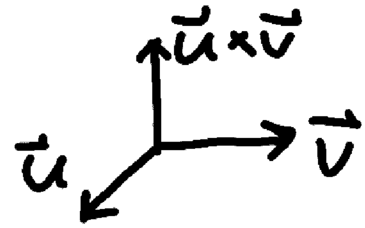
MAT 203 L3 Mon Feb 2

<u>Recall</u>: • Dot product.

$\vec{u} = \langle u_1, u_2, u_3 \rangle, \ \vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

• Cross product.



$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

$$= (u_2 v_3 - v_2 v_3)\vec{i} - (u_1 v_3 - v_1 u_3)\vec{j}$$
$$+ (u_1 v_2 - v_1 u_2)\vec{k}$$

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§2.5 Lines and planes in space

Recall that a $\underbrace{\ }_{\text{non-vertical}}$ line in the

Plane has the equ $y = kx + m$
where k = slope, m = y-intercept.

A vertical line has the equ
$x = a$.

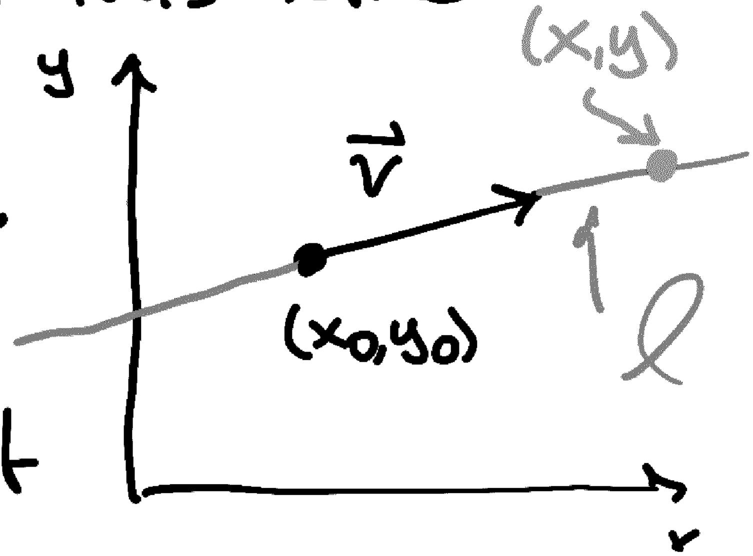
A more general way of writing it: $\boxed{ax + by + c = 0}$ $(a,b) \neq (0,0)$.

<u>If $b \neq 0$</u> : $y = \underbrace{-\frac{a}{b}}_{=k} x \underbrace{- \frac{c}{b}}_{=m}$ (non-vertical line)

<u>If $b = 0$</u> : $x = -\frac{c}{a}$ (vertical line)

Will now describe this line using vectors.

We now describe the line passing through the point



(x_0, y_0) and is parallel to the vector $\vec{v} = \langle a, b \rangle$. If (x, y) is any point on the line ℓ, then

$\langle x - x_0, y - y_0 \rangle$ is parallel to $\vec{v}$, so it's a scalar multiple of $\vec{v}$:
So there is some scalar t such that $\langle x - x_0, y - y_0 \rangle = t\vec{v} = t\langle a, b \rangle$

$$\begin{cases} x - x_0 = ta \\ y - y_0 = tb \end{cases} \Rightarrow \begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

or:

$$\boxed{\vec{\gamma}(t) = \langle x_0, y_0 \rangle + t\langle a, b \rangle}$$

This is the <u>parametric form</u> of the line. The scalar t is called the parameter.

This also works in the same way in space:

The line passing through (x_0, y_0, z_0) that is parallel to $\vec{v} = \langle a, b, c \rangle$ is given by

$$\boxed{\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle.}$$

<u>Ex</u>: Find the parametric equ of the line passing through the points $(1, -3, 2)$ and $(5, -2, 8)$.

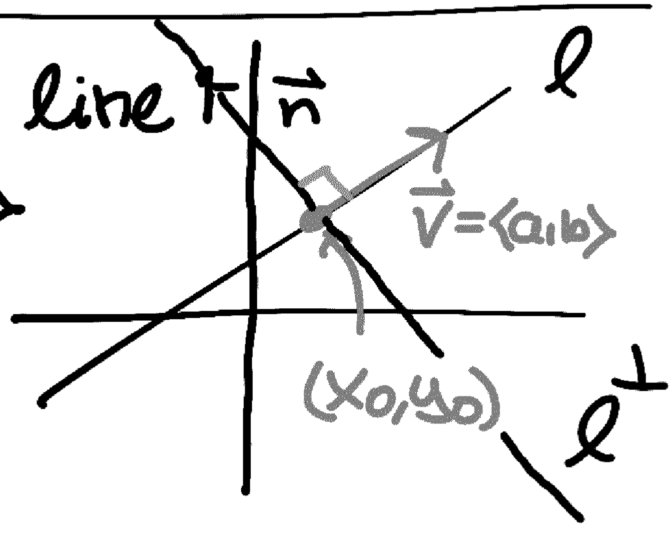
<u>Sol</u>: Let $\vec{v}$ be the vector from $(1, -3, 2)$ to $(5, -2, 8)$.

$$\vec{v} = \langle 5-1, -2-(-3), 8-2 \rangle = \langle 4, 1, 6 \rangle.$$

So the line is described by

$$\boxed{\vec{r}(t) = \langle 1, -3, 2 \rangle + t \langle 4, 1, 6 \rangle}$$

Now consider the line ℓ parallel to $\vec{v} = \langle a, b \rangle$ that passes through (x_0, y_0).



