

§ 4.1 Vector spaces

So far in this course we have been discussing vectors in $\mathbb{R}^n$.

We defined a subspace V of $\mathbb{R}^n$ to be a set of vectors satisfying some axioms.

We will now discuss and define "vectors" more abstractly.

What is it that makes $\mathbb{R}^n$ a "space of vectors"?

Def (Vector space)

A vector space is a set V equipped with a rule for addition

(so if f and g are in V , then $f+g$ is in V),

and also a rule for scalar mult

(So if k is a scalar, and f is in V , then kf is in V)
Such that the following 8 axioms are satisfied:

(1) $(f+g)+h = f+(g+h)$

(2) $f+g = g+f$

(3) There exists a neutral element e such that $f+e = f$ for all f in V . This neutral element is unique and is denoted by 0 .

(4) For each f in V , there exists some g in V such that $f+g = 0$. This g is unique, and is denoted by $(-f)$.

(5) $k(f+g) = kf + kg$ (k scalar)

(6) $(c+k)f = cf + kf$ (c, k scalars)

(7) $c(kf) = (ck)f$

(8) $1f = f$.

Ex: $\mathbb{R}^n$ is a vector space.
The neutral element is $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

The main reason we introduce this abstract notion of a vector space is because in mathematics we care about other vector spaces than just $\mathbb{R}^n$.

Ex: Let $F(\mathbb{R}, \mathbb{R}) = \{\text{functions } f: \mathbb{R} \rightarrow \mathbb{R}\}$.

With the following rules of addition & scalar multiplication:

$$\begin{cases} (f+g)(x) = f(x) + g(x) \\ (kf)(x) = k f(x) \end{cases}$$

The neutral element is the zero function $f(x) = 0$ for all x .

With all of this $F(\mathbb{R}, \mathbb{R})$ is a vector space!

Ex: $\mathbb{R}^{n \times m} = \{ (n \times m)\text{-matrices} \}$

The rules for addition and scalar multiplication are the usual addition & scalar mult for matrices $A+B$ and kA .

The neutral element is the zero matrix: $O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

Ex: $\mathbb{R}^\infty = \{ \text{infinite sequences} \}$

$(x_0, x_1, x_2, \dots)$

We equip $\mathbb{R}^\infty$ with the following rules for addition & scalar multiplication:

$$\begin{cases} (x_0, x_1, \dots) + (y_0, y_1, \dots) = (x_0 + y_0, x_1 + y_1, \dots) \\ k(x_0, x_1, x_2, \dots) = (kx_0, kx_1, kx_2, \dots) \end{cases}$$

The neutral element is $(0, 0, 0, \dots)$

Since

$$(x_0, x_1, \dots) + (0, 0, \dots) = (x_0 + 0, x_1 + 0, \dots)$$

$$= (x_0, x_1, \dots).$$

Ex: $P_m(\mathbb{R}) = \{ \text{polynomials of degree at most } m \}.$

$$f(x) = a_0 + a_1 x + \dots + a_m x^m$$

$$g(x) = b_0 + b_1 x + \dots + b_m x^m$$

The rules for addition and scalar multiplications are

$$(f+g)(x) = f(x) + g(x)$$

$$= (a_0 + a_1 x + \dots + a_m x^m)$$

$$+ (b_0 + b_1 x + \dots + b_m x^m)$$

$$= (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_m + b_m)x^m$$

The concept of subspace now generalizes to any vector space.
Let us repeat:

Def: Suppose V is a vector space.

A subspace U of V is a

Subset of V such that

(1) U contains the neutral element 0 of V

(2) if f and g are in U , then $f+g$ is in U

(3) if k is a scalar, and f is in U , then kf is in U .

Ex $P_n(\mathbb{R})$ is a subspace of $F(\mathbb{R}, \mathbb{R})$.

EX: Consider $\mathbb{R}^{n \times n}$. The trace of a matrix $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$

is the sum of the diagonal elements
 $\text{tr } A = a_{11} + a_{22} + \dots + a_{nn}$.

Then let $sl(n) = \{A \text{ in } \mathbb{R}^{n \times n} \mid \text{tr } A = 0\}$

Then $sl(n)$ is a subspace of $\mathbb{R}^{n \times n}$.

