MAT203 L19 Wed Apr 22

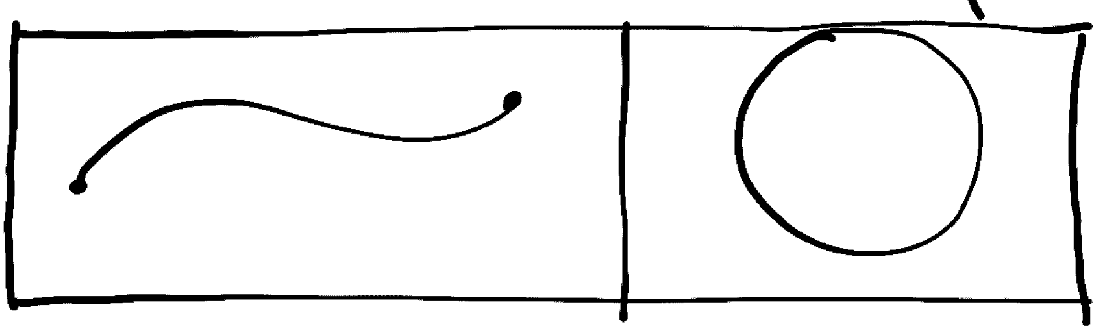
Recall: • Let $\vec{F}$ be a conservative vector field with potential f. If C is a curve with parametrization $\vec{r}(t)$, $t_0 \le t \le t_1$ then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(t_1)) - f(\vec{r}(t_0)).$$

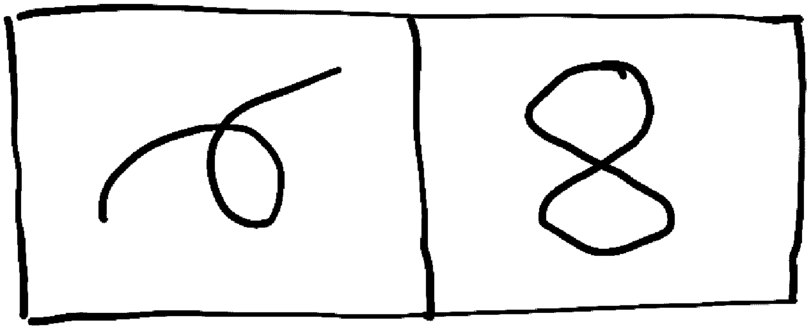
§6.4 Green's theorem

Def: A curve C in the plane is simple if it never intersects itself.

Ex: These curves are simple:



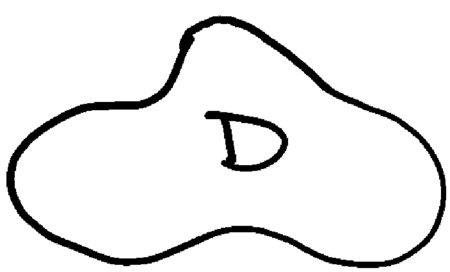
These are not:



<u>Def</u>: A domain in the xy-plane is called <u>simply</u> <u>connected</u> if "there are no holes."

<u>Remark</u>: The correct definition is that D is simply connected if any closed loop in D can be continuously shrunk to a point while remaining completely in D.

<u>Ex</u>: This domain is simply connected:



This is <u>not</u>:

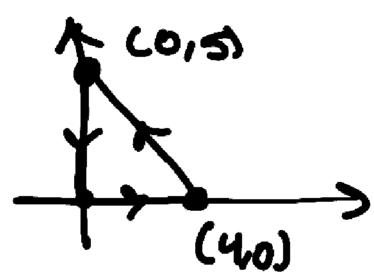
Theorem (Green's theorem)

Let D be a simply connected domain, with boundary curve C that is simple, and oriented counterclockwise. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field. Then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy}$$

Ex: Consider $\int_C \vec{F}\cdot d\vec{r}$ where

C is the curve



and $\vec{F}(x,y) = \langle y^2, 2xy+1\rangle$.

Last time we computed $\int_C \vec{F}\cdot d\vec{r} = 0$.

Let's now compute it using

Green's theorem instead.

$$\langle y^2, 2xy+1\rangle = \langle P(x,y), Q(x,y)\rangle$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 2y, \text{ so}$$

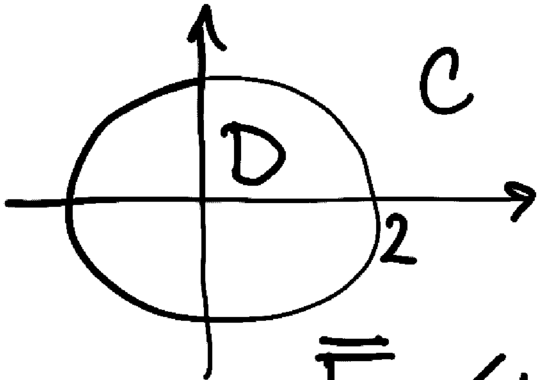
$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0} \, dx\,dy = 0.$$

<u>Ex</u>: Let $\vec{F}(x,y) = \langle y + \sin x, e^y + x^2\rangle$ and let C be the unit circle with radius 2, oriented counter-clockwise.

