MAT 203 L18 Mon Apr 20

<u>Recall</u>: • A vector field $\vec{F}(x,y,z)$ is called <u>conservative</u> if there is a function $f(x,y,z)$ such that

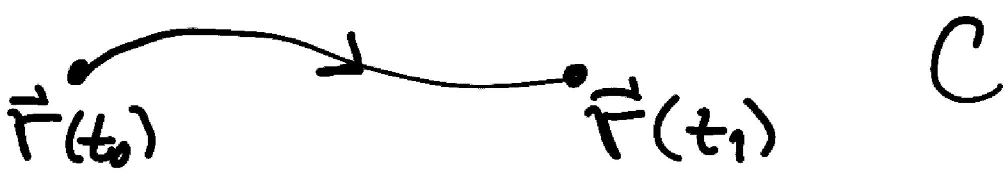
$$\vec{F}(x,y,z) = \nabla f(x,y,z)$$

• If C is a curve parametrized by $\vec{r}(t)$, $t_0 \le t \le t_1$, and $\vec{F}(x,y,z)$ is a vector field, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

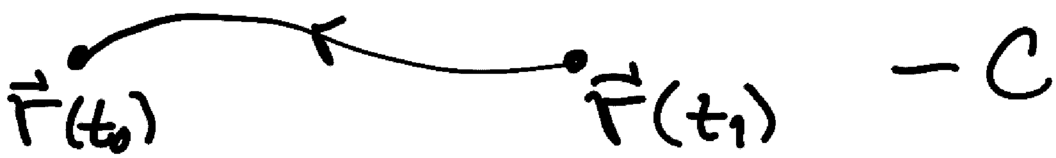
Recall that a curve w/ a parametrization has an <u>orientation</u>.

$\vec{r}(t)$, $t_0 \le t \le t_1$



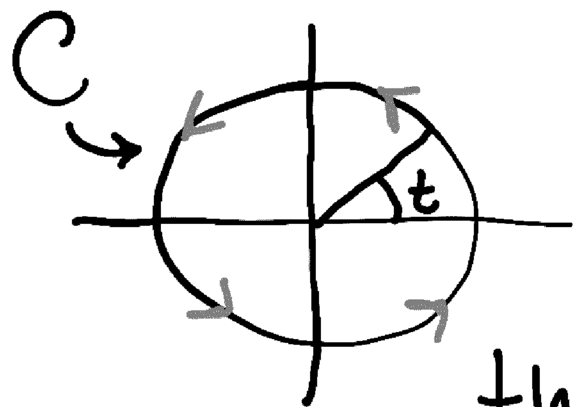
We can reverse t, to get the same curve, but with opposite ori.

$\vec{r}(-t)$, $t_0 \leq t \leq t_1$



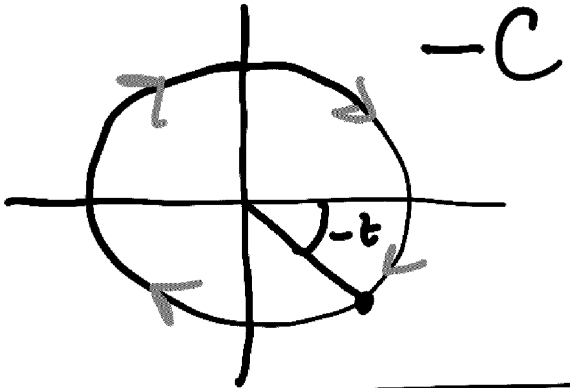
The curve $-C$ is C with the opposite orientation.

<u>Ex</u>: $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$



orientation is counterclockwise (ccw) since t measures the angle, and as t increases, the rotation goes CCW.