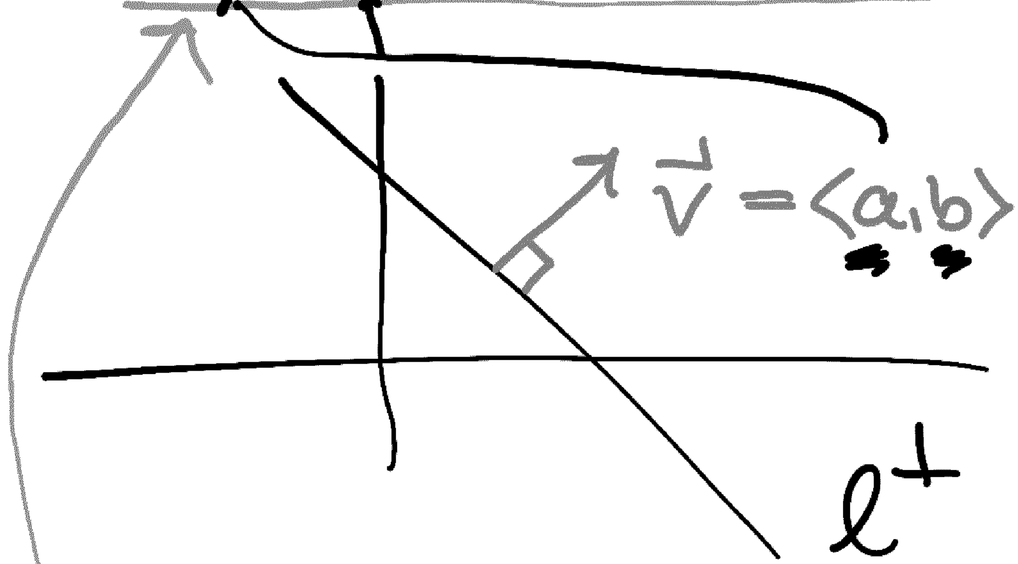
Let us find the line $\ell^{\perp}$ through (x_0, y_0) that is perpendicular to ℓ.

The direction vector $\vec{n}$ is orthogonal to $\vec{v}$, so $\vec{n} \cdot \vec{v} = 0$

If (x, y) is a point on $\ell^{\perp}$, then $\vec{n} = \langle x - x_0, y - y_0 \rangle$, $\vec{v} = \langle a, b \rangle$ so

$$\langle x - x_0, y - y_0 \rangle \cdot \langle a, b \rangle = a(x - x_0) + b(y - y_0) = 0$$

$$\Longleftrightarrow \boxed{ax + by - ax_0 - by_0 = 0}$$



equation for the line $\ell^{\perp}$ with normal vector $\langle a, b \rangle$ passing through (x_0, y_0).

Q: How to change between a parametric & linear form (meaning $ax+by+c=0$) of a line?

Ex: Consider the line $2x+3y+6=0$
Find a parametric form for it.

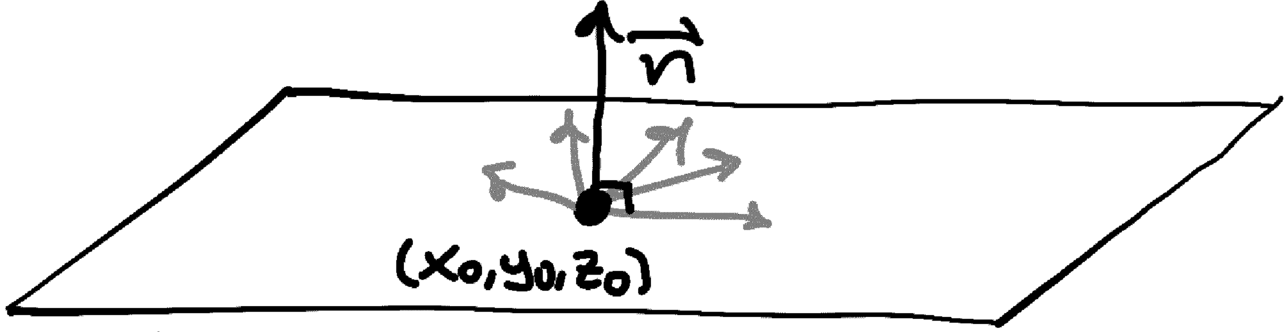
Sol: $y = -\frac{2}{3}x - 2$.

