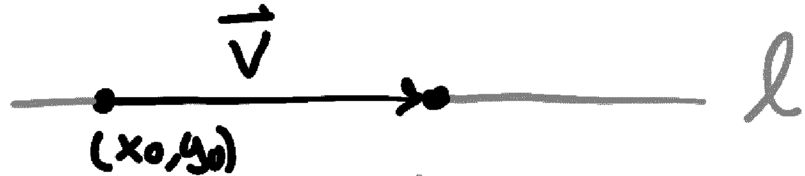
MAT 203 L4 Wed Feb 4

Recall: • Line in the plane:

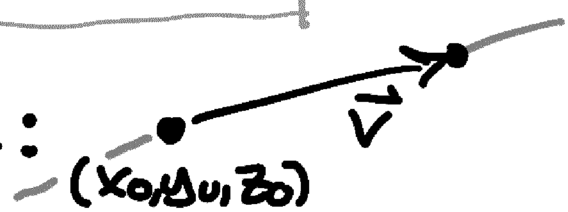
Parametric:



$$\boxed{\vec{\gamma}(t) = \langle x_0, y_0\rangle + t\vec{v}}$$

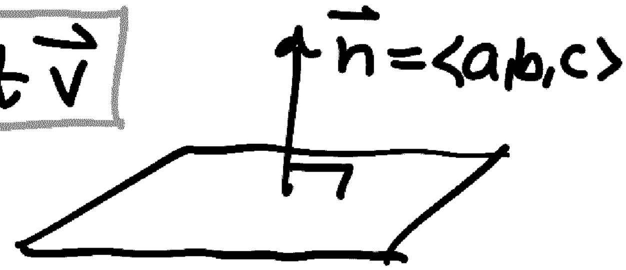
Linear: $\boxed{ax+by+c=0}$ $(a,b)\neq(0,0)$

• Line in space:



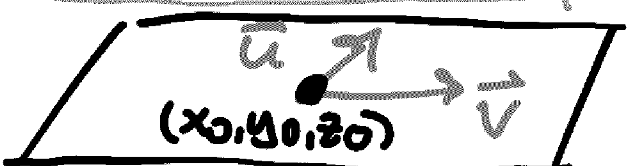
$$\boxed{\vec{\gamma}(t) = \langle x_0, y_0, z_0\rangle + t\vec{v}}$$

• Plane in space:



Linear: $\boxed{ax+by+cz+d=0}$ $(a,b,c)\neq(0,0,0)$

Parametric:



$\vec{u},\vec{v}$ non-parallel in the plane,

$$\boxed{\vec{r}(s,t) = \langle x_0,y_0,z_0\rangle + s\vec{u}+t\vec{v}}$$

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§1.1, 1.2 Parametrizing curves

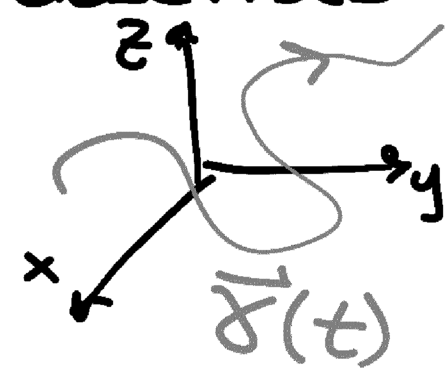
The parametric form of a line in direction $\vec{v} = \langle a, b, c \rangle$ is

$$\vec{\gamma}(t) = \langle x_0, y_0, z_0 \rangle + t\vec{v}$$
$$= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$
$$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

It's a function where the variable (parameter) t is a scalar, and the output is the vector $\vec{\gamma}(t)$.