If we change the parametrization to $\vec{s}(t) = \vec{r}(-t) = \langle \cos(-t), \sin(-t) \rangle$ we get the same curve but with the opposite orientation



Ex: $\vec{F}(x,y) = \langle -y, x \rangle$

$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le \pi.$

Last time: $\int_C \vec{F} \cdot d\vec{r} = \pi.$

Now let's compute $\int_{-C} \vec{F} \cdot d\vec{r}.$

The curve $-C$ is parametrized by

$$\vec{s}(t) = \vec{r}(-t) = \langle \cos(-t), \sin(-t) \rangle$$
$$= \langle \cos t, -\sin t \rangle.$$

$0 \le t \le \pi$ becomes $0 \le -t \le \pi$

$\Leftrightarrow \; 0 \ge t \ge -\pi.$

$$\int_{-C} \vec{F}(\vec{s}(t)) \cdot \vec{s}\,'(t)\,dt$$

$$= \int_{-\pi}^{0} \vec{F}(\cos t, -\sin t) \cdot \langle -\sin t, -\cos t \rangle \, dt$$

$$= \int_{-\pi}^{0} \langle \sin t, \cos t \rangle \cdot \langle -\sin t, -\cos t \rangle \, dt$$

$$= \int_{-\pi}^{0} -\sin^2 t - \cos^2 t \, dt = \int_{-\pi}^{0} -1 \, dt = -\pi.$$

Properties of line integrals:

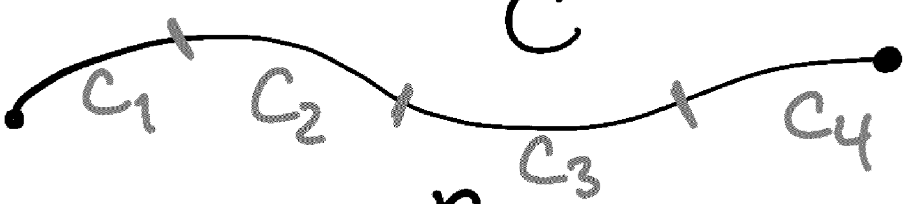
(1) $\displaystyle\int_C (\vec{F}+\vec{G}) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C \vec{G} \cdot d\vec{r}$

(2) $\displaystyle\int_C k\vec{F} \cdot d\vec{r} = k \int_C \vec{F} \cdot d\vec{r}$

Scalar

(3) $\displaystyle\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$

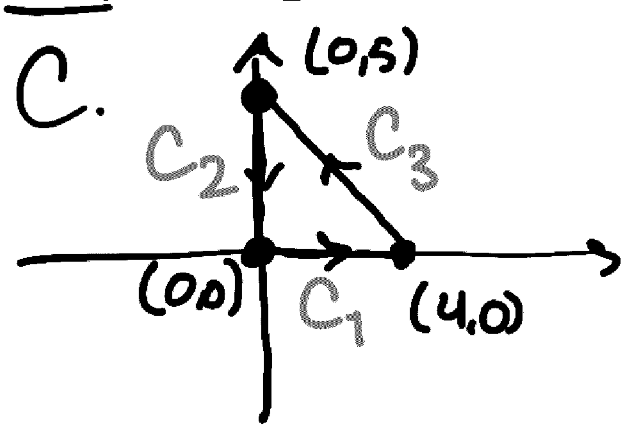
(4) If C is the concatenation of the curves $C_1, \ldots, C_n$



$$\int_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n} \int_{C_i} \vec{F} \cdot d\vec{r} .$$

<u>Ex</u>: Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle y^2, 2xy+1 \rangle$, and C is the triangle in the plane with vertices $(0,0)$, $(4,0)$ and $(0,5)$ oriented counterclockwise.

<u>Sol</u>: We first have to parametrize C.



