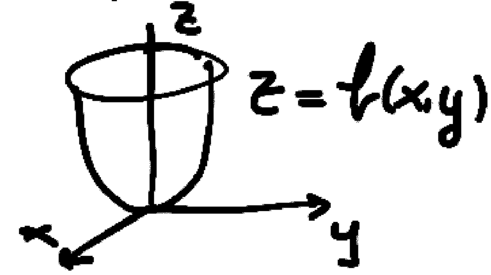
§6.1 Vector fields

<u>Definition</u>: A vector field is an assignment of a vector at every point in the plane, or in space.

<u>Ex</u>: Gradient vector field of a function $f(x,y)$.

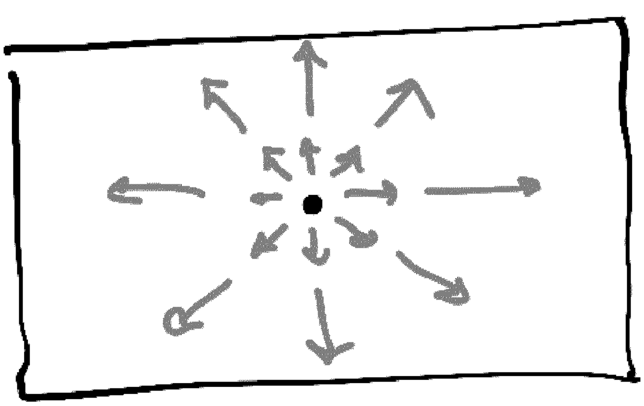


e.g. $f(x,y)=x^2+y^2$

$\nabla f(x,y)=\langle 2x,2y\rangle$.

So in the plane we draw the vector $\langle 2x,2y\rangle$ at position (x,y):

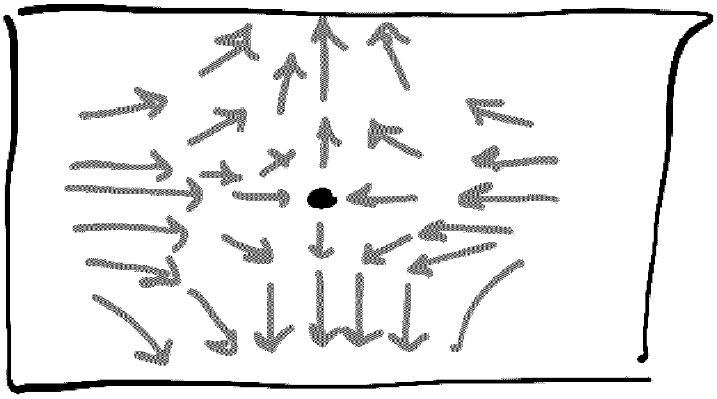
xy-plane



e.g. $f(x,y) = x^2 - y^2$

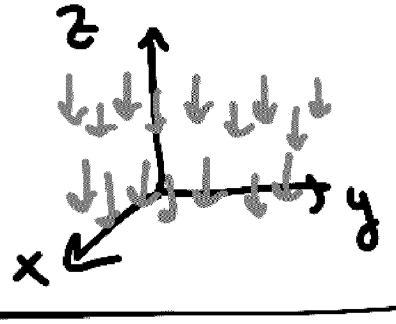
$\nabla f(x,y) = \langle 2x, -2y \rangle$

saddle



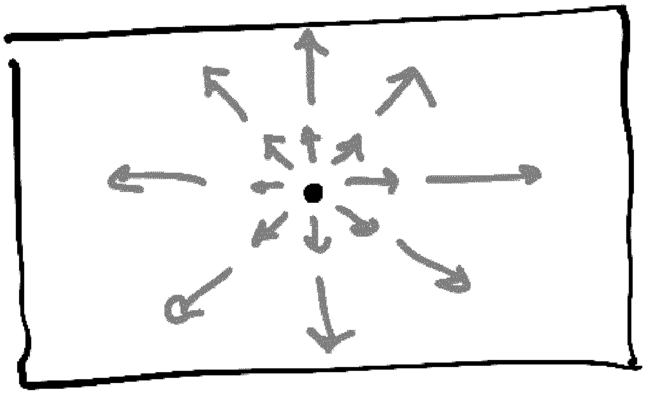
xy-plane

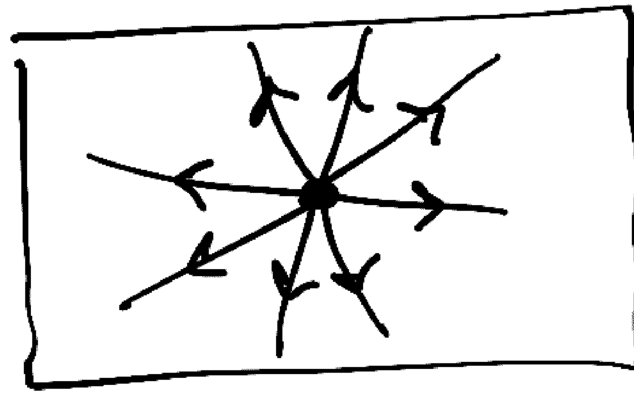
<u>Ex</u>: Gravitational vector field:
It's a vector field in space that assigns every pt to the vector
$\langle 0, 0, -g \rangle$ $(g \approx 9.82\ m/s^2)$

