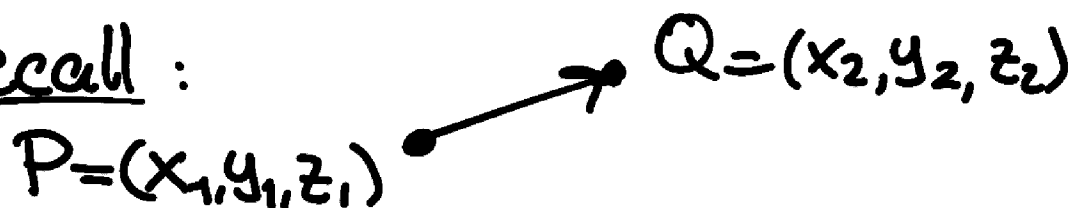


Recall:

$$\vec{v} = \overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

$$= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

$$\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle, \vec{k} = \langle 0, 0, 1 \rangle$$

"Standard unit vectors".

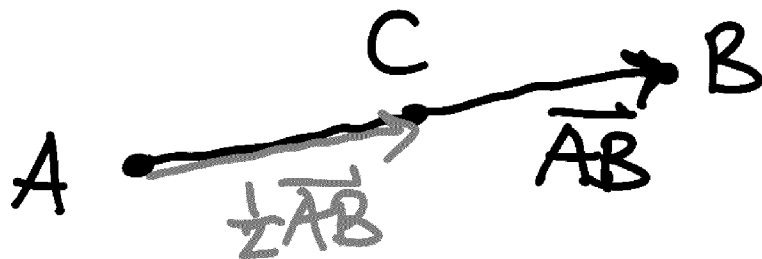
$$\|\vec{v}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EX: Let $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$.

Suppose C is the midpoint of

A, B . Express the coordinates of C in terms of those of A and B .

Sol:



$C = (x, y, z)$. We have

$$\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

$$\vec{AC} = \langle x - x_1, y - y_1, z - z_1 \rangle.$$

Note $\vec{AC} = \frac{1}{2} \vec{AB} = \left\langle \frac{x_2 - x_1}{2}, \frac{y_2 - y_1}{2}, \frac{z_2 - z_1}{2} \right\rangle$

$$\begin{cases} x - x_1 = \frac{x_2 - x_1}{2} \\ y - y_1 = \frac{y_2 - y_1}{2} \\ z - z_1 = \frac{z_2 - z_1}{2} \end{cases}$$

Solve for (x, y, z) :

$$\begin{cases} x = \frac{x_2 - x_1}{2} + x_1 = \frac{x_2 + x_1}{2} \\ y = \frac{y_2 + y_1}{2} \\ z = \frac{z_2 + z_1}{2} \end{cases}$$

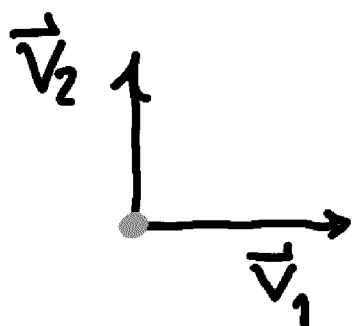
Ans: $C = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2} \right)$

§2.3 Dot product

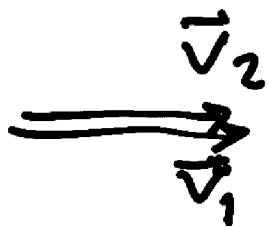
The dot product of two vectors $\vec{v}_1, \vec{v}_2$ is a real number $\vec{v}_1 \cdot \vec{v}_2$.

Roughly tells us "how much" in the same direction two vectors point.

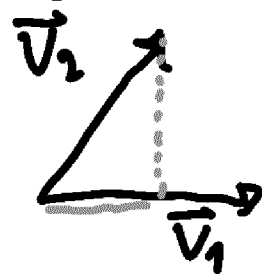
$\vec{v}_1, \vec{v}_2$ unit vectors ($\|\vec{v}_1\| = \|\vec{v}_2\| = 1$).



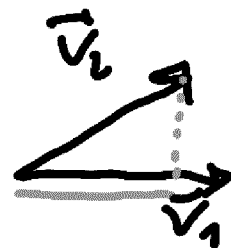
$$\vec{v}_1 \cdot \vec{v}_2 = 0$$



$$\vec{v}_1 \cdot \vec{v}_2 = 1$$



$$\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{2}$$



$$\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{4}$$

Definition: Let $\vec{v}_1 = \langle x_1, y_1, z_1 \rangle$,
 $\vec{v}_2 = \langle x_2, y_2, z_2 \rangle$.

Then $\vec{v}_1 \cdot \vec{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$.