In general, if $x(t), y(t), z(t)$ are functions depending on t,

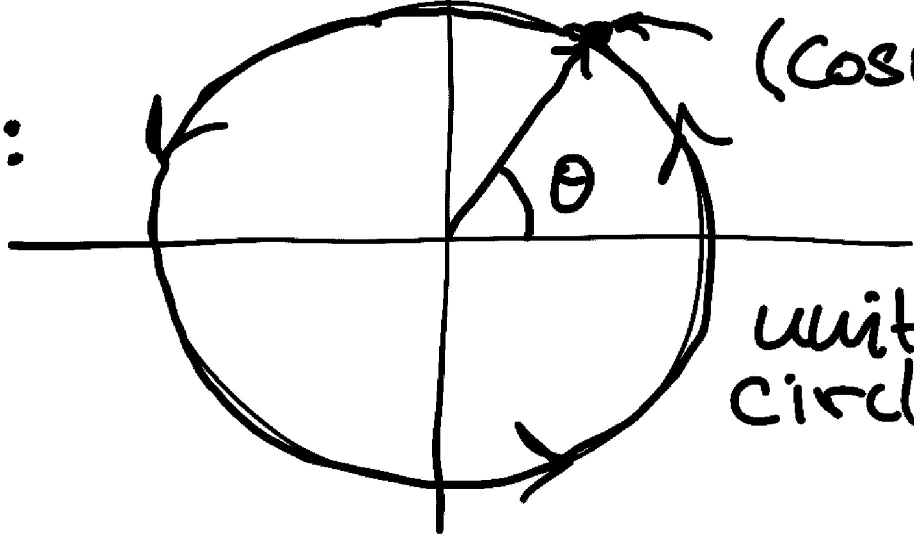
$\vec{\gamma}(t) = \langle x(t), y(t), z(t) \rangle$ describes a curve in space.



$\vec{\gamma}(t) = \langle x(t), y(t) \rangle$

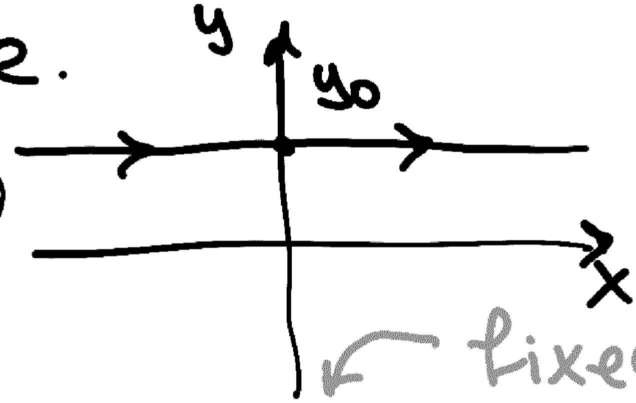
parametrized curve in the plane.

Ex:



$\vec{\gamma}(\theta) = \langle \cos\theta, \sin\theta \rangle$ is a parametrization of the unit circle.

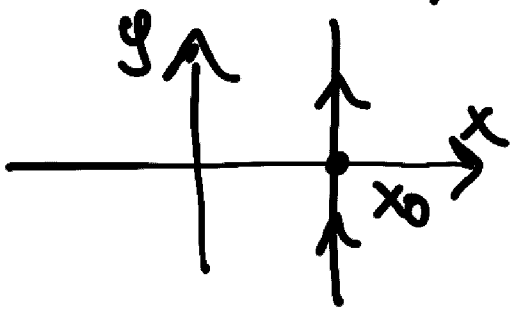
Ex: ①



fixed number

$\vec{\gamma}(t) = \langle t, y_0 \rangle$

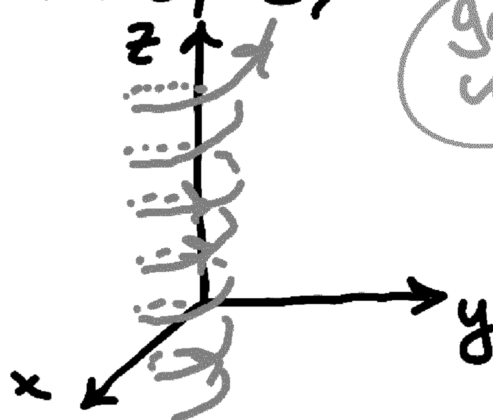
②



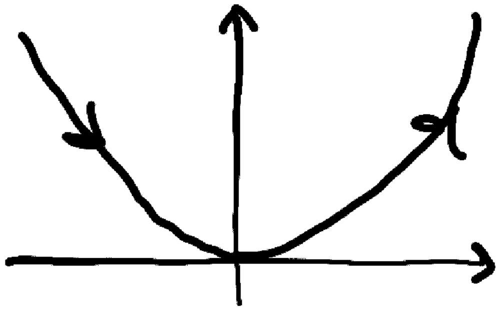
$\vec{\gamma}(t) = \langle x_0, t \rangle$.