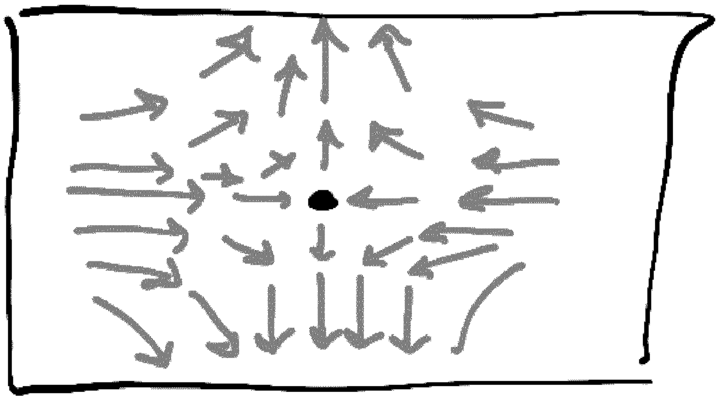


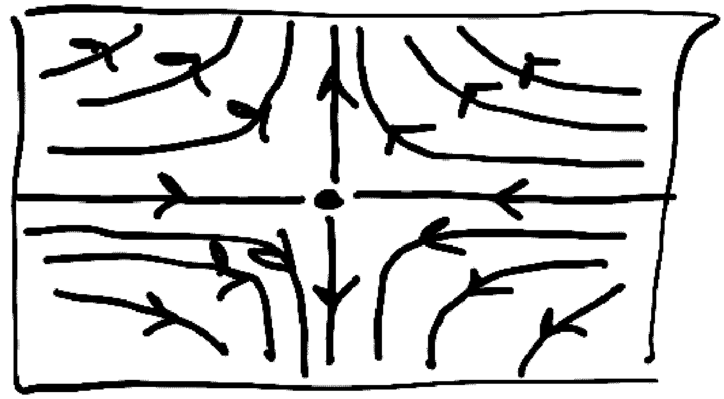
<u>Definition</u>: A flow line of a vector field is a curve that is tangent to the vector field.

Ex. Vector field $\qquad$ flow lines









Conservative vector fields:

A vector field $\vec{F}$ is called "conservative" if

$$\boxed{\vec{F} = \nabla f}$$ for some function f.

If $\vec{F}$ is conservative, such a function is called a potential function.

We will later see why this is important.

<u>Ex:</u> $f(x,y,z) = x^2yz - \sin(xy)$ is a potential function for

$$\vec{F}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y \rangle$$

Since we may compute ∇f, and we get

$$\nabla f = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y \rangle = \vec{F}.$$

<u>Theorem:</u> If f and g are potential functions for a vector field $\vec{F}$, then $f = g + \text{Constant}$.

<u>Ex.</u> Find a potential function for $\vec{F}(x,y,z) = \langle 0, 0, -g \rangle$

By inspection we see that

$f(x,y,z) = -gz$ is a potential.

So every potential is $-gz+C$, where C is a constant.

Ex. Find a potential for

$\vec{F}(x,y) = \langle y, x+\cos(y)\rangle$.

Sol: If there's a potential $f(x,y)$, then

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = \langle y, x+\cos(y)\rangle$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = y \\[2ex] \dfrac{\partial f}{\partial y} = x+\cos(y) \end{cases}$$

We now solve this system.

1st eq gives $f(x,y) = xy + g(y)$

then $\dfrac{\partial f}{\partial y} = x + g'(y) = x+\cos(y)$

$\Leftrightarrow g'(y) = \cos(y) \Leftrightarrow g(y) = \sin(y) + C$

So $f(x,y) = xy + \sin(y) + C$ is a potential.

<u>Ex</u>, Find a potential for $\vec{F}(x,y,z) = \langle 7, -2, x^3 \rangle$.

<u>Sol</u>: Like above we try to solve $\nabla f = \vec{F}$.

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 7, -2, x^3 \rangle$$

$$\begin{cases} \partial f/\partial x = 7 \\ \partial f/\partial y = -2 \\ \partial f/\partial z = x^3 \end{cases}$$

1^{st} eq $\Rightarrow f(x,y,z) = 7x + g(y,z)$.