Ex: $\langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 1 \cdot 0 + 0 \cdot 1 = 0$

$$\langle 1, 0 \rangle \cdot \langle 1, 0 \rangle = 1 \cdot 1 + 0 \cdot 0 = 1$$

$$\langle 1, 0 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = 1 \cdot \frac{1}{2} + 0 \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}$$

$$\langle 1, 0 \rangle \cdot \left\langle \frac{3}{4}, \frac{\sqrt{7}}{4} \right\rangle = 1 \cdot \frac{3}{4} + 0 \cdot \frac{\sqrt{7}}{4} = \frac{3}{4}$$

Properties :

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and let c be a scalar.

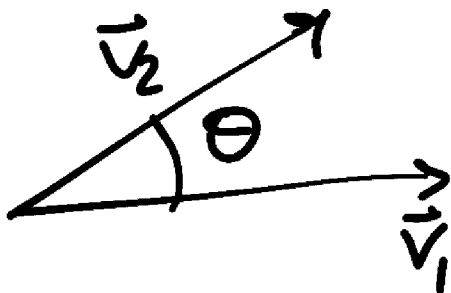
$$(i) \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$(ii) \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$(iii) c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$$

$$(iv) \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

The dot product is, as illustrated above, related to the angle between two vectors.



If $\vec{v}_1, \vec{v}_2$ are any two vectors, and $0 \leq \theta \leq \pi$ is the smallest angle between them, then

$$\boxed{\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \cos \theta.}$$

This allows us to compute the angle:

$$\Theta = \arccos \left(\frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|} \right).$$

Ex: $\vec{v}_1 = \langle 1, 1, 1 \rangle$, $\vec{v}_2 = \langle 2, -1, -3 \rangle$ then we find the angle b/w them:

$$\vec{v}_1 \cdot \vec{v}_2 = 1 \cdot 2 + 1 \cdot (-1) + 1 \cdot (-3) = 2 - 1 - 3 = -2$$

$$\|\vec{v}_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}, \|\vec{v}_2\| = \sqrt{2^2 + (-1)^2 + (-3)^2} \\ = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$\cos \Theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|} = \frac{-2}{\sqrt{3} \cdot \sqrt{14}} = -\frac{2}{\sqrt{42}}$$

$$\Theta = \arccos \left(-\frac{2}{\sqrt{42}} \right) (\approx 108^\circ).$$

Def: We say that $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$. $\vec{v} \perp \vec{u}$

(Corresponds to angle $\Theta = \pi$.)

Ex, For what value of x is

$\vec{u} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{v} = \langle x, -1, 2 \rangle$?

Sol: They are orthogonal if $\vec{u} \cdot \vec{v} = 0$

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \langle 2, 8, -1 \rangle \cdot \langle x, -1, 2 \rangle = 2x + 8(-1) + (-1) \cdot 2 \\ &= 2x - 8 - 2 = 2x - 10 = 0\end{aligned}$$

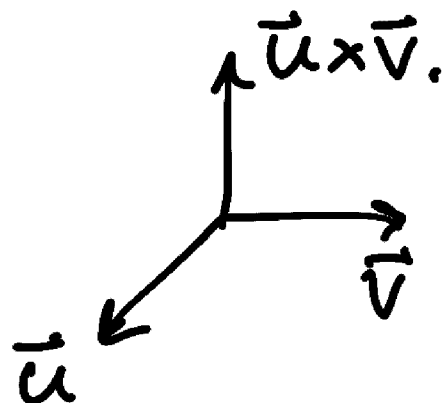
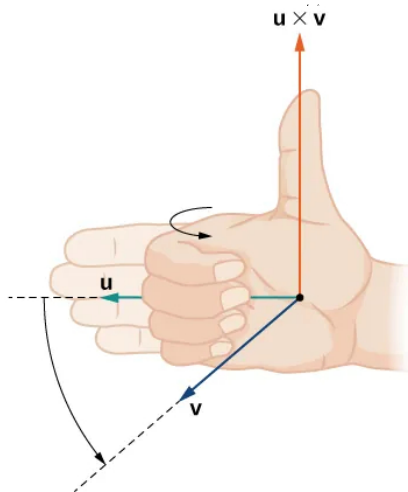
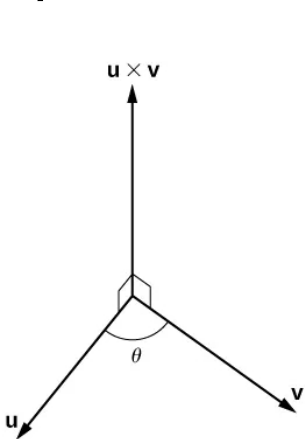
$$\Leftrightarrow 2x = 10 \Leftrightarrow \boxed{x = 5}$$

§2.4 Cross product

This only works in space and not in the plane.

The cross product of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ is a vector

$\vec{u} \times \vec{v}$ that is orthogonal to both $\vec{u}$ and $\vec{v}$.



