Name: _____ ID: _____

Answer each question completely. You must justify your answers to get credit. Even a correct answer with no justification will get no credits. Each problem is worth 5 points.

1. Consider the following vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}a\\b\\c\end{bmatrix}.$$

The vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ form a basis of $\mathbb{R}^3$ for any values of a, b, and c (you do not need to prove this). Find the values of a, b and c so that $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is an orthonormal basis.

Solution. We can check that $\vec{v}_1$ and $\vec{v}_2$ are both unit vectors, and orthogonal to each other. In order for $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ to be an orthonormal basis, we need $||\vec{v}_3|| = 1$, $\vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$. This gives a system of equations

$$\begin{cases} \sqrt{a^2+b^2+c^2} = 1 \\ \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 0 \\ \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = 0 \end{cases}.$$

We see that we can set $a = b = 0$, and then the first equation implies $c = 1$. So when $(a, b, c) = (0, 0, 1)$, $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is an orthonormal basis. □

2. Let A be an $(n \times n)$-matrix.

(a) (1 pt) Write down (in terms of matrix operations) what it means for A to be orthogonal. No justification needed.

Solution. $A^T = A^{-1}$ □

(b) (1 pt) Write down (in terms of matrix operations) what it means for A to be symmetric. No justification needed.

Solution. $A^T = A$ □

(c) (3 pts) Suppose A is both orthogonal and symmetric. Compute A^2.

Solution. Since A is orthogonal, it means $A = (A^T)^{-1}$. Then

$$A^2 = AA = (A^T)(A^T)^{-1} = I.$$

We used symmetry in the first factor, and orthogonality in the second factor. □