Inputting it into 2^{nd} eq gives

$$\frac{\partial g}{\partial y} = -2 \Rightarrow g(y,z) = -2y + h(z)$$

Now inserting it into 3rd equ

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(7x-2y+h(z)\right) = \boxed{h'(z) = x^3}$$

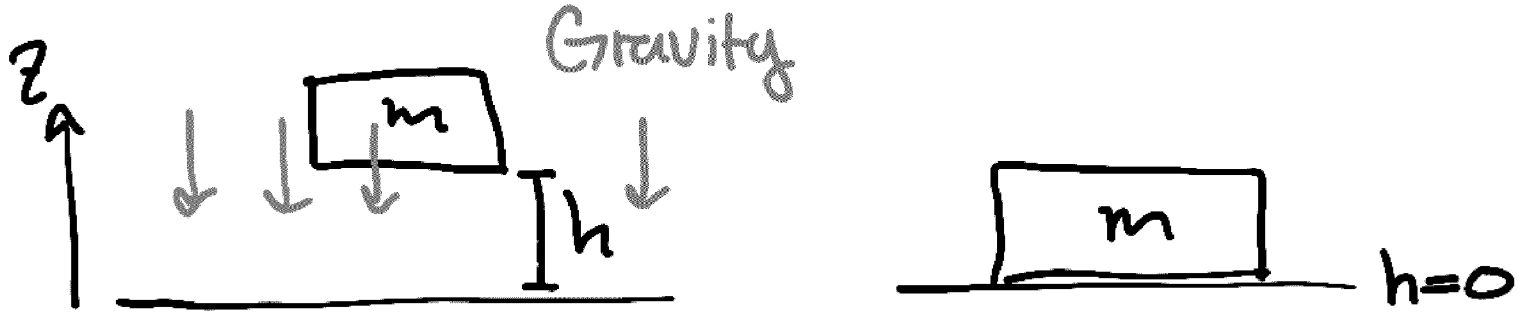
Since left hand side only depends on z, and right hand side only depends on x, they can not be equal for all (x,y,z).

Conclusion: There is <u>no</u> potential!

---

§6.2 <u>Line integrals</u>

Energy/Work in physics.



box w/ mass m

The work done by gravity on the box is

$$W = \text{Force} \cdot \Delta\text{height}$$
$$= \langle 0,0,-mg\rangle \cdot \langle 0,0,0-h\rangle$$
$$= mgh$$

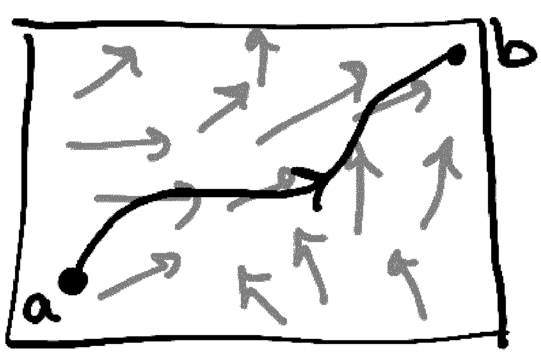
Another way to phrase this is as an integral:

$$\int_0^h mg\,dz = \int_0^h \text{Force}\,dz$$

This kind of integral is the special case of a more general integral:

If $\vec{F}$ is a vector field in the plane (let's say),

$\vec{F}$



then if C is a parametrized curve in the plane

one can compute the work done requires to walk from a to b.

How do we do it?

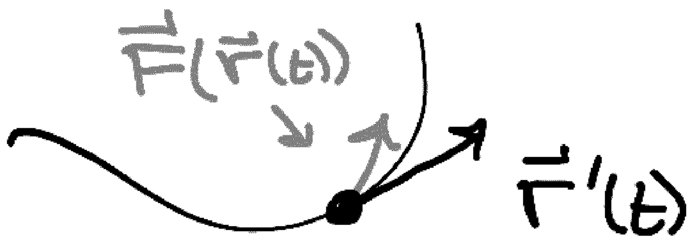
Let's assume the parametrization of C is given by:

$$\vec{r}(t), \quad t_0 \le t \le t_1$$

So that $\begin{cases} \vec{r}(t_0) = a \\ \vec{r}(t_1) = b \end{cases}$

Then $\text{Energy} = \displaystyle\int_{t_0}^{t_1} \text{Force}(\vec{r}(t))\,dt$

but what is Force at $\vec{r}(t)$?



$\text{Force}(\vec{r}(t)) = \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)$. So

$$\boxed{\text{Energy} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\,dt.}$$

Now, we can write this as

$$\boxed{\text{Energy} = \int_C \vec{F} \cdot d\vec{r}}$$

This is an example of a *line integral*.

<u>Def</u>: Let C be a parametrized curve in space, and let $\vec{F}$ be a vector field. Suppose the parametrization of C is $\vec{r}(t)$ for $t_0 \leq t \leq t_1$. Then

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

<u>Ex</u>: Consider the curve C parametrized by $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq \pi$, and consider the vector field $\vec{F}(x,y) = \langle -y, x \rangle$.

Let's compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}(t)\, dt \quad (*)$$

First $\vec{F}(\vec{r}(t)) = \vec{F}(\cos t, \sin t)$

$= \langle -\sin t, \cos t \rangle$, and

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$, so

$$(*) = \int_0^{\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{\pi} (-\sin t)^2 + (\cos t)^2\, dt = \int_0^{\pi} dt = \pi.$$