The right hand rule determines which direction $\vec{u} \times \vec{v}$ should point.

Definition: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$
 $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2) \vec{i} - (u_1 v_3 - u_3 v_1) \vec{j} + (u_1 v_2 - u_2 v_1) \vec{k}.$$

Let's check that $\vec{u}$ is orthogonal to $\vec{u} \times \vec{v}$, namely that $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

$$\begin{aligned} \vec{u} \cdot (\vec{u} \times \vec{v}) &= u_1(u_2 v_3 - u_3 v_2) - u_2(u_1 v_3 - u_3 v_1) \\ &\quad + u_3(u_1 v_2 - u_2 v_1) = \underline{u_1 u_2 v_3} - \underline{u_1 u_3 v_2} \\ &\quad - \underline{u_1 u_2 v_3} + \underline{u_2 u_3 v_1} + \underline{u_1 u_3 v_2} - \underline{u_2 u_3 v_1} = 0 \end{aligned}$$

Can similarly check $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Ex: Let $\vec{p} = \langle 5, 1, 2 \rangle$, $\vec{q} = \langle -2, 0, 1 \rangle$ then

$$\begin{aligned} \vec{p} \times \vec{q} &= (1 \cdot 1 - 2 \cdot 0) \vec{i} - (5 \cdot 1 - 2 \cdot (-2)) \vec{j} \\ &\quad + (5 \cdot 0 - 1 \cdot (-2)) \vec{k} = \vec{i} - 9 \vec{j} + 2 \vec{k} = \langle 1, -9, 2 \rangle \end{aligned}$$

Properties: $\vec{u}, \vec{v}, \vec{w}$ vectors, c scalar

$$(i) \vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$$(ii) \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$(iii) c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$$

$$(iv) \vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$$

$$(v) \vec{v} \times \vec{v} = \vec{0}$$

$$(vi) \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

Quick guide to determinants:

The formula for the cross product is hard to memorize but is easier when using "determinants."

2x2 determinants:

a, b, c, d scalars, then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \text{ by def.}$$

3x3. $a, b, \dots, i$ scalars, then

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

We can then reformulate the formula for the cross product as follows:

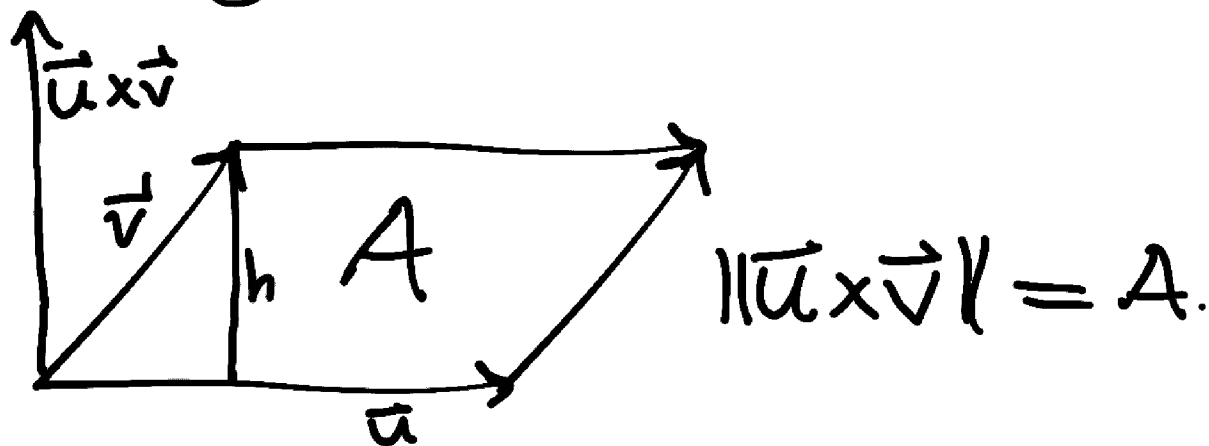
$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \\ &+ \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = (u_2 v_3 - v_2 u_3) \vec{i} \\ &- (u_1 v_3 - v_1 u_3) \vec{j} + (u_1 v_2 - v_1 u_2) \vec{k} \end{aligned}$$

Ex: $\vec{p} = \langle -1, 2, 5 \rangle$, $\vec{q} = \langle 4, 0, -3 \rangle$

$$\vec{p} \times \vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix}$$

$$+ \vec{k} \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} = (-6 - 0)\vec{i} - (3 - 20)\vec{j} + (0 - 8)\vec{k} \\ = -6\vec{i} + 17\vec{j} - 8\vec{k}$$

Geometrically, it turns out that $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.



Also: $\|\vec{u} \times \vec{v}\| = \underbrace{\|\vec{u}\|}_{\text{base}} \cdot \underbrace{\|\vec{v}\| \sin \theta}_{\text{height of parallelogram}}.$
