MAT 203 FINAL EXAM

WEDNESDAY DECEMBER 10, 2025
2:15–5:00PM

Name: ______________________________ ID: ______________

Instructions.

1. Fill in your name and Stony Brook ID number.

2. This exam is closed-book; no electronic devices. You are only allowed to have one (1) sheet of your own notes.

3. You have 2 hours and 30 minutes to complete this exam.

4. You must justify all your answers and show all your work. Even a correct answer without any justification will result in no credit.

1. (a) (5 pts) Compute $(\vec{u}+\vec{v})\cdot(\vec{u}\times\vec{v})$, where $\vec{u}=\langle 0,1,2\rangle$ and $\vec{v}=\langle -2,1,1\rangle$.

Solution. First we have $\vec{u}+\vec{v}=\langle -2,2,3\rangle$. Then

$$\vec{u}\times\vec{v}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ -2 & 1 & 1 \end{vmatrix}=\langle 1-2,-4,-(-2)\rangle=\langle -1,-4,2\rangle .$$

Then we have

$$(\vec{u}+\vec{v})\cdot(\vec{u}\times\vec{v})=\langle -2,2,3\rangle\cdot\langle -1,-4,2\rangle=2-8+6=0.$$

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(b) (5 pts) Consider the vector $\vec{w}=\langle a,b,c\rangle$ and compute $\vec{w}\times\vec{w}$.

Solution. We have

$$\vec{w}\times\vec{w}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ a & b & c \end{vmatrix}=\langle bc-bc,ac-ac,ab-ab\rangle=\langle 0,0,0\rangle .$$

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2. In each of the following problems, either compute the limit, or show that it does not exist.

(a) (5 pts) $\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2-y^2}$

Solution. We have $x^2 - y^2 = (x-y)(x+y)$, so

$$\frac{x+y}{x^2-y^2} = \frac{x+y}{(x-y)(x+y)} = \frac{1}{x-y},$$

and the limit as $(x, y) \to (0, 0)$ does not exist. □

(b) (5 pts) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$

Solution. Similar to part (a), we have

$$\frac{x^2-y^2}{x+y} = x - y \longrightarrow 0$$

as $(x, y) \to (0, 0)$. □

3. Consider the function $f(x,y,z) = z + e^{2x-y}$.

(a) (2 pts) Compute the gradient $\nabla f(x,y,z)$.

Solution. We compute:

$$\nabla f(x,y,z) = \left\langle 2e^{2x-y}, -e^{2x-y}, 1 \right\rangle .$$

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(b) (3 pts) Compute the divergence $\nabla \cdot \nabla f(x,y,z)$.

Solution. From part (a) we have $\nabla f(x,y,z) = \langle 2e^{2x-y}, -e^{2x-y}, 1\rangle$, so the divergence is

$$\nabla \cdot \nabla f(x,y,z) = \frac{\partial}{\partial x}(2e^{2x-y}) + \frac{\partial}{\partial y}(-e^{2x-y}) + \frac{\partial}{\partial z}(1) = 4e^{2x-y} + e^{2x-y} = 5e^{2x-y}.$$

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(c) (5 pts) Compute the curl $\nabla \times \nabla f(x,y,z)$.

Solution. From part (a) we have $\nabla f(x,y,z) = \langle 2e^{2x-y}, -e^{2x-y}, 1\rangle$, so the curl is

$$\begin{aligned}\nabla \times \nabla f(x,y) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2e^{2x-y} & -e^{2x-y} & 1 \end{vmatrix} \\ &= \left\langle \frac{\partial(1)}{\partial y} - \frac{\partial(-e^{2x-y})}{\partial z}, \frac{\partial(2e^{2x-y})}{\partial z} - \frac{\partial(1)}{\partial x}, \frac{\partial(-e^{2x-y})}{\partial x} - \frac{\partial(2e^{2x-y})}{\partial y} \right\rangle \\ &= \langle 0, 0, -2e^{2x-y} - (-2e^{2x-y})\rangle = \langle 0,0,0\rangle \end{aligned}$$

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4. Consider the surface S in space given by the equation $z = x^2 + xy + y^2$.

(a) (7 pts) Find the equation for the plane that is tangent to the surface S at the point $(x, y, z) = (2, -1, 3)$.

