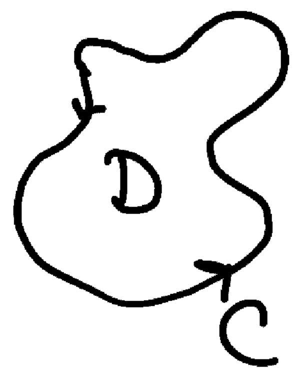
MAT 203 L20 Mon Apr 27

Recall:

- Green's theorem:

Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field in the plane.

Let $C$ be a simple, closed (no self-intersections) curve bounding a domain $D$ oriented CCW. Then




$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy}$$

- $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

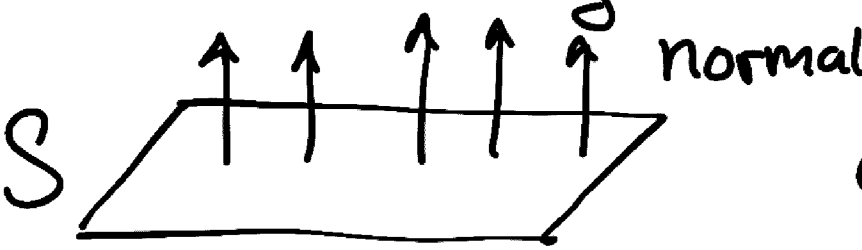
Divergence:

$$(\nabla \cdot \vec{F})(x,y,z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

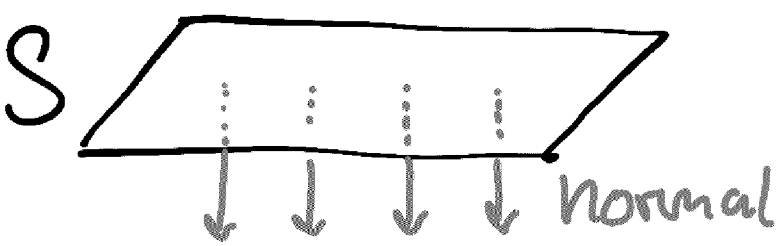
Curl: $$(\nabla \times F)(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

# §6.6 Surface integrals

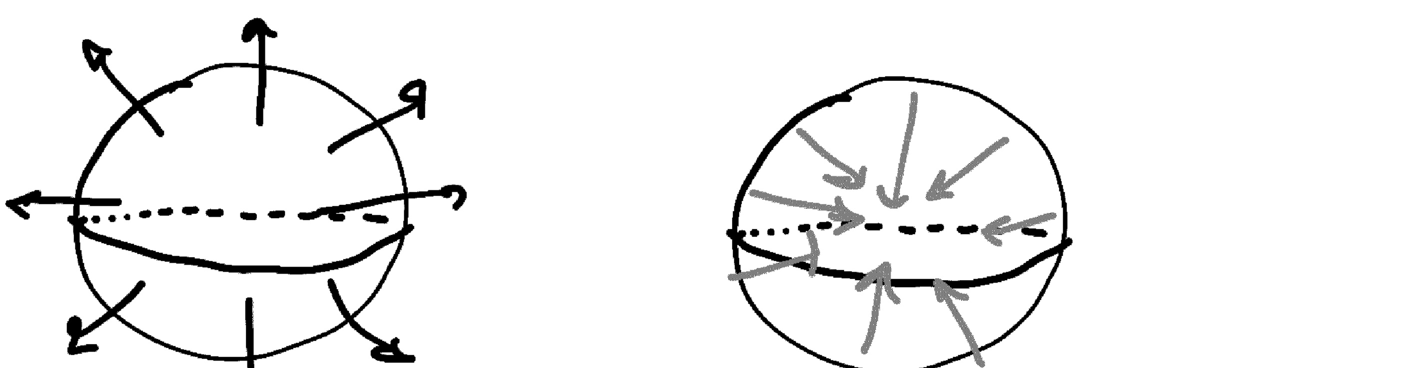
Let S be a surface in space. Much like a curve has a direction, or an *orientation*, a surface also has an orientation. The orientation of a surface is determined by a normal vector




orientation 1




orientation 2



"outwards normal"

"inwards normal"

Same surface but different orientations.

