

§6.1 Determinants.

Let's first consider (2×2) -matrices

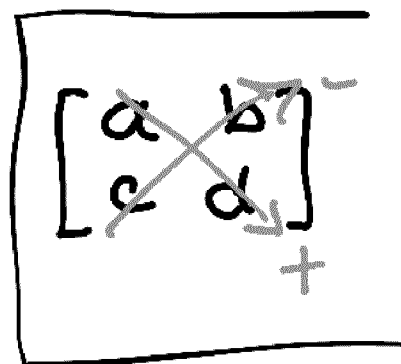
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant is a

number that's supposed to be able to tell us when/if A is invertible.

Def: The determinant of

a (2×2) -matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



Theorem

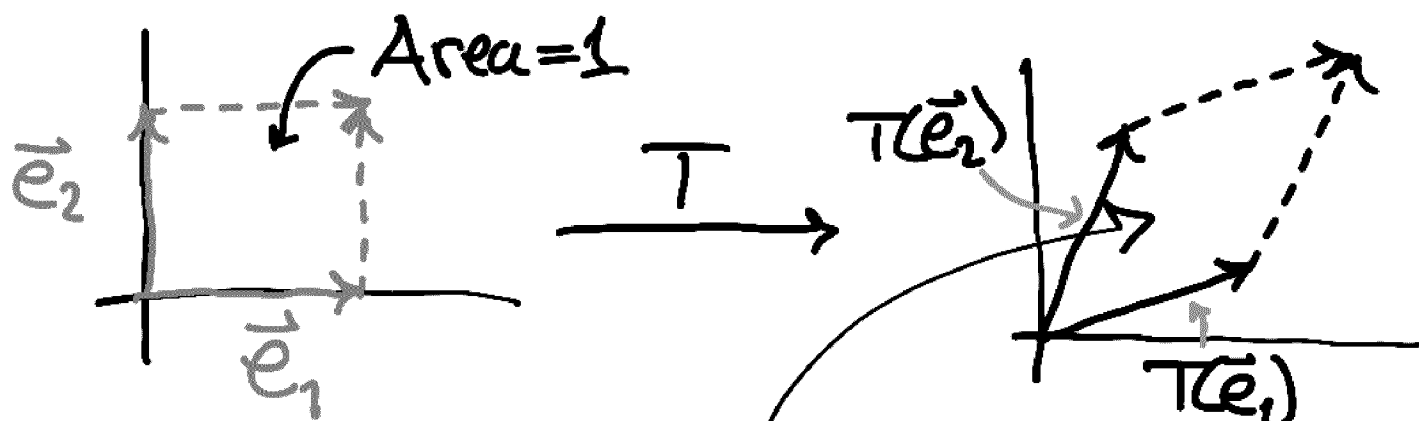
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $\det A \neq 0$.

In this case the inverse is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The determinant is actually a geometric quantity.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(\vec{x}) = A\vec{x}$.



Columns of $A = [T(\vec{e}_1), T(\vec{e}_2)]$

Area of parallelogram spanned by $T(\vec{e}_1), T(\vec{e}_2)$
 $= |\det A|$.

Ex: For which values on k is $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$ invertible?

$$\det A = (1-k)(3-k) - 8 = 3 - k - 3k + k^2 - 8$$

$$= k^2 - 4k - 5 = (k+1)(k-5) = 0$$

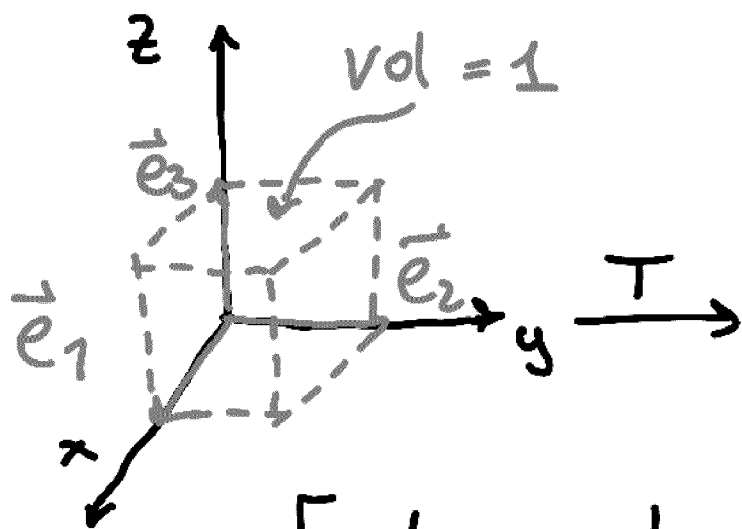
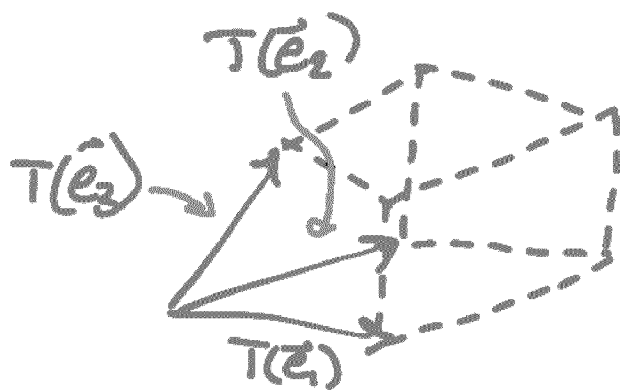
So $\det A = 0 \Leftrightarrow k = -1$ or $k = 5$.