Solution. If the equation for the surface is $f(x, y) = 0$, the equation for the tangent plane at the point (x_0, y_0, z_0) is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) - (z - z_0) = 0.$$

With the function $f(x, y) = x^2 + xy + y^2$, we find

$$\frac{\partial f}{\partial x}(2, -1, 3) = 2x + y|_{(x,y)=(2,-1)} = 3$$
$$\frac{\partial f}{\partial y}(2, -1, 3) = x + 2y|_{(x,y)=(2,-1)} = 0$$

and therefore the tangent plane is given by

$$3(x - 2) - 0(y + 1) - (z - 3) = 0 \iff 3x - z = 3.$$

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(b) (3 pts) Find a unit normal vector to the surface S at the point $(2, -1, 3)$.

Solution. Since a normal vector to the plane $ax + by + cz = d$ is given by $\vec{n} = \langle a, b, c\rangle$, we read off from the tangent plane in (a) that a normal vector is $\vec{n} = \langle 3, 0, -1\rangle$. To find a unit normal vector we normalize it, and get

$$\frac{\vec{n}}{||\vec{n}||} = \frac{\langle 3, 0, -1\rangle}{||\langle 3, 0, -1\rangle||} = \frac{1}{\sqrt{10}} \langle 3, 0, -1\rangle.$$

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5. Consider the vector field $\vec{F}(x,y) = \langle 3x^2+6xy, 3x^2+6y\rangle$.

(a) (7 pts) Prove that $\vec{F}$ is conservative by finding a potential function ϕ.

Solution. To find a potential function ϕ, we solve the system of equations

$$\begin{cases} \frac{\partial \phi}{\partial x} = 3x^2 + 6xy \\ \frac{\partial \phi}{\partial y} = 3x^2 + 6y \end{cases} . \tag{1}$$

Integrating the first equation with respect to x gives $\phi(x,y) = x^3 + 3x^2y + \psi(y)$. Differentiating this function with respect to y gives $\frac{\partial \phi}{\partial y} = 3x^2 + \psi'(y)$. Plugging this into the second equation in (1) yields

$$3x^2 + \psi'(y) = 3x^2 + 6y,$$

which gives $\psi(y) = 6y$, and consequently $\psi(y) = 3y^2 + C$ after integrating with respect to y. Plugging this back into the above expression for ϕ gives a potential $\phi(x,y) = x^3 + 3x^2y + 3y^2 + C$ for any constant C. □

(b) (3 pts) Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the graph of the function $f(x) = x^3$ for $-1 \leq x \leq 1$, by using your result in part (a).

Solution. From part (a), we found a potential $\phi(x,y)$ for the vector field $\vec{F}(x,y)$. The curve C stats at the point $(-1, f(-1)) = (-1,-1)$, and ends at the point $(1, f(1)) = (1,1)$. By the fundamental theorem for line integrals, we therefore have

$$\int_C \vec{F} \cdot d\vec{r} = \phi(1,1) - \phi(-1,-1) = (1+3+3) - (-1-3+3) = 8.$$

□

6. (10 pts) Consider the region R in space determined by the inequalities $0 \leq z \leq x+y$, $0 \leq y \leq x$, and $1 \leq x \leq 2$. Compute the triple integral

$$\iiint_R 2e^z \, dxdydz.$$

Solution. By the bounds, the triple integral is

$$\iiint_R 2e^z \, dxdydz = \int_1^2 \int_0^x \int_0^{x+y} 2e^z \, dzdydx.$$

We then compute this iterated integral as follows:

$$\begin{aligned}
\int_1^2 \int_0^x \int_0^{x+y} 2e^z \, dxdydz &= \int_1^2 \int_0^x [2e^z]_0^{x+y} dydx \\
&= \int_1^2 \int_0^x 2e^{x+y} - 2 \, dydx \\
&= \int_1^2 [2e^{x+y} - 2y]_0^x \, dx = \int_1^2 (2e^{2x} - 2x) - (2e^x - 0) \, dx \\
&= \left[e^{2x} - 2e^x - x^2\right]_1^2 = \left(e^4 - 2e^2 - 4\right) - \left(e^2 - 2e - 1\right) \\
&= e^4 - 3e^2 + 2e - 3.
\end{aligned}$$

□

7. (10 pts) Consider the vector field $\vec{F}(x, y) = \langle e^x, y + x^2 + xy \rangle$, and let C be the curve in the plane consisting of the upper half of the unit circle and the x-axis, oriented counterclockwise. Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