$(0,-2)$, $(3,-4)$ are two pts on the line, so a direction vector is $\vec{v} = \langle 3-0, -4-(-2) \rangle = \langle 3, -2 \rangle$.
Therefore a parametric form is given by

$$\boxed{\vec{\gamma}(t) = \langle 0,-2 \rangle + t\langle 3,-2 \rangle}$$

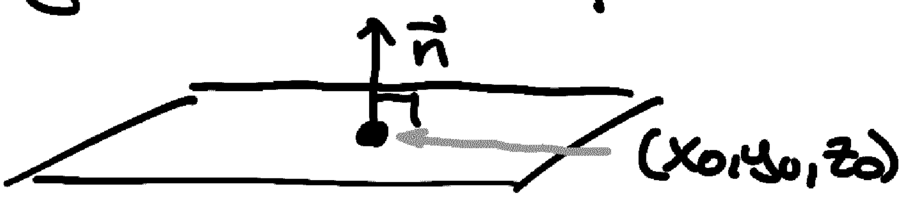
Lines are best described by a direction vector.

A plane has many directions since it's 2-dimensional:

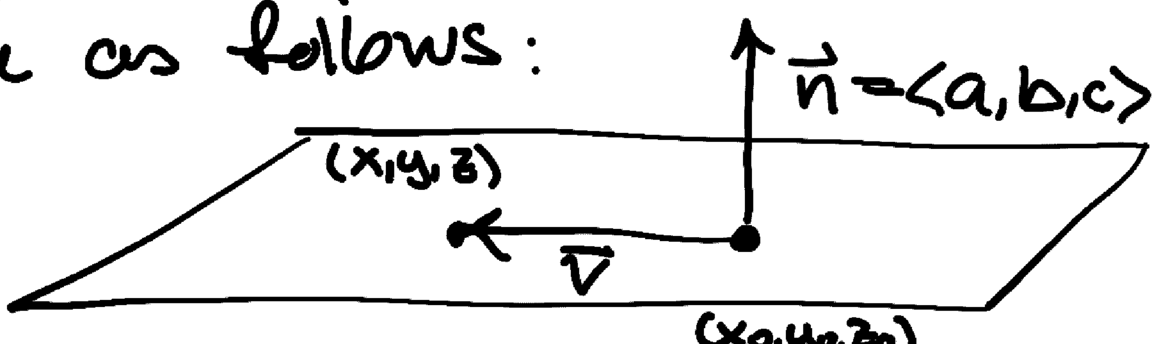


All of these vectors point along the plane.

The best way to describe a plane is by a normal vector: a vector that is orthogonal to any vector pointing along the plane.



If $\vec{n} = \langle a, b, c \rangle$ and the plane passes through (x_0, y_0, z_0), then we can describe any other point (x, y, z) in the plane as follows:



$\vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$ ← in the plane

$\vec{n} = \langle a, b, c \rangle$ ← normal to the plane

$\vec{n} \cdot \vec{v} = 0 \iff \langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle$
$= a(x-x_0) + b(y-y_0) + c(z-z_0)$
$= ax+by+cz-ax_0-by_0-cz_0 = 0$

Linear form of a plane:

$$\boxed{ax+by+cz+d=0} \qquad (a,b,c) \neq (0,0,0)$$

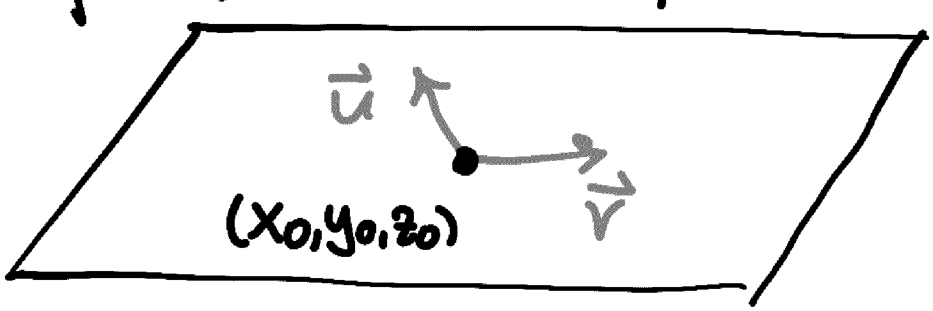
Normal to the plane:

$$\boxed{\vec{n} = \langle a,b,c\rangle.}$$

<u>Parametric form of a plane</u>

As mentioned above, a plane doesn't have one direction, since it is 2-dimensional.

In fact if $\vec{u}, \vec{v}$ are two vectors that are not parallel to each other, but parallel to a plane



then any other vector in the plane

$\vec{w}$ can be written as $\vec{w} = t\vec{u} + s\vec{v}$ for two scalars t and s. So if the plane passes through (x_0, y_0, z_0), the parametric form is given by:

$$\boxed{\vec{r}(s,t) = \langle x_0, y_0, z_0 \rangle + s\vec{u} + t\vec{v}}$$

There are two parameters s and t since the plane is 2-dimensional.
