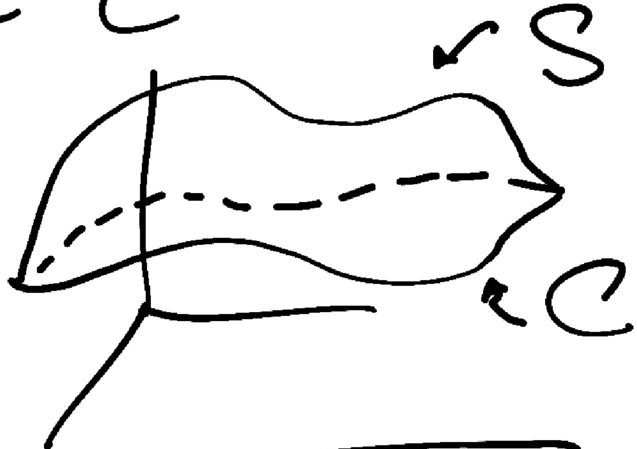
MAT 203 L22 Mon May 4

Recall: • $\vec{F}(x,y,z)$ vector field,

S surface with boundary curve C



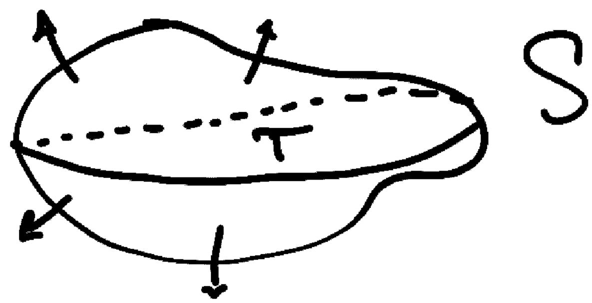
$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \vec{F}\cdot d\vec{r}}$$

3D version of Green's theorem.

§6.8 Divergence theorem

The divergence theorem is sometimes called "Gauss' theorem."

suppose S is a surface, with no boundary curve, in space that encloses a domain T.

eg. 

We assume S is oriented with the outwards normal.

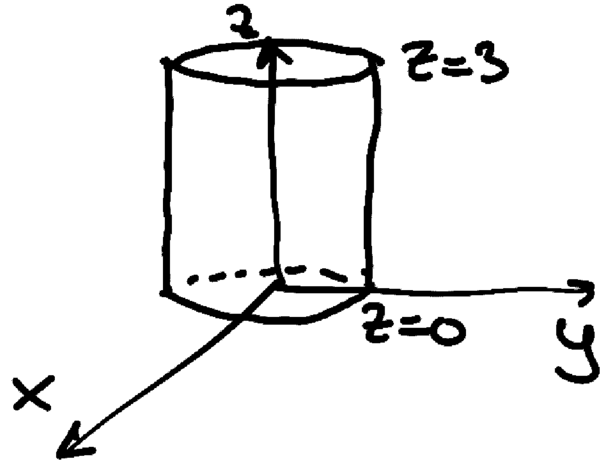
If $\vec{F}$ is a vector field, then

$$\boxed{\iint_S \vec{F}\cdot d\vec{r} = \iiint_T \nabla\cdot\vec{F}\, dx\,dy\,dz}$$