<u>Def</u>: Let $S$ be a surface in space with parametrization $\vec{r}(s,t)$, $(s,t) \in D$ for some domain $D$. Let $\vec{F}$ be a vector field.

The surface integral of $\vec{F}$ over $S$ is:

$$\boxed{\iint_S \vec{F} \cdot d\vec{r} = \iint_D \vec{F}(\vec{r}(s,t)) \cdot \vec{N}(s,t)\, ds\, dt}$$

where $\vec{N}(s,t) = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$

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For a parametrized surface $\vec{r}(s,t)$, the vector $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ is normal to the surface.

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<u>Ex</u>: Let $\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$, and let $S$ be the surface parametrized

by $\vec{r}(u,v) = \langle u, v^2-u, u+v \rangle$ $0 \leq u \leq 1$, $0 \leq v \leq 1$. Compute $\iint_S \vec{F} \cdot d\vec{r}$.

Sol: We first compute the normal

$$\frac{\partial \vec{r}}{\partial u} = \langle 1, -1, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial v} = \langle 0, 2v, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 0 & 2v & 1 \end{vmatrix} = \langle -1-2v, -1, 2v \rangle$$

Then

$$\iint_S \vec{F} \cdot d\vec{r} = \int_0^1 \int_0^1 \vec{F}(\vec{r}(u,v)) \cdot \left( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du\,dv$$

$$= \int_0^1 \int_0^1 \langle -v^2+u, u, 0 \rangle \cdot \langle -1-2v, -1, 2v \rangle du\,dv$$

$$= \int_0^1 \int_0^1 (-v^2+u)(-1-2v) + u(-1)\, du\,dv$$

$$= \int_0^1 \int_0^1 v^2 + 2v^3 - u - 2uv - u \, du\, dv$$

$$= \int_0^1 \left[ (v^2+2v^3)u - (1+v)u^2 \right]_0^1 dv$$

$$= \int_0^1 (v^2+2v^3) - (1+v) \, dv$$

$$= \left[ v^3 + \frac{6v^4}{4} - 9v - \frac{9v^2}{2} \right]_0^1$$

$$= 1 + \frac{3}{2} - 9 - \frac{9}{2} = -8 - 3 = -11$$

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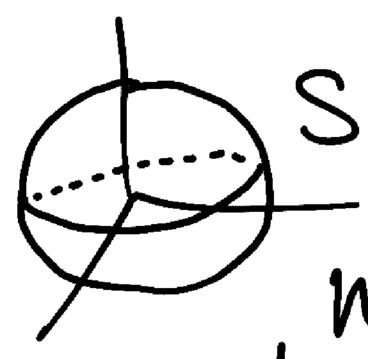
<u>Ex</u>: Consider the sphere $S = \{x^2+y^2+z^2=1\}$ oriented with the outwards normal, and compute $\iint_S \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$

<u>Sol</u>:




Sphere of radius 1.

We parametrize using spherical coords.

$$\begin{cases} x = \cos\theta\sin\varphi & 0 \le \theta \le 2\pi \\ y = \sin\theta\sin\varphi & 0 \le \varphi \le \pi. \\ z = \cos\varphi & \end{cases}$$

$$\vec{r}(\theta,\varphi) = \langle \cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi \rangle$$

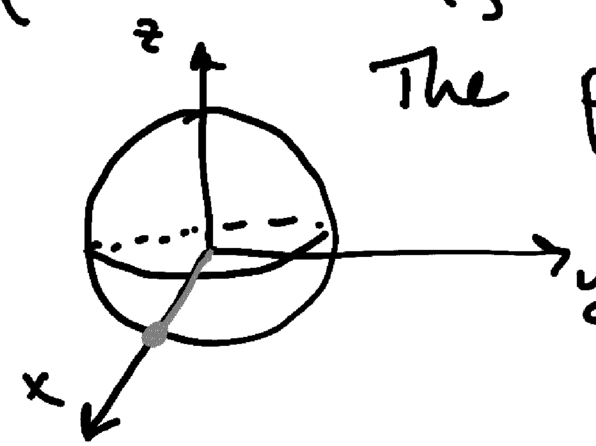
$$\vec{N}(\theta,\varphi) = \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\varphi}$$

$$= \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \end{array}$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix}$$

$$= -\cos\theta\sin^2\varphi\ \hat{\imath} - \left(\sin\theta\sin^2\varphi\right)\hat{\jmath}$$
$$+ \left(-\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi\right)\hat{k}$$

$$= \langle -\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi \rangle$$




The parameters
$\theta = 0,\ \varphi = \pi/2$
corresponds to
the point $(1,0,0)$

and the normal there is $N(0,\frac{\pi}{2}) = \langle -1, 0, 0 \rangle$ which points inwards. For our problem we therefore use $-\vec{N}(\theta,\varphi)$ which will point outwards.

