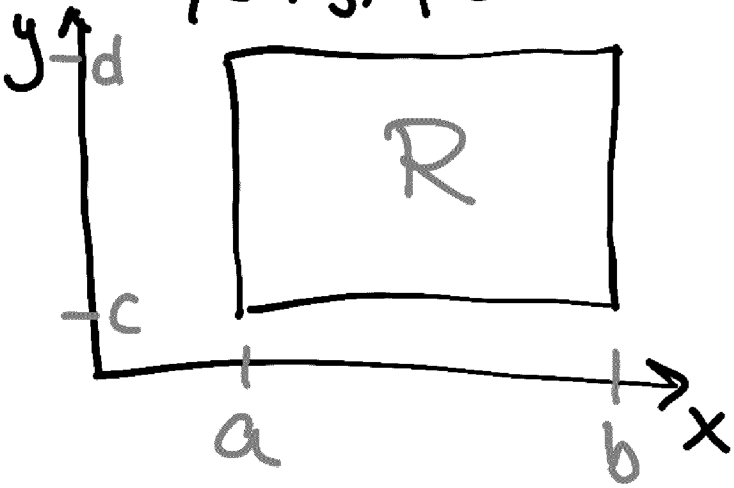
MAT 203 L15 Wed Apr 1

Recall: **Double integrals**:

If $R = [a,b] \times [c,d]$

$= \{(x,y) \mid a \le x \le b \text{ and } c \le y \le d\}$,



then $\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\,dx \right) dy$

$= \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\,dy \right) dx$

Ex: $\int_0^1 \int_1^2 \frac{x}{x^2+y^2}\,dy\,dx$. Integrate in x

first. $\int_0^1 \frac{x}{x^2+y^2}\,dx = \left[\begin{array}{l} u = x^2+y^2 \\ du = 2x\,dx \\ u=0 \Rightarrow x=y^2 \\ u=1 \Rightarrow x=1+y^2 \end{array} \right]$

$$= \frac{1}{2}\int_{y^2}^{1+y^2} \frac{1}{u}\,du = \frac{1}{2}\Big[\ln(u)\Big]_{y^2}^{1+y^2}$$

$$= \frac{1}{2}\left(\ln(1+y^2) - \ln(y^2)\right) = \frac{1}{2}\ln\left(\frac{1+y^2}{y^2}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1}{y^2}+1\right).$$ So

$$\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dy\,dx = \int_1^2\left(\int_0^1 \frac{x}{x^2+y^2}\,dx\right)dy$$

$$= \int_1^2 \frac{1}{2}\underbrace{\ln\left(1+\frac{1}{y^2}\right)}_{u}\,\underbrace{dy}_{v}$$. Partial integration:

$$du = \frac{1}{1+\frac{1}{y^2}}\cdot\left(-\frac{2}{y^3}\right)dy,$$ so we get

$$\frac{1}{2}\left[\ln\left(1+\frac{1}{y^2}\right)y\right]_1^2 - \frac{1}{2}\int_1^2 y\left(-\frac{2}{y^3}\right)\frac{1}{1+\frac{1}{y^2}}\,dy$$

$$= \frac{1}{2}\left(2\ln(5/4) - \ln(2)\right) - \frac{1}{2}\int_1^2 \frac{-2}{y^2+1}\,dy$$

$$= \ln(5/4) - \ln(\sqrt{2}) + \int_1^2 \frac{1}{y^2+1}\,dy$$

$$= \ln\left(\frac{5}{4\sqrt{2}}\right) + \Big[\arctan y\Big]_1^2 = \ln\left(\frac{5}{4\sqrt{2}}\right)$$

$+ \arctan(2) - \frac{\pi}{4}.$

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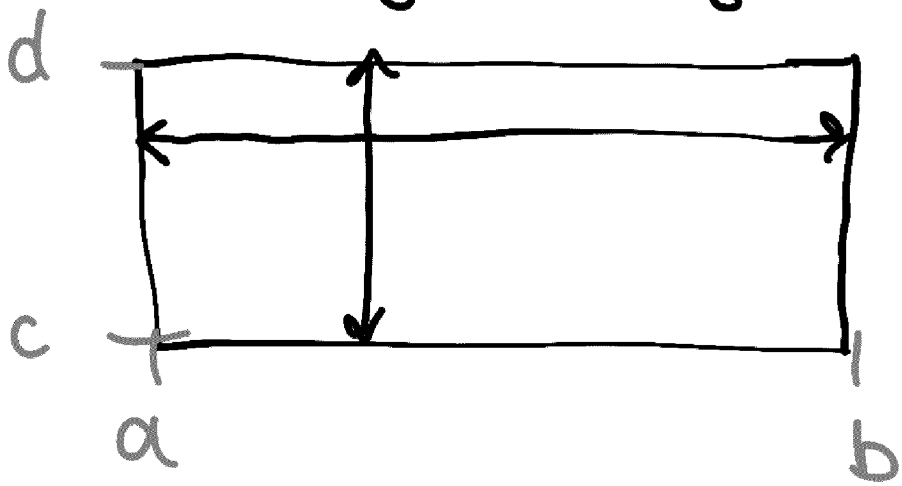
§5.2 More general regions

