal{L}(V,W)$. The <u>dual map</u> is $T'\in\mathcal{L}(W',V')$ defined for each $\varphi\in W'$ by

$$T'(\varphi):=\varphi\circ T: V\xrightarrow{T}W\xrightarrow{\varphi}\mathbb{F}$$

<u>Remark</u>: That T' is linear follows from properties of the composition:

$$T'(\varphi+\psi)=(\varphi+\psi)\circ T=\varphi\circ T+\psi\circ T$$
$$=T'(\varphi)+T'(\psi)$$

Homogeneity is similar.

<u>Ex</u>: Consider $D:\mathcal{P}(\mathbb{R})\longrightarrow\mathcal{P}(\mathbb{R})$ $Dp=p'(x)$. Let's find D' (the dual map) for some linear functional on

$\mathcal{P}(\mathbb{R})$. Take e.g. $\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}$
$\varphi(p) = p(0)$. Then

$D'(\varphi) = \varphi \circ D : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}$ is
def as $(D'(\varphi))(p) = \varphi(Dp) = \varphi(p'(x))$
$= p'(0)$.

If we instead take
$\varphi(p) = \int_0^1 p(x)\,dx$ we get

$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(p'(x))$
$= \int_0^1 p'(x)\,dx = p(1) - p(0)$.

<u>Prop</u>: Let $T \in \mathcal{L}(V,W)$.

(a) $(S+T)' = S' + T'$ $\forall S \in \mathcal{L}(V,W)$

(b) $(\lambda T)' = \lambda T'$ $\forall \lambda \in \mathbb{F}$

(c) $(ST)' = T'S'$ $\forall S : \mathcal{L}(W,U)$.

<u>Proof</u>: (a) and (b): Exercise.

(c) $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(W,U)$

and $T' \in \mathcal{L}(W'; V')$, $S' \in \mathcal{L}(U', W')$.
Let $\varphi \in U'$. Then

$$(ST)'(\varphi) = \varphi(ST) = (\varphi S)T$$
$$= T'(\varphi S) = T'(S'(\varphi)) = (T'S')(\varphi) \quad \square$$

<u>Remark</u>: Items (a) and (b) above gives that we can define a linear map

$$\mathcal{L}(V, W) \longrightarrow \mathcal{L}(W', V')$$
$$T \longmapsto T'.$$

Let's now study the null space & range of the dual map.

<u>Def</u>: Let $U \subset V$ be a subspace. The <u>annihilator</u> of U, denoted U^0, is

$$U^0 := \{\varphi \in V' \mid \varphi(u) = 0 \ \forall u \in U\}$$

<u>Ex</u>: Let $U = \{x^2 p(x) \mid p \in \mathcal{P}(\mathbb{R})\} \subset \mathcal{P}(\mathbb{R})$.
Any $q \in U$ is of the form

$$q(x) = x^2 p(x) = x^2(a_0 + a_1 x + \cdots + a_n x^n)$$
$$= a_0 x^2 + a_1 x^3 + \cdots + a_n x^{n+2};$$ So for example

$$\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}, \; \varphi(p) = p(0)$$

is in U^0 because

$$\varphi(q) = q(0) = 0 \text{ for any } q \in U$$

$$\psi : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}, \; \psi(p) = p'(0)$$

also belongs to U^0.

<u>Ex</u>: Let e_1, e_2, e_3 be Std basis for $\mathbb{R}^3$, and let $\varphi_1, \varphi_2, \varphi_3$ be the dual basis. Let

$$U = \text{Span}(e_1, e_2) = \{(x_1, x_2, 0 \mid x_1, x_2 \in \mathbb{R}\}.$$

Then let's show $U^0 = \text{span}(\varphi_3)$

Recall
$$\varphi_j(x_1, x_2, x_3) = x_j \text{ for } j = 1, 2, 3.$$

First if $\varphi = \text{span}(\varphi_3)$, i.e., $\varphi = \lambda \varphi_3$ for some $\lambda \in \mathbb{R}$, and $u \in U$ i.e.

$u = c_1 e_1 + c_2 e_2$, then

$(\lambda \varphi_3)(u) = \lambda \varphi_3(u) = \lambda \cdot 0 = 0$
because u has no 3rd coord.
So $\mathrm{span}(\varphi_3) \subset U^0$. To show the other inclusion, assume
$\varphi \in U^0$, i.e., $\varphi(u) = 0 \; \forall u \in U$.
Write $\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$. Then

$(c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3)(u)$

$= c_1\varphi_1(u) + c_2\varphi_2(u) + c_3\varphi_3(u) = 0$
$\forall u$ of the form $u = (x_1, x_2, 0)$.
In particular $\varphi_3(u) = 0$ is always true,
so we get

$$c_1\varphi_1(u) + c_2\varphi_2(u) = 0.$$

If $u = e_1$ then it leads to $c_1 = 0$
If $u = e_2$ — " ———— $c_2 = 0$
& c_3 is arbitrary. So $\varphi = c_3\varphi_3$ and this shows $U^0 \subset \mathrm{span}(\varphi_3)$ and therefore $U^0 = \mathrm{span}(\varphi_3)$.

Prop: Let $U \subset V$ be a subspace. Then $U^0 \subset V$ is a subspace.

Proof: Exercise. $\square$

Prop: If V is fin dim and $U \subset V$ is a subspace, then
$\dim U^0 = \dim V - \dim U$

Proof idea: Consider the linear map $i : U \longrightarrow V$, $i(u) = u$. Apply fundamental thm for linear maps to i' (its dual map). $\square$

Remark: This proposition implies $U^0 \cong V/U$.