__Ex__: Let's verify the divergence theorem in the following example:

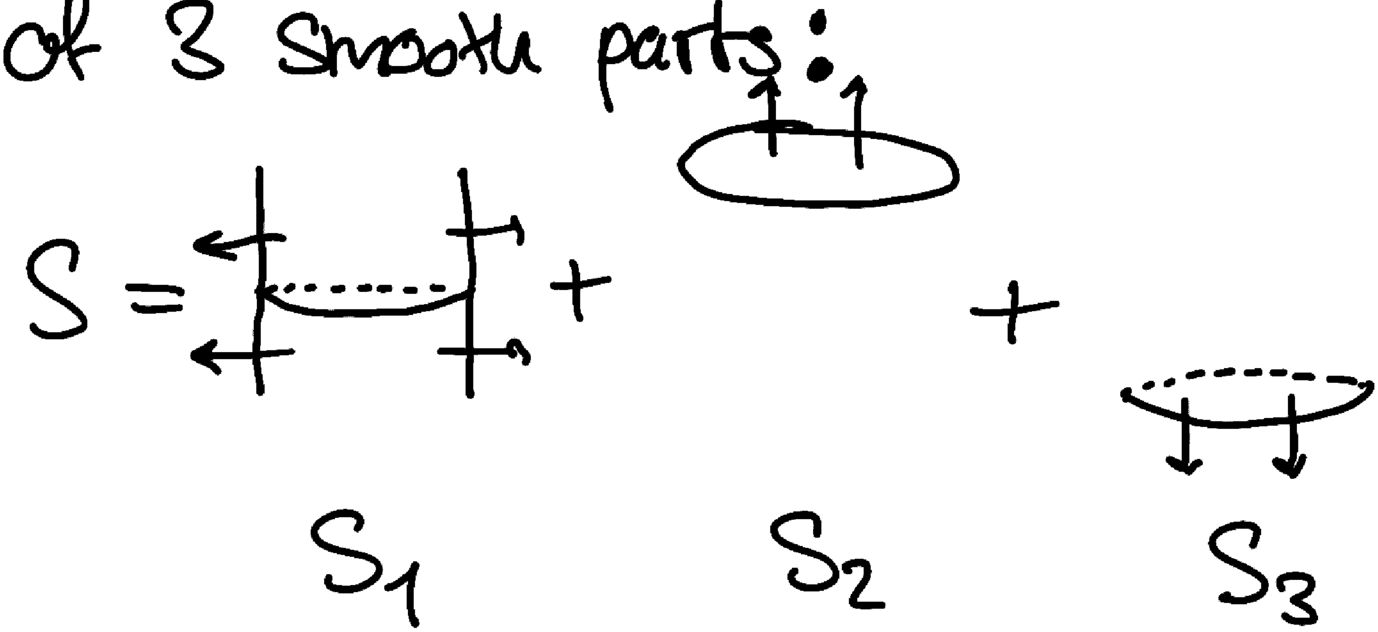
$\vec{F}(x,y,z) = \langle x+y+z,\ y,\ 2x-y\rangle$, and the surface S is the cylinder

$x^2+y^2=1$, $0 \le z \le 3$ plus the top & the bottom:



We first compute the surface integral:

Because the surface consists of 3 smooth parts:



We need to split the integral up into pieces:

$$\boxed{\iint_{S_1} \vec{F} \cdot d\vec{r}}$$

We parametrize the cylinder:

$r = 1$ on the surface, so

$$\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$$

$$0 \leq \theta \leq 2\pi, \ 0 \leq z \leq 3.$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\iint_{S_1} \vec{F}(\vec{r}(\theta,z)) \cdot \left(\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} \right) d\theta \, dz$$

$$= \int_0^3 \int_0^{2\pi} \langle \cos\theta + \sin\theta + z, \sin\theta, 2\cos\theta - \sin\theta \rangle$$

$$\cdot \langle \cos\theta, \sin\theta, 0 \rangle \, d\theta \, dz$$

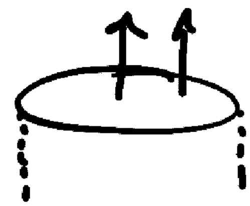
$$= \int_0^3 \int_0^{2\pi} \cos^2\theta + \sin\theta\cos\theta + z\cos\theta + \sin^2\theta \, d\theta \, dz$$

$$= \int_0^3 \int_0^{2\pi} 1 + \tfrac{1}{2}\sin(2\theta) + z\cos\theta \quad d\theta \, dz$$

$$= \int_0^3 \left[\theta - \frac{\cos(2\theta)}{4} + z\sin\theta \right]_0^{2\pi} dz$$