Ex: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ is a helix in space

going up

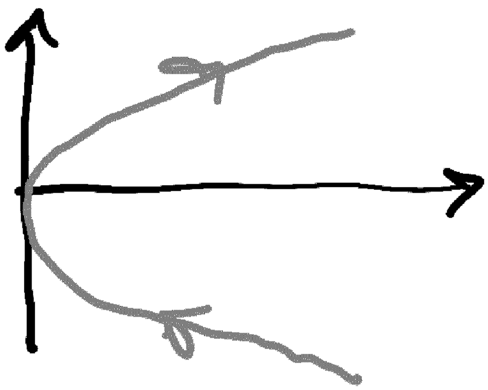


<u>Ex</u>: $\vec{r}(t) = \langle t, t^2 \rangle$ is a parametrization of a parabola:



$$\begin{cases} x = t \\ y = t^2 \end{cases} \quad \boxed{y = x^2}$$

<u>Ex</u>: $\vec{r}(t) = \langle t^2, t \rangle$ is also a param. of a parabola:



$$\begin{cases} x = t^2 \\ y = t \end{cases} \quad \boxed{y^2 = x}$$

For $y > 0$, $y = \sqrt{x}$

$y < 0$, $y = -\sqrt{x}$

<u>Ex</u>: $\vec{\gamma}(t) = \langle \sqrt{2t+4}, 2t+1 \rangle$. $(2t+4 \geq 0)$

How to draw this? We eliminate the parameter as above:

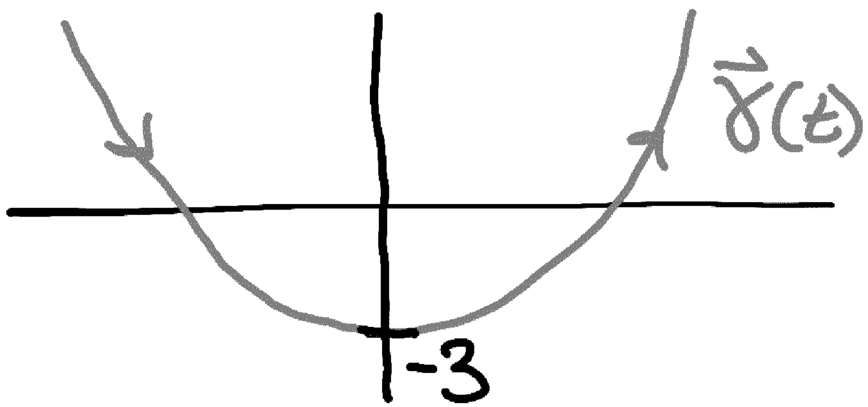
$$\begin{cases} x = \sqrt{2t+4} \\ y = 2t+1 \iff y+3 = 2t+4 \end{cases}$$

$$x = \sqrt{2t+4} = \sqrt{y+3}$$

$x^2 = y+3 \Leftrightarrow y = x^2 - 3.$

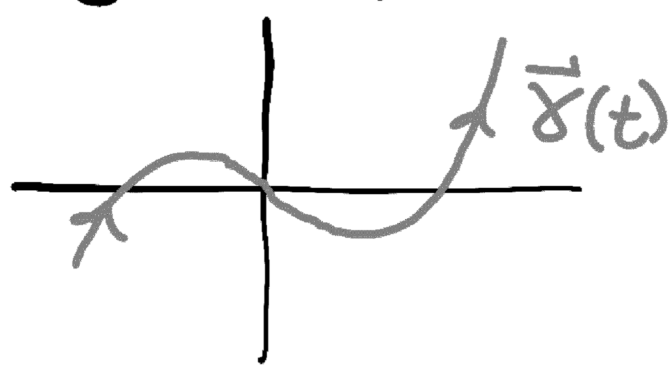
Since $2t+4 \geq 0$ we have $y+3 \geq 0$