Solution. Since we are integrating over a curve C that bounds the upper part of the unit disk D in the plane, we use Green's theorem that says that

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dxdy,$$

where $P(x, y) = e^x$ and $Q(x, y) = y + x^2 + xy$. We calculate

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x + y.$$

We parametrize D with polar coordinates:

$$D = \{(r, \theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi\},$$

and we get

$$\begin{aligned} \iint_D 2x + y \, dxdy &= \int_0^\pi \int_0^1 (2r\cos(\theta) + r\sin(\theta)) r \, drd\theta \\ &= \left(\int_0^1 r^2 \, dr \right) \left(\int_0^\pi 2\cos(\theta) + \sin(\theta) \, d\theta \right) \\ &= \frac{1}{3} \left[2\sin(\theta) - \cos(\theta) \right]_0^\pi = \frac{2}{3}. \end{aligned}$$

□

8. (10 pts) Find the global minimum and global maximum of the function $f(x,y) = \frac{x^3}{3} + y^2$ on the unit disk in the xy-plane.

Solution. We first find all critical points in the interior of the disk. Namely,

$$\nabla f(x,y) = \langle x^2, 2y \rangle = \langle 0, 0 \rangle\,,$$

which gives the only critical point $(x,y) = (0,0)$. The boundary is the unit circle, which is described by the equation $x^2 + y^2 = 1 \Leftrightarrow g(x,y) = 0$ where $g(x,y) = x^2 + y^2 - 1$. To find critical points along the boundary we use Lagrange multipliers. Namely, we solve the system of equations

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} \langle x^2, 2y \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} x^2 = 2x\lambda \\ 2y = 2y\lambda \\ x^2 + y^2 = 1 \end{cases}.$$

In the second equation we have two cases. Either $y = 0$, or $y \neq 0$ and $\lambda = 1$. In the first case we get $x = \pm 1$ by the last equation. In the second case we get $x^2 = 2x \Leftrightarrow x(x-2) = 0$ from the first equation which has the solutions $x = 0$ and $x = 2$. Plugging $x = 0$ into the last equation gives $y = \pm 1$, and $x = 2$ does not give any solutions when plugged into the last equation. To summarize, there is one interior critical point at $(0,0)$, and four critical points along the boundary $(0, \pm 1)$, and $(\pm 1, 0)$. We compare their values

$$f(0,0) = 0,\ f(0,\pm 1) = 1,\ f(\pm 1, 0) = \pm \frac{1}{3},$$

and we get that the global minimum is $-\frac{1}{3}$ at the point $(-1, 0)$, and the global maximum is 1 at the two points $(0, \pm 1)$. ☐

9. (10 pts) Consider the surface P parametrized by $\vec{r}(u,v) = \langle u, v, -u+v-1\rangle$ for $0 \le u \le 2$ and $0 \le v \le 1$. Consider the vector field $\vec{F}(x,y,z) = \langle 2x-y, z, y\rangle$ and compute the surface integral $\iint_P \vec{F}\cdot d\vec{r}$.

Solution. Let $D = [0,2]\times[0,1]$. Then by definition the surface integral is given by

$$\int_0^2 \int_0^1 \vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v)\, dvdu,$$

where $\vec{n}(u,v) = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is a normal to the surface. We have $\frac{\partial \vec{r}}{\partial u} = \langle 1,0,-1\rangle$ and $\frac{\partial \vec{r}}{\partial v} = \langle 0,1,1\rangle$, and then we compute

$$\vec{n}(u,v) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \langle 1,-1,1\rangle\,.$$

Since

$$\vec{F}(\vec{r}(u,v)) = \langle 2u-v, -u+v-1, v\rangle\,,$$

we get

$$\begin{aligned} \vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v) &= \langle 2u-v, -u+v-1, v\rangle \cdot \langle 1,-1,1\rangle \\ &= (2u-v) - (-u+v-1) + v = 3u - v + 1 \end{aligned}$$

and therefore the integral is

$$\int_0^2 \int_0^1 3u - v + 1\, dvdu = \int_0^2 \left[3uv - \frac{v^2}{2} + v\right]_0^1 du = \int_0^2 3u + \frac{1}{2}\, du = \left[\frac{3u^2}{2} + \frac{u}{2}\right]_0^2 = 7.$$

□