$$= \int_0^3 2\pi \, dz = 6\pi$$

Second part: $\boxed{\iint_{S_2} \vec{F} \cdot d\vec{r}.}$

Parametrization:  $z = 3$

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 3 \rangle$$

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

$$\text{Normal} = \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \langle 0, 0, r \rangle$$

$$\iint_{S_3} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \int_0^1 \vec{F}(\vec{r}(r,\theta)) \cdot \langle 0, 0, r \rangle \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \langle r\cos\theta + r\sin\theta + 3, r\sin\theta, 2r\cos\theta - r\sin\theta \rangle \cdot \langle 0, 0, r \rangle \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^2 (2\cos\theta - \sin\theta) \, dr \, d\theta$$

$$= \left(\int_0^{2\pi} 2\cos\theta - \sin\theta \, d\theta \right) \underbrace{\left(\int_0^1 r^2 \, dr \right)}_{= 1/3}$$

$$= \frac{1}{2}\underbrace{\left[-2\sin\theta - \cos\theta\right]_0^{2\pi}}_{=0} = 0$$

A very similar computation shows

$$\iint_{S_3} \vec{F}.d\vec{r} = 0 .$$

So $\displaystyle\iint_S \vec{F}.d\vec{r} = \iint_{S_1} \vec{F}.d\vec{r} + \iint_{S_2} \vec{F}.d\vec{r} + \iint_{S_3} \vec{F}.d\vec{r}$

$$= 6\pi + 0 + 0 = 6\pi$$

Next, let's compute

$\displaystyle\iiint_B \nabla.\vec{F}\, dxdydz$, where

B = inside of the cylinder

$$(\nabla\cdot\vec{F})(x,y,z) = \frac{\partial(x+y+z)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(2x-y)}{\partial z}$$

$$= 1+1 = 2.$$

So $\iiint\limits_B \nabla\cdot\vec{F}\,dx\,dy\,dz = 2\cdot \text{Vol(Cylinder)}$.

$= 2\cdot(\pi\cdot 1^2\cdot 3) = 6\pi$.

<u>Ex</u>: Two lectures ago we computed the surface integral

$\iint\limits_S \vec{F}\cdot d\vec{F}$, $S = \{x^2+y^2+z^2=1\}$

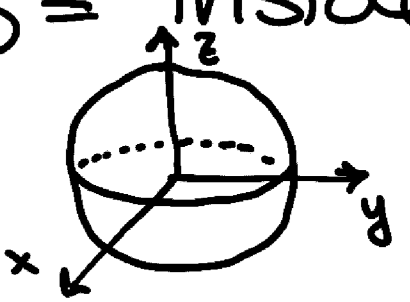
where $\vec{F}(x,y,z) = \langle 0,0,z\rangle$.

Let's now compute it using the divergence theorem.

<u>Sol</u>: $\nabla\cdot\vec{F} = 1$, so

$\iint\limits_S \vec{F}\cdot d\vec{F} = \iiint\limits_B \nabla\cdot\vec{F}\,dV$, where

$B =$ inside the sphere S



But now

$$\iint_S \vec{F}\cdot d\vec{r} = \iiint_B \nabla\cdot\vec{F}\, dV = \iiint_B dV$$

$$= \mathrm{Vol}(B) = \frac{4\pi\cdot 1^3}{3} = \frac{4\pi}{3}.$$

Ex: Let S be the surface of the box

$B = \{0 \leq x \leq 2,\ 1 \leq y \leq 4,\ 0 \leq z \leq 1\}$

Compute $\iint_S \vec{F}\cdot d\vec{r}$ where

$\vec{F} = \langle x^2+yz,\ y-z,\ 2x+2y+2z\rangle$.

Sol: We use the divergence theorem:

$$\iint_S \vec{F}\cdot d\vec{r} = \iiint_B \nabla\cdot\vec{F}\, dV$$

$$= \int_0^1\int_1^4\int_0^2 2x+1+2\, dx\,dy\,dz$$

$$= 3\cdot 1\cdot\int_0^2 2x+3\, dx = 3\left[x^2+3x\right]_0^2 = 13$$