MAT 203 L21 Wed Apr 29

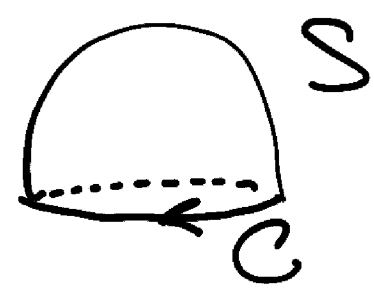
Recall:

- S surface with parametrization $\vec{r}(s,t)$, $(s,t)\in D$, then

$$\boxed{\iint_S \vec{F}\cdot d\vec{r} = \iint_D \vec{F}(\vec{r}(s,t))\cdot\left(\frac{\partial \vec{r}}{\partial s}\times\frac{\partial \vec{r}}{\partial t}\right) ds\,dt}$$

§6.7 Stokes' theorem

Let S be a surface with boundary curve C.



Assume S is parametrized by $\vec{r}(s,t)$, and that $\vec{F}$ is a vector field. Then

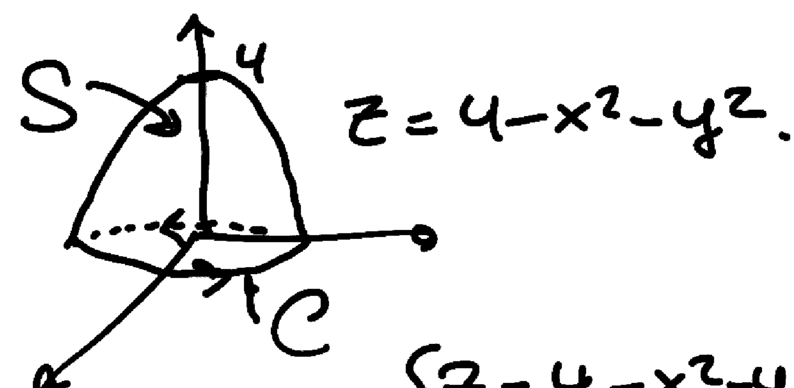
$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{r}}$$

Ex: Consider the vector field $\vec{F}(x,y,z) = \langle y, 2z, x^2 \rangle$. Let S be the part of the paraboloid

$$z = 4 - x^2 - y^2 \quad \text{with } z \geq 0,$$

with outwards normal. Let's verify Stokes' theorem in this case.

Sol:



Boundary curve: $\begin{cases} z = 4 - x^2 - y^2 \\ z = 0 \end{cases}$

$\Leftrightarrow \{x^2 + y^2 = 4, \ z = 0\}$

This is the circle centered at O with radius 2, with counterclockwise orientation by the right hand rule.

Parametrize the curve by

$$\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0 \rangle, \quad 0 \leq \theta \leq 2\pi$$

Then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}(\vec{r}(\theta))\cdot \vec{r}\,'(\theta)\,dt$$

$$= \int_0^{2\pi} \langle 2\sin\theta, 0, 4\cos^2\theta\rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} -4\sin^2\theta \, d\theta = -4\int_0^{2\pi} \frac{1-\cos(2\theta)}{2} d\theta$$

$$= -4\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{2\pi}$$

$$= -4\left(\frac{2\pi}{2}\right) = -4\pi.$$

Now let's compute the surface integral.

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{r}.$$

$$(\nabla\times\vec{F})(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & x^2 \end{vmatrix}$$

$$= \langle -2, -2x, -1 \rangle .$$

Parametrize the surface by

$$\vec{r}(x,y) = \langle x, y, 4-x^2-y^2 \rangle,$$
$$(x,y) : 4-x^2-y^2 \geq 0 \iff x^2+y^2 \leq 4.$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{r} = \iint_D (\nabla \times \vec{F})(\vec{r}(x,y)) \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right) dx\,dy$$

$$= \iint_D \langle -2, -2x, -1 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx\,dy$$

$$= \iint_D -4x - 4xy - 1 \, dx\,dy = (*)$$

$D = \{ (x,y) \mid x^2+y^2 \leq 4 \}$. Let's Change to polar coords:

$$D = \{ (r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi \}.$$

$$(*) = \int_0^{2\pi} \int_0^{2} (-4r\cos\theta - 4r^2\sin\theta\cos\theta - 1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[-\frac{4r^3}{3}\cos\theta - r^4\sin\theta\cos\theta - \frac{r^2}{2} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} -\frac{32}{3}\cos\theta - 16\sin\theta\cos\theta - 2\ d\theta$$

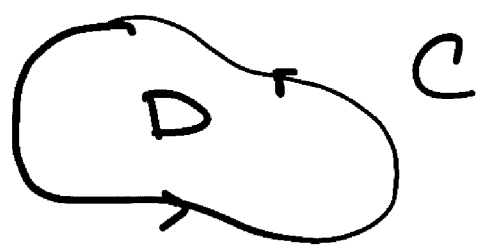
$$= \left[-\frac{32}{3}\sin\theta - 2\theta \right]_0^{2\pi} - 16 \int_0^{2\pi} \sin\theta\cos\theta\,d\theta$$

$$= [-4\pi] - 16 \underbrace{\left[\sin^2\theta \right]_0^{2\pi}}_{=0} = -4\pi.$$

So it matches the line integral computation above!

Green's theorem

$\vec{F} = \langle P, Q \rangle$



$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$

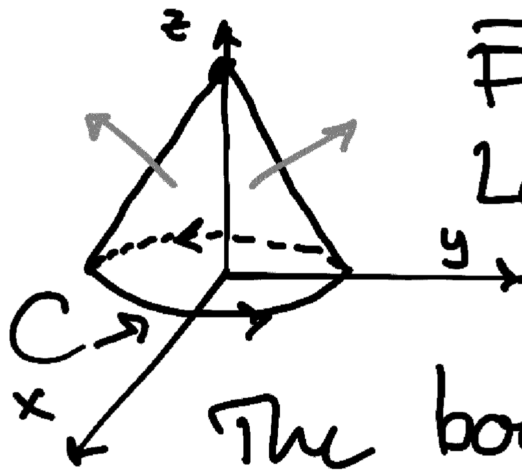
is in fact a special case of Stokes' theorem, since

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the last component of the curl of $\langle P, Q, 0\rangle$.

Ex. Consider the cone

$K = \{ z = 1 - \sqrt{x^2+y^2},\ z \geq 0 \}$

from last time.



$\vec{F}(x,y,z) = \langle 0,0,1\rangle$.

Let's compute it via Stokes' theorem.

The boundary curve is

$C = \{ x^2+y^2 = 1,\ z = 0 \}$

& it's oriented $\underline{\underline{CCW}}$, since the surface is oriented with the outwards normal.

The cone was parametrized by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1-r \end{cases},$$

and setting $z = 1-r = 0$

gives the parametrization of

$$C: \quad \vec{r}(\theta) = \langle \cos\theta, \sin\theta, 0\rangle$$

$$0 \le \theta \le 2\pi.$$

To use Stokes' this way we need to find $\vec{G}$ such that

$$\nabla \times \vec{G} = \vec{F} = \langle 0, 0, 1\rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_x & G_y & G_z \end{vmatrix} = \left\langle \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \right.$$

$$, -\left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z}\right), \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \rangle$$

Can take eg. $\vec{G} = \langle -\frac{y}{2}, \frac{x}{2}, 0 \rangle$

Thu

$$\iint_S \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle -\frac{\sin\theta}{2}, \frac{\cos\theta}{2}, 0 \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} \frac{\sin^2\theta}{2} + \frac{\cos^2\theta}{2} \, d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi.$$

It matches the computation from last time.