Let's calculate $\int_C \vec{F}\cdot d\vec{r}$.

You can try to calculate it directly via the definition, but it seems close to impossible.

We will use Green's theorem instead:



$D = \{(x,y) \mid x^2+y^2 \le 4\}$.

$\vec{F} = \langle y + \sin x,\ e^y + x^2 \rangle = \langle P, Q \rangle$

$\dfrac{\partial Q}{\partial x} = 2x \qquad \dfrac{\partial P}{\partial y} = 1$

so $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_D 2x - 1 \, dx\,dy$.

Change to polar coords:

$D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$$\iint_D 2x-1 \, dx\,dy = \int_0^{2\pi}\int_0^2 (2r\cos\theta - 1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{2r^3}{3}\sin\theta - \frac{r^2}{2}\right]_0^2 d\theta = \int_0^{2\pi} \frac{16}{3}\sin\theta - 2\,d\theta$$

$$= \left[-\frac{16}{3}\cos\theta - 2\theta\right]_0^{2\pi}$$

$$= -\frac{16}{3}(1-1) - 4\pi = -4\pi$$

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§6.5 Divergence & Curl

If f is a scalar function, its gradient is $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$, and ∇f is a vector field.

If we instead have a vector field $\vec{F}$, we can construct a scalar function that measures how much a flow is "flowing outwards".

Def: If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field the divergence of $\vec{F}$ is

$$\boxed{\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

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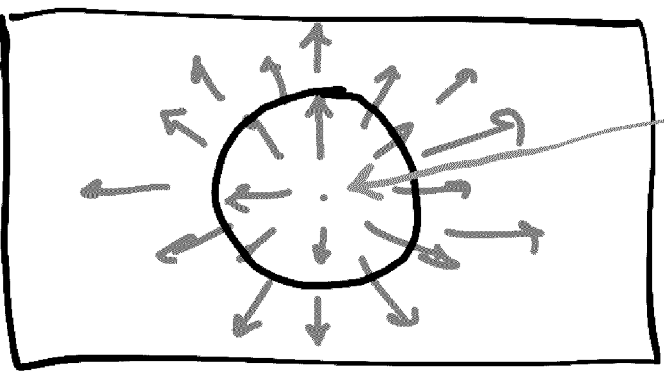
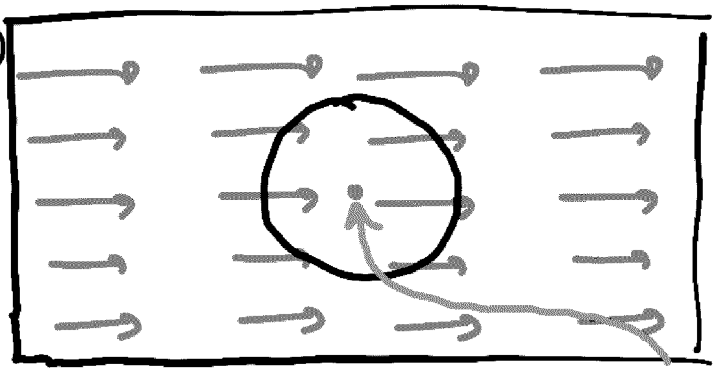
1 

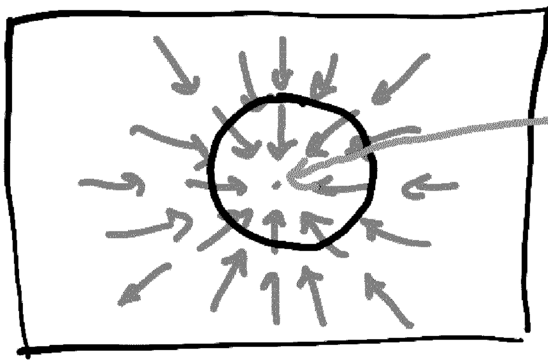
Divergence is >0 here more of the vector field flows outwards at this point.

(2)



Divergence at this point is zero since the vector field flows in & out an "equal amount."

(3)



Divergence at this point is <0 since more of the vector field is flowing in to the center.

<u>Ex.</u>: The actual vector fields in 1, 2, 3 above are:

(1) $\vec{F}(x,y) = \langle x, y \rangle$, and we have $(\nabla \cdot \vec{F})(x,y) = 1 + 1 = 2$.