$\boxed{C_1}$ $\vec{r}_1(t) = \langle t, 0 \rangle \quad 0 \le t \le 4$

$\boxed{C_2}$ $\vec{r}_2(t) = \langle 0, -t \rangle \quad -5 \le t \le 0$

$\boxed{C_3}$ Direction vector:

$\langle 0,5 \rangle - \langle 4,0 \rangle = \langle -4, 5 \rangle$

Start pt: $(4,0)$

$\vec{r}_3(t) = \langle 4,0 \rangle + t\langle -4,5 \rangle, \quad 0 \le t \le 1$

$= \langle 4-4t, 5t \rangle, \quad 0 \le t \le 1.$

<u>Integrals</u>:

① $\vec{F}(\vec{r}_1(t)) = \vec{F}(\langle t,0 \rangle) = \langle 0,1 \rangle$

$\int_{C_1} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt = \int_0^4 \langle 0,1 \rangle \cdot \langle 1,0 \rangle\,dt$

$= 0$

② $\vec{F}(\vec{r}_2(t)) = \vec{F}(\langle 0,-t \rangle) = \langle t^2, 1 \rangle$

$\int_{C_2} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt = \int_{-5}^{0} \langle t^2, 1 \rangle \cdot \langle 0,-1 \rangle\,dt$

$$= \int_{-5}^{0} -1\,dt = \int_{0}^{-5} dt = -5$$

③ $\vec{F}(\vec{r}_3(t)) = \vec{F}(\langle 4-4t, 5t\rangle)$

$$= \langle 25t^2, 2(5t)(4-4t)+1\rangle = \langle 25t^2, 40t-40t^2+1\rangle$$

$$\int_{C_3} \vec{F}(\vec{r}_3(t)) \cdot \vec{r}_3\,'(t)\,dt = \int_0^1 \langle 25t^2, 40t-40t^2+1\rangle \cdot \langle -4, 5\rangle\,dt$$

$$= \int_0^1 -100t^2 + 5(40t - 40t^2 + 1)\,dt$$

$$= \int_0^1 -100t^2 + 200t - 200t^2 + 5\,dt$$

$$= \left[-100t^3 + 100t^2 + 5t\right]_0^1 = 5$$

SO: $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$

$$+ \int_{C_3} \vec{F} \cdot d\vec{r} = 0 - 5 + 5 = 0.$$

<u>EX</u>: Let $\vec{F}(x,y,z) = \langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz\rangle$

and C is a curve with parametrization $\vec{r}(t) = \langle t^2, t, t \rangle$, $1 \le t \le e$.

Then let's compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^e \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

$\vec{F}(\vec{r}(t)) - \vec{F}(t^2, t, t)$

$= \langle 2t^2 \ln(t), t^3 + t^2, 2t^2 \rangle$

$\vec{r}\,'(t) = \langle 2t, 1, 1 \rangle$. Then

$$\int_1^e \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_1^e \langle 2t^2 \ln(t), t^3 + t^2, 2t^2 \rangle \cdot \langle 2t, 1, 1 \rangle\, dt$$

$$= \int_1^e 4t^3 \ln(t) + t^3 + t^2 + 2t^2\, dt$$

$$= \left[\frac{t^4}{4} + t^3 \right]_1^e + \int_1^e 4t^3 \ln(t)\, dt$$

$$= \left[\frac{e^4 - 1}{4} + e^3 - 1 \right] + \left[t^4 \ln(t) \right]_1^e$$

$$-\int_1^e t^4 \cdot \frac{1}{t}\,dt = \left(\frac{e^4-1}{4}+e^3-1\right)$$

$$+\left(e^4 \underbrace{\ln(e)}_{=1}\right) - \left[\frac{t^4}{4}\right]_1^e$$

$$=\left(\frac{e^4-1}{\cancel{4}}+e^3-1\right)+e^4-\left(\frac{e^4-1}{\cancel{4}}\right)$$

$$= e^4+e^3-1.$$

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§6.3 Conservative Vector Fields

Recall that a vector field $\vec{F}$ is Conservative if $\vec{F}=\nabla f$ for some function f called a potential.