This is because

If $B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}$ then

clearly

$$\text{tr}(A+B) = \text{tr} \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \dots & a_{nn}+b_{nn} \end{bmatrix}$$

$$= (a_{11}+b_{11}) + \dots + (a_{nn}+b_{nn})$$

$$= (a_{11} + \dots + a_{nn}) + (b_{11} + \dots + b_{nn})$$

$$= \text{tr} A + \text{tr} B.$$

So if $\text{tr} A = \text{tr} B = 0$ then

$\text{tr}(A+B) = 0$, so $A+B$ is in $\mathfrak{sl}(n)$

if A and B are in $\mathfrak{sl}(n)$. Similarly

$$\begin{aligned} \text{tr}(kA) &= ka_{11} + \dots + ka_{nn} = k(a_{11} + \dots + a_{nn}) \\ &= k \text{tr} A, \end{aligned}$$

so if A is in $\mathfrak{sl}(n)$, meaning $\text{tr} A = 0$,
then $\text{tr}(kA) = 0$ so kA is in $\mathfrak{sl}(n)$.

§5.5 Inner product spaces

The dot product in $\mathbb{R}^n$ is an example of a so-called inner product in a general vector space.

Def. Let V be a vector space.

An inner product is a rule that assigns to two vectors f and g , a scalar $\langle f, g \rangle$ such that the following properties hold for all f, g, h in V and all scalars k :

$$(1) \langle f, g \rangle = \langle g, f \rangle \quad (\text{symmetry})$$

$$(2) \langle f+h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$$

$$(3) \langle kf, g \rangle = k \langle f, g \rangle$$

$$(4) \langle f, f \rangle > 0 \quad \text{for all } f \neq 0$$

(positive definiteness)

A vector space that is equipped

with an inner product is called an inner product space.

Ex: The dot product in $\mathbb{R}^n$
 $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n$
is an inner product in $\mathbb{R}^n$.

Ex: Consider $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$,
define

$$\langle \vec{x}, \vec{y} \rangle = 2x_1 y_1 + x_2 y_2.$$

Then $\langle \cdot, \cdot \rangle$ is an inner product.

Note that b/c of the 2, $\langle \cdot, \cdot \rangle$ is not the dot product.

Ex: Consider the vector space
 $C([0,1], [0,1])$ which consists of

all continuous functions
 $f: [0,1] \rightarrow [0,1]$

Then we can consider
 $\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$

This in fact defines an inner product.

Ex. Consider the vector space
 $\mathbb{R}^{n \times m}$ of $(n \times m)$ -matrices.

Defining

$$\langle A, B \rangle = \text{tr}(A^T B)$$

is in fact an inner product.

The symmetry & positive definiteness axioms need careful checks:

+ the observation $\text{tr}(C) = \text{tr}(C^T)$
for any $(n \times n)$ -matrix C .

Symmetry:

$$\begin{aligned}\langle A, B \rangle &= \text{tr}(A^T B) = \text{tr}((A^T B)^T) \\ &= \text{tr}(B^T (A^T)^T) = \text{tr}(B^T A) \\ &= \langle B, A \rangle\end{aligned}$$

Positive definiteness:

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix}, \quad A^T = \begin{bmatrix} - & \vec{v}_1^T & - \\ & \vdots & \\ - & \vec{v}_m^T & - \end{bmatrix}$$

$$\begin{aligned}A^T A &= \begin{bmatrix} - & \vec{v}_1^T & - \\ & \vdots & \\ - & \vec{v}_m^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} \vec{v}_1^T \vec{v}_1 & \dots & \vec{v}_1^T \vec{v}_m \\ \vdots & \ddots & \vdots \\ \vec{v}_m^T \vec{v}_1 & \dots & \vec{v}_m^T \vec{v}_m \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{tr}(A^T A) &= \vec{v}_1^T \vec{v}_1 + \dots + \vec{v}_m^T \vec{v}_m \\ &= \|\vec{v}_1\|^2 + \dots + \|\vec{v}_m\|^2 > 0\end{aligned}$$

if $A \neq 0$ (so not all $\vec{v}_i = 0$).