(2) $\vec{G}(x,y) = \langle 1, 0 \rangle$, and we have $(\nabla \cdot \vec{G})(x,y) = 0 + 0 = 0$

(3) $\vec{H}(x,y) = \langle -x, -y \rangle$, and we have $(\nabla \cdot \vec{H})(x,y) = -1 - 1 = -2$.

<u>Ex</u>: Let $\vec{F} = \langle e^x, yz, -yz^2 \rangle$, and let's compute the divergence at $(0,2,-1)$:

$$(\nabla \cdot \vec{F})(0,2,-1) = \frac{\partial e^x}{\partial x}(0,2,-1) + \frac{\partial (yz)}{\partial y}(0,2,-1)$$

$$+ \frac{\partial(-yz^2)}{\partial z}(0,2,-1) = e^0 + (-1) + \underbrace{(-2(2)(-1))}_{=4}$$

$$= 4.$$

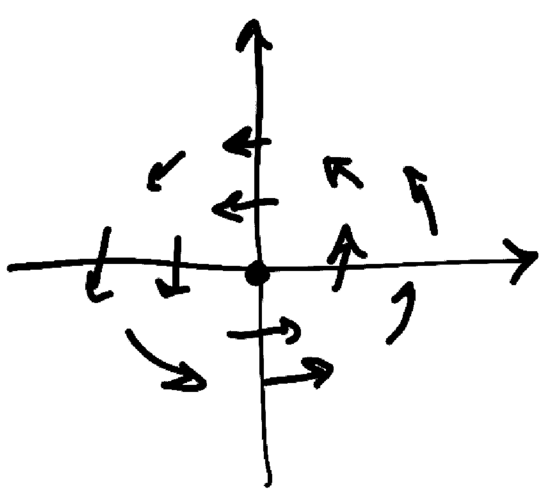
<u>Def</u>: Let $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ be a vector field in space. The <u>curl of $\vec{F}$</u> is the vector field

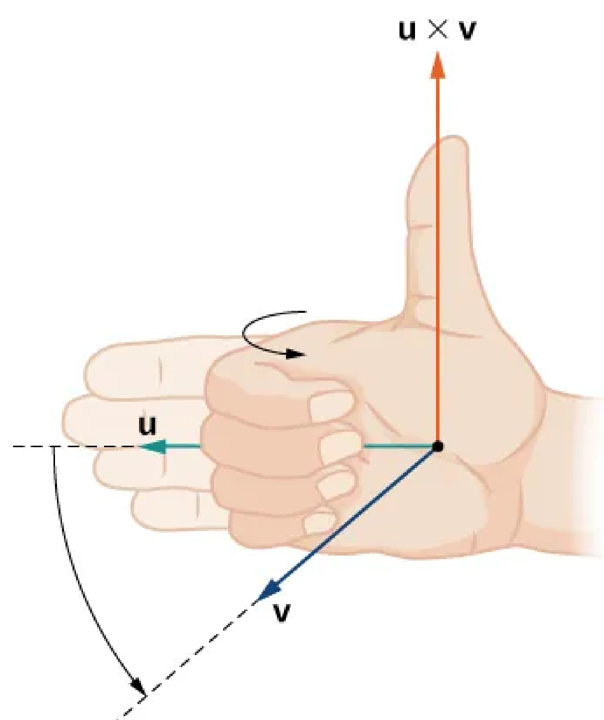
$$(\nabla \times \vec{F})(x,y,z) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

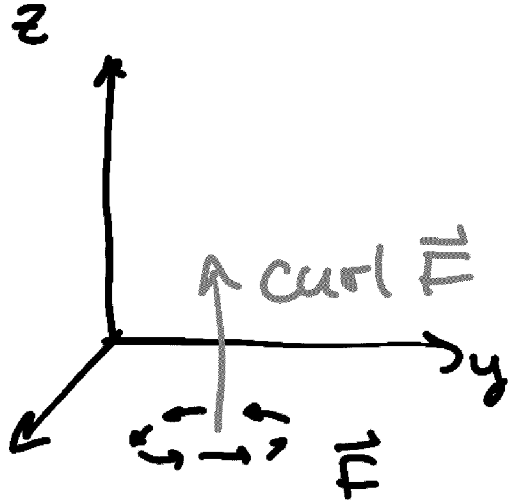
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j}$$
$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

The curl roughly measures how much the vector field makes a floating object rotate.

If we consider the vector field $\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$ then in each slice $z=c$ in space, $\vec{F}$ looks like:







Ex: Let $\vec{F}(x,y,z) = \langle x^2 z, e^y + xz, xyz \rangle$.

Then we compute the curl:

$$(\nabla \times F)(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & e^y + xz & xyz \end{vmatrix}$$

$$= \langle xz - x, x^2 - yz, z \rangle$$