$\boxed{y \geq -3}$



Ex: If $y = x^3 - x$, then we can parametrize its graph by letting $x = t$. Then $y = t^3 - t$, so

$\vec{\gamma}(t) = \langle t, t^3 - t \rangle$



Calculus of parametric curves:

If $\vec{\gamma}(t) = \langle x(t), y(t) \rangle$, and $x'(t)$, $y'(t)$ exists, we can calculate the derivative $\frac{dy}{dx}$ assuming $x'(t) \neq 0$ by

$$\boxed{\frac{dy}{dx} = \frac{y'(t)}{x'(t)}}.$$

This is a consequence of the chain rule.

<u>Ex</u>: $\vec{\gamma}(t) = \langle t^2-3, 2t-1\rangle$

$x(t) = t^2-3, \quad y(t) = 2t-1$

$x'(t) = 2t \quad , \quad y'(t) = 2$

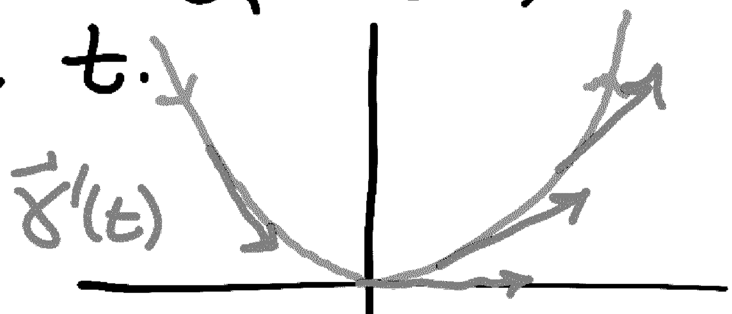
$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2}{2t} = \frac{1}{t}.$$

Derivative of $y = f(x)$ at x-value $x(t)$, depending on t.

<u>Ex</u>: If $\vec{\gamma}(t) = \langle t, t^2\rangle$,

$x'(t) = 1, \quad y'(t) = 2t$

$\vec{\gamma}'(t) = \langle 1, 2t\rangle$ is a tangent vector of $\vec{\gamma}(t)$ at parameter value t.



If $\vec{\gamma}(t) = \langle x(t), y(t), z(t)\rangle$ we can therefore find the tangent line at time t_0

$$\vec{\gamma}(t_0) = \langle x'(t_0), y'(t_0). z'(t_0)\rangle$$

tangent vector, at point $(x(t_0), y(t_0), z(t_0))$ on the curve.

Parametrization of tangent line:

$$\vec{r}(s) = \langle x(t_0), y(t_0), z(t_0)\rangle + s\vec{\gamma}'(t_0).$$

<u>Ex</u>: $\vec{\gamma}(t) = \langle \cos t, \sin t, t\rangle$.
Find tangent line to the curve at $t = \pi/4$.

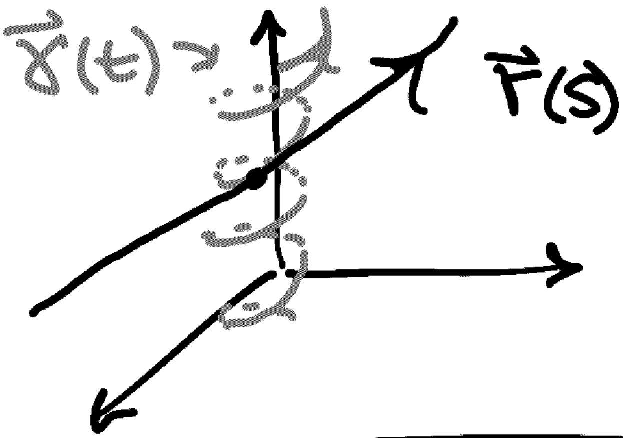
<u>Sol</u>: $\vec{\gamma}'(t) = \langle -\sin t, \cos t, 1\rangle$

$\vec{\gamma}'(\frac{\pi}{4}) = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\rangle$. It passes through the point

$$(\cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}, \tfrac{\pi}{4}) = (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, \tfrac{\pi}{4})$$

Tangent line is parametrized as:

$$\vec{r}(s) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4} \right\rangle + s\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \right\rangle$$



<u>Ex</u>: Find tangent line in linear form of the curve $\vec{\gamma}(t) = \langle t^2-4t, 2t^3-6t \rangle$ at $t=5$.

