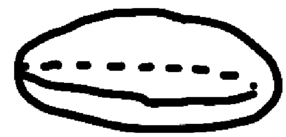
MAT 203 L6 Wed Feb 11

Recall: Quadratic surfaces:

Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

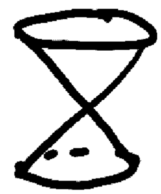


Hyperboloids

one-sheeted: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

two-sheeted: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

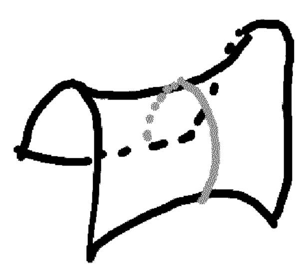
Cone: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$



Paraboloid,

elliptic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$

hyperbolic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$



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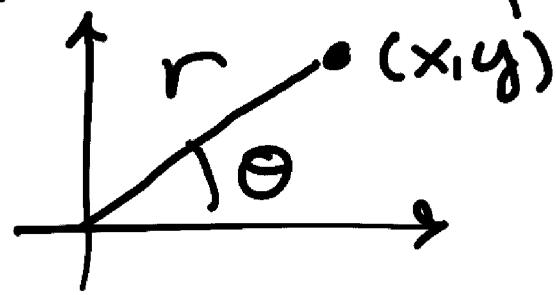
§2.7 Cylindrical and Spherical coordinates

To parametrize conic sections, it's usually convenient to use polar coords (r,θ) in the plane

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



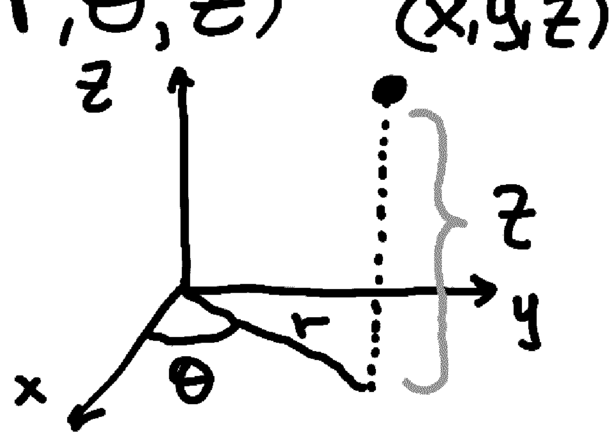
In space, there are two convenient coordinate systems.

<u>Cylindrical coordinates</u>:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$r \geq 0, \quad 0 \leq \theta \leq 2\pi$

(r,θ,z)



Can solve for r, θ, z in terms of x,y,z:

$$\begin{cases} r^2 = x^2+y^2 \\ \tan\theta = y/x \\ z = z \end{cases}$$

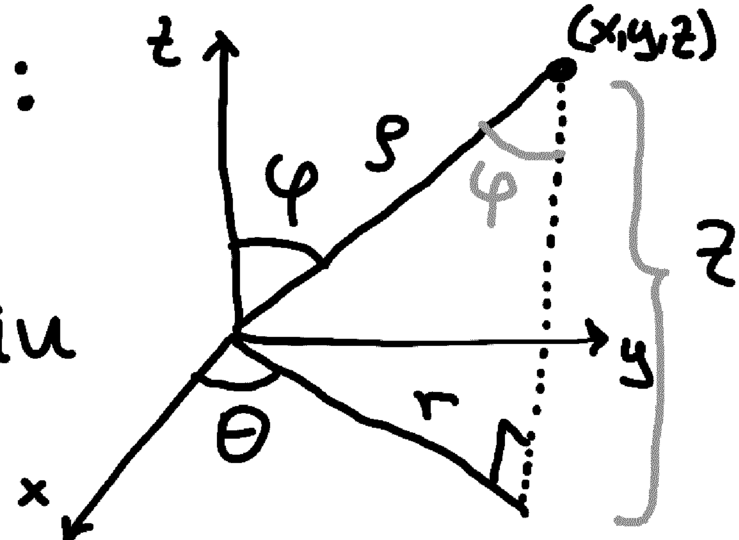
Ex: Parabola $x^2+y^2=z$

In cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{then } z = x^2+y^2 = r^2$$

$$\vec{\gamma}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$$

Spherical coords:



We derive the spherical coords in two steps:

1. Cylindrical coords

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

2. Now $\begin{cases} r = \rho\sin\varphi \\ z = \rho\cos\varphi \end{cases}$

Which gives spherical coords: (ρ, θ, φ)

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi$$

Can also solve for x, y, z:

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\theta = y/x \\ \varphi = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{cases}$$

<u>Ex</u>: <u>Ellipsoid</u>:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{then}$$

$$\begin{cases} x = a \sin\varphi \cos\theta \\ y = b \sin\varphi \cos\theta \\ z = c \cos\varphi \end{cases}$$

$$\vec{\gamma}(\theta,\varphi) = \langle a\sin\varphi\cos\theta,\ b\sin\varphi\cos\theta,\ c\cos\varphi\rangle$$

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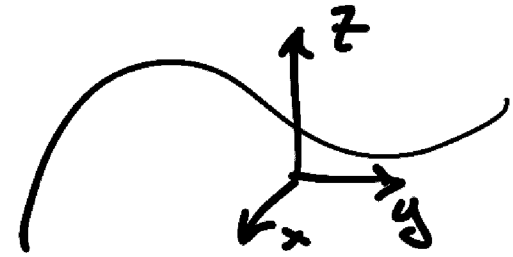
§3.1 Vector-valued functions

We have already discussed parametrizations of curves in space:

$$\vec{\gamma}(t) = \langle x(t), y(t), z(t)\rangle$$



this is a vector-valued function input is a scalar, but the output is a vector.

$$\langle x(t), y(t), z(t)\rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

We will now discuss limits and continuity.