$\Rightarrow A$ is invertible if $k \neq -1, 5$

Now let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

the (absolute value of the) determinant of A is a volume.

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3, T(\vec{x}) = A\vec{x}$$



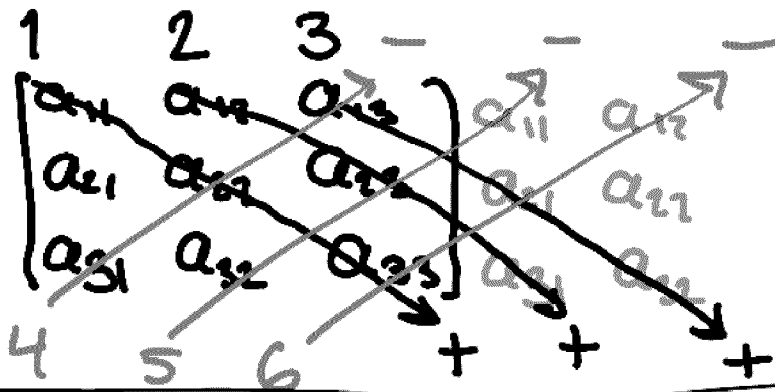
$$A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix}$$

$$\text{Vol} = |\det A|.$$

Sarrus rule to compute:

"Extend" the matrix by 2 columns to the right:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$



$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Ex: $A = \begin{bmatrix} 1 & -2 & -3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ then

$$\begin{bmatrix} 1 & -2 & -3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{array}{cc} 1 & -2 \\ 1 & 1 \\ 3 & 2 \end{array}$$

$$\begin{aligned}\det A &= (1)(1)(1) + (-2)(1)(3) + (-3)(1)(2) \\ &\quad - (3)(1)(-3) - (2)(1)(1) - (1)(1)(-2) \\ &= 1 - 6 - 6 + 9 - 2 + 2 = -2.\end{aligned}$$

The determinant is defined for any $(n \times n)$ -matrix, but Sarrus rule does not generalize, so you should not try to use it for any other matrices but 3×3 .

Thm: For an $(n \times n)$ -matrix A , A is invertible if and only if $\det A \neq 0$.

A matrix of the form

$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ is called upper triangular.

Similarly

$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ is called lower triangular.

Thm: If A is upper or lower triangular, then

$\det A = \text{product of diagonal elements.}$

Proof: We just know how to compute $\det A$ for 2×2 and (3×3) -matrices, so we prove it in those cases.

$$\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac - 0 \cdot b = ac$$

$$\begin{vmatrix} a & 0 \\ b & c \end{vmatrix} = ac - b \cdot 0 = ac$$

$$\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} \begin{vmatrix} a & b \\ 0 & d \\ 0 & 0 \end{vmatrix}$$

$$= adf + be \cdot 0 + c \cdot 0 \cdot 0$$

$$- 0 \cdot dc - 0 \cdot ea - f \cdot 0 \cdot b = adf$$

lower triangular is similar. $\square$

Ex: For which values of λ is $A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & 1 & \lambda \end{bmatrix}$ invertible?

A invertible $\Leftrightarrow \det A \neq 0$

$$\begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & 1 & \lambda \end{vmatrix} \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \\ 1 & 1 \end{vmatrix} = \lambda^3 - 1 + 1 - \lambda + \lambda - \lambda$$

$$= \lambda^3 - \lambda = \lambda(\lambda^2 - 1)$$

$$= \lambda(\lambda + 1)(\lambda - 1) \neq 0$$

$\Leftrightarrow \lambda \neq 0, \pm 1$. So for any value

$\lambda \neq 0, \pm 1$, then A is invertible.

Thm: $\det(A^T) = \det(A)$.

Proof: We show it in the case of 2×2 and (3×3) -matrices.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

$$\det A^T = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

We compare & pair together the

terms & see they are equal. $\square$