<u>Sol</u>: $\vec{\gamma}'(t) = \langle 2t-4, 6t-6 \rangle$, so

$\to \vec{\gamma}'(5) = \langle 6, 24 \rangle$

Tangent vector at $(x(5), y(5)) = (5, 220)$

Parametrization of tangent:

$$\vec{r}(s) = \langle 5, 220 \rangle + s\langle 6, 25 \rangle$$

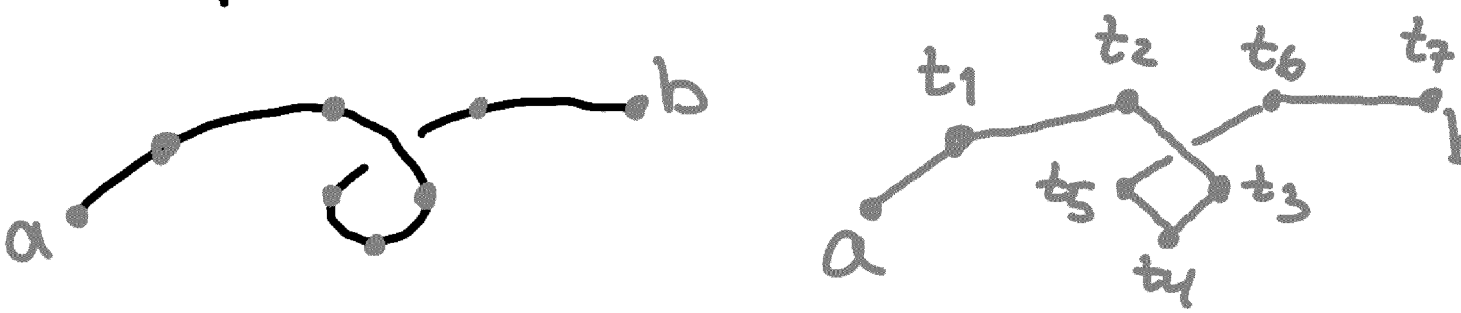
$$= \langle 5+6s, 220+25s \rangle$$

$$\begin{cases} x = 5+6s \iff s = \dfrac{x-5}{6} \\ y = 220+25s \end{cases}$$

$$\boxed{y = 220 + 25\left(\frac{x-5}{6}\right) = \frac{25x}{6} + \left(220 - \frac{125}{6}\right)}$$

Arc length.

If $\vec{\gamma}(t) = \langle x(t), y(t), z(t)\rangle$ is a parametrized curve in space, we would like to calculate its arc length between two parameter values $a \le t \le b$



length of curve for $t_i \le t \le t_{i+1}$

$\approx$ distance between endpoints of $\vec{\gamma}(t_i)$ and $\vec{\gamma}(t_{i+1})$

$$= \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2}$$

If $\Delta t_i = t_{i+1} - t_i$ is very small:

$$\begin{cases} x(t_{i+1}) - x(t_i) \approx x'(t_i)\,\Delta t_i \\ y(t_{i+1}) - y(t_i) \approx y'(t_i)\,\Delta t_i \end{cases}$$

$z(t_{i+1}) - z(t_i) \approx z'(t_i)\,\Delta t_i$

length of curve for $t_i \le t \le t_{i+1}$

$\approx \sqrt{(x'(t_i))^2 + (y'(t_i))^2 + (z'(t_i))^2}\,\Delta t_i$

$= \|\vec{\gamma}\,'(t_i)\|\,\Delta t_i$.

Therefore, the arc length of $\vec{\gamma}(t)$ between $a \le t \le b$ is given

by $\boxed{\displaystyle\int_a^b \|\vec{\gamma}\,'(t)\|\,dt}$.