Def: A vector-valued function $\vec{\gamma}(t)$ approaches the limit $\vec{L}$ as $t \to a$

if $\boxed{\lim_{x \to a} \|\vec{r}(t) - \vec{L}\| = 0.}$

In this case we write

$$\lim_{x \to a} \vec{r}(t) = \vec{L}.$$

Note that $\|\vec{r}(t) - \vec{L}\|$ is a function w/ scalars as input and scalars as output. So the limit $\lim_{x \to a} \|\vec{r}(t) - \vec{L}\|$ is exactly the kind we studied in Calculus I!

In practice, to compute limits we will use the following:

<u>Theorem</u>: If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle$$

provided that the limits

$\lim_{t \to a} x(t)$, $\lim_{t \to a} y(t)$, and $\lim_{t \to a} z(t)$ exist. (If one of them doesn't, the limit $\lim_{t \to a} \vec{\gamma}(t)$ doesn't exist)

<u>Ex</u>: ①

$$\lim_{t \to 3} \langle t^2 - 3t + 4, 4t + 3 \rangle = \langle 9 - 9 + 4, 12 + 3 \rangle$$
$$= \langle 4, 15 \rangle$$

② $\lim_{t \to 3} \left\langle \frac{2t-4}{t+1}, \frac{t}{t^2+1}, 4t-3 \right\rangle$

$$= \left\langle \frac{1}{2}, \frac{3}{10}, 9 \right\rangle$$

Remember that $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$.

<u>Def</u>: $\vec{\gamma}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous at $t = t_0$ if

1) $\vec{\gamma}(t_0)$ exists

2) $\lim_{t \to t_0} \vec{\gamma}(t)$ exists

3) $\lim_{t \to t_0} \vec{\gamma}(t) = \vec{\gamma}(t_0)$

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§3.2 Calculus of vector-valued functions

We have already computed derivatives of parametrizations before to find tangents.
Let us articulate this "officially" now:

<u>Def</u>: The derivative of $\vec{\gamma}(t)$ is given by the limit

$$\lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h} = \vec{\gamma}'(t)$$

If it exists, $\vec{\gamma}(t)$ is differentiable at t.

Thm: Assume $\vec{\gamma}(t) = \langle x(t), y(t), z(t) \rangle$, is differentiable at $t = t_0$. Then

$$\vec{\gamma}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

Ex: 1) $\vec{\gamma}(t) = \langle 3\cos t, 2\sin t \rangle$

$$\vec{\gamma}'(t) = \langle -3\sin t, 2\cos t \rangle$$

2) $\vec{\gamma}(t) = \langle e^t \sin t, e^t \cos t \rangle$

$$\vec{\gamma}'(t) = \langle e^t \sin t + e^t \cos t, \; e^t \cos t - e^t \sin t \rangle$$

3) $\vec{\gamma}(\theta) = \langle \sin\theta \cos\theta, \sin\theta \sin\theta \rangle$

$$\vec{\gamma}'(\theta) = (\cos^2\theta - \sin^2\theta, \cos\theta \sin\theta + \sin\theta \cos\theta \rangle$$

$$= \langle \cos(2\theta), \sin(2\theta) \rangle.$$

Properties of the derivative:

(i) $\frac{d}{dt}(c\vec{r}(t)) = c\vec{r}\,'(t)$

(ii) $\frac{d}{dt}\left(\vec{r}(t) \pm \vec{u}(t)\right) = \vec{r}'(t) \pm \vec{u}'(t)$

(iii) $\frac{d}{dt}\left(f(t)\,\vec{u}(t)\right) = f'(t)\,\vec{u}(t)$
$+ f(t)\,\vec{u}'(t)$

(iv) $\frac{d}{dt}\left(\vec{r}(t) \bullet \vec{u}(t)\right) = \vec{r}'(t) \bullet \vec{u}(t)$

$+ \vec{r}(t) \bullet \vec{u}'(t)$

(v) $\frac{d}{dt}\left(\vec{r}(t) \times \vec{u}(t)\right) = \vec{r}'(t) \times \vec{u}(t)$

$+ \vec{r}(t) \times \vec{u}'(t)$

(vi) $\frac{d}{dt}\left(\vec{r}(f(t))\right) = \vec{r}'(f(t)) \cdot f'(t)$

(vii) If $\vec{r}(t) \bullet \vec{r}(t) = c$ (a Scalar)
then $\vec{r}(t) \bullet \vec{r}'(t) = 0$.