Also recall from Calc II:

$$\int_a^b \frac{df}{dx}\,dx = f(b)-f(a)$$

(fundamental theorem of Calculus)

Thm (Fundamental theorem of line integrals)

Let C be a curve with parametrization $\vec{r}(t)$, $a \leq t \leq b$, and let f be a differentiable function. Then

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

Ex: Let's consider the previous example with

$$\vec{F}(x,y,z) = \langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz \rangle.$$

Is this vector field conservative?

$$\vec{F} = \nabla f \Leftrightarrow \begin{cases} \frac{\partial f}{\partial x} = 2x\ln(y) & (1) \\ \frac{\partial f}{\partial y} = \frac{x^2}{y} + z^2 & (2) \\ \frac{\partial f}{\partial z} = 2yz & (3) \end{cases}$$

Integrate 1st eq: $f(x,y,z) = x^2 \ln(y) + g(y,z)$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} + \frac{\partial g}{\partial y} \overset{(2)}{=} \frac{x^2}{y} + z^2$$

$$\Rightarrow \frac{\partial g}{\partial y} = z^2 \Rightarrow g(y,z) = yz^2 + h(z).$$

So $f(x,y,z) = x^2 \ln(y) + yz^2 + h(z)$.

$$\frac{\partial f}{\partial z} = 2yz + h'(z) \overset{(3)}{=} 2yz$$

$$\Leftrightarrow h'(z) = 0 \Rightarrow h(z) = C$$

So $f(x,y,z) = x^2 \ln(y) + yz^2 + C$

is a potential:

$$\vec{F} = \nabla f.$$

Therefore

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

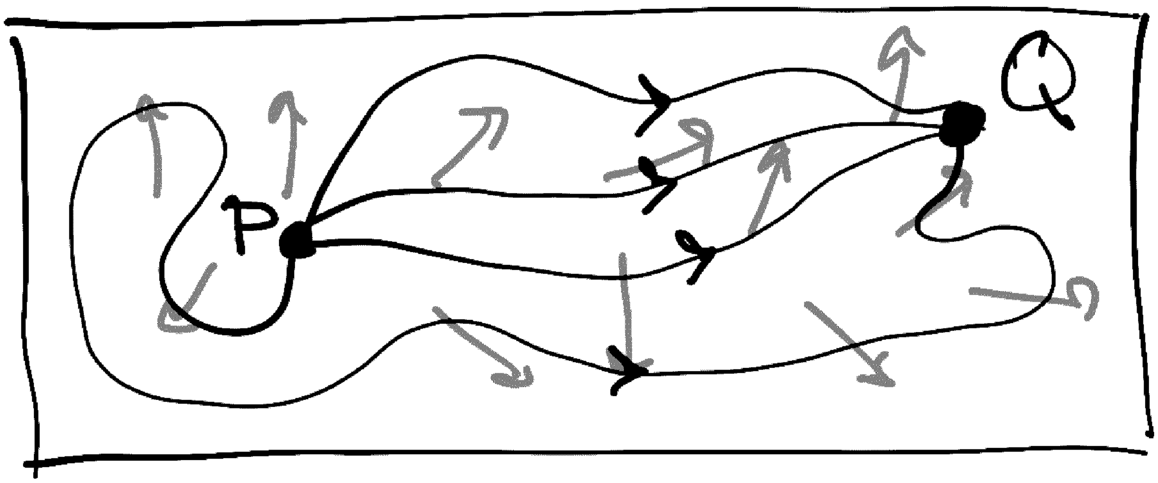
$$= f(\vec{r}(e)) - f(\vec{r}(1))$$

$$= f(e^2, e, e) - f(1,1,1)$$

$$= \left(e^4 \ln(e) + e^3 + C\right) - \left(1 \underbrace{\ln(1)}_{=0} + 1 + C\right)$$

$$= e^4 + e^3 - 1.$$

If $\vec{F}$ is conservative, it's only the endpoints of the path that matters for $\int_C \vec{F} \cdot d\vec{r}$.



Only the value of the potential f of $\vec{F}$ at endpts P and Q matter for $\int_C \nabla f \cdot d\vec{r}$