$$\iint_S \vec{F} \cdot d\vec{r} = \int_0^{\pi}\int_0^{2\pi} \langle 0, 0, \cos\varphi \rangle \cdot (-\vec{N}(\theta,\varphi))\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin\varphi \cos^2\varphi \, d\theta\, d\varphi = 2\pi \int_0^{\pi} \sin\varphi \cos^2\varphi \, d\varphi$$

$$= \left[\begin{array}{l} u = \cos\varphi \\ du = -\sin\varphi\, d\varphi \\ \varphi = 0 \Rightarrow u = 1 \\ \varphi = \pi \Rightarrow u = -1 \end{array}\right] = -2\pi \int_1^{-1} u^2\, du$$

$$= 2\pi \left[\frac{u^3}{3}\right]_{-1}^{1} = 2\pi\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) = \frac{4\pi}{3}.$$

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Ex: Consider the surface $K$ described by $z = 1-\sqrt{x^2+y^2}$ for $z \geq 0$ with the outwards normal.

Let $\vec{F}(x,y,z) = \langle 0,0,1 \rangle$ & compute $\iint_K \vec{F} \cdot d\vec{r}$.

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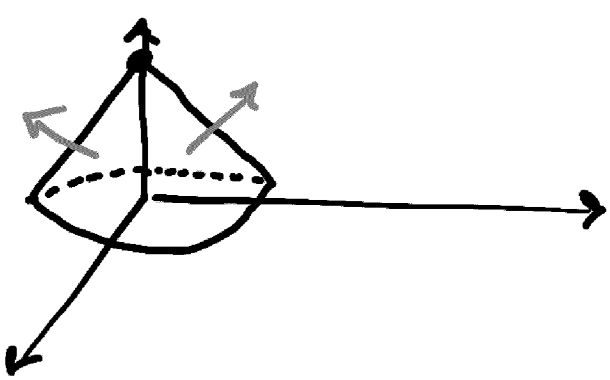
Sol: $z - 1 = -\sqrt{x^2+y^2}$

$\Leftrightarrow (1-z)^2 = x^2+y^2$ This is a cone.



We parametrize it via polar coords:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1-\sqrt{x^2+y^2} = 1-r \end{cases}$$

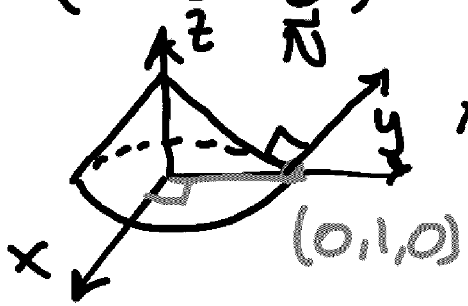
$$\vec{r}_1(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 1-r\rangle$$

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

$$\vec{N}_1(r,\theta) = \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \langle r\cos\theta,\ r\sin\theta,\ r\cos^2\theta + r\sin^2\theta\rangle$$

$$= \langle r\cos\theta,\ r\sin\theta,\ r\rangle$$




At $r=1$, $\theta = \frac{\pi}{2}$ we're at the point $(0,1,0)$ & there the normal is $\vec{N}_1(1,\frac{\pi}{2}) = \langle 0, 1, 1\rangle$ which points outwards.

$$\iint_K \vec{F}\, d\vec{r} = \iint_K \vec{F}(\vec{r}_1(r,\theta)) \cdot \vec{N}_1(r,\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \langle 0,0,1\rangle \cdot \langle r\cos\theta,\ r\sin\theta,\ r\rangle\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r \quad dr\, d\theta = 2\pi \left[\frac{r^2}{2}\right]_0^1 = \pi.$$