For rectangular regions:

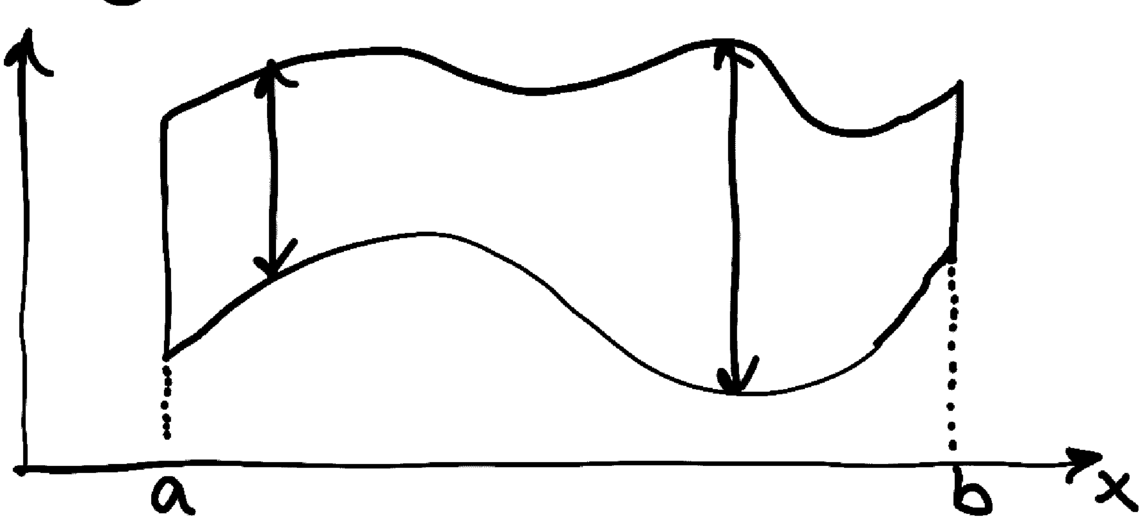


For every $y \in [c,d]$, the x integration bounds are $a \le x \le b$.

Similarly: for every $x \in [a,b]$, the y integration bounds are $c \le y \le d$.

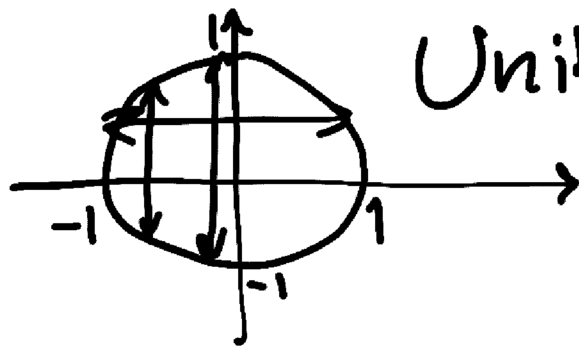
More general regions:

Ex:



For each $x \in [a,b]$ the y integration bounds <u>depend on</u> x.

<u>Ex</u>:



Unit circle

For every $y \in [-1,1]$, the x integration bounds depend on x.

We focus on two kinds:

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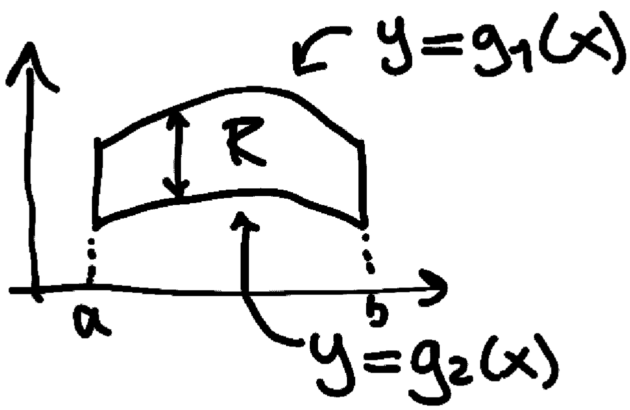
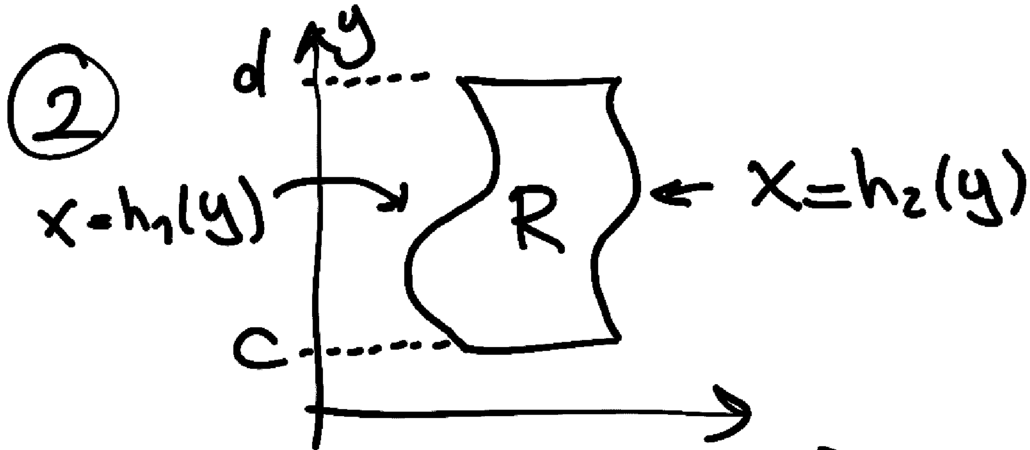


$$R = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

Then:

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

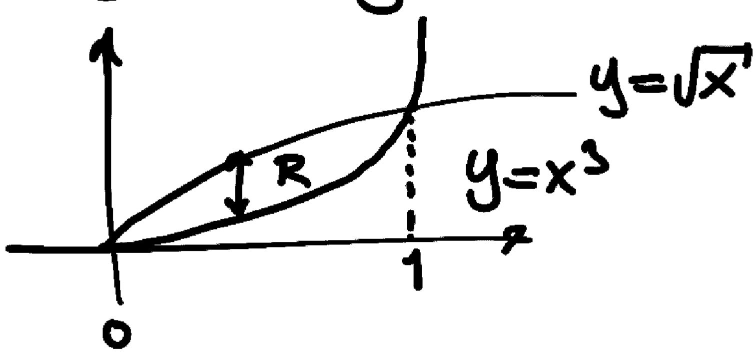
⚠ You can <u>not</u> change integration order.

② 

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

⚠ You can not change integration order.

Ex: Compute $\iint_R 2xy\,dx\,dy$, where R is the region in the plane bounded by the curves $y=\sqrt{x}$ and $y=x^3$.

Sol: 

$g_1(x)=\sqrt{x}$, $g_2(x)=x^3$.

$R=\{(x,y) \mid 0\le x\le 1,\ x^3\le y\le \sqrt{x}\}$.

So

$$\iint_R xy\,dx\,dy = \int_0^1 \left(\int_{x^3}^{\sqrt{x}} 2xy\,dy \right) dx$$

$$= \int_0^1 [xy^2]_{x^3}^{\sqrt{x}}\,dx = \int_0^1 (x \cdot |x| - x^7)\,dx$$

$$= \int_0^1 x^2 - x^7\,dx = \left[\frac{x^3}{3} - \frac{x^8}{8}\right] = \frac{1}{3} - \frac{1}{8}$$

$$= \frac{8-3}{24} = \frac{5}{24}.$$

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§5.3 Polar coords:

Recall that we described the area element dA in terms of dx and dy: $dA = dx\,dy$.

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy.$$

Now let's change to polar coords:

$$\begin{cases} x = x(r,\theta) = r\cos\theta \\ y = y(r,\theta) = r\sin\theta \end{cases}$$

$$\begin{cases} dx = \cos\theta\,dr - r\sin\theta\,d\theta \\ dy = \sin\theta\,dr + r\cos\theta\,d\theta \end{cases}$$

$$dA = dx\,dy = (\cos\theta\,dr - r\sin\theta\,d\theta)(\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$= \sin\theta\cos\theta\,(dr)^2 + r\cos^2\theta\,dr\,d\theta$$

$$- r\sin^2\theta\,d\theta\,dr - r^2\sin\theta\cos\theta\,(d\theta)^2 \quad (*)$$

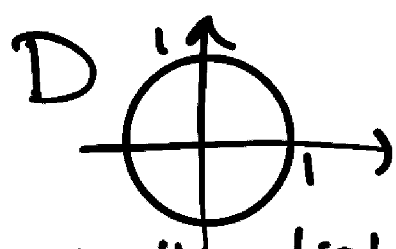
Two properties for "infinitesimal differentials": $\begin{cases} (dx)^2 = 0 \\ dx\,dy = -dy\,dx \end{cases}$

$$(*) = r(\cos^2\theta + \sin^2\theta)\,dr\,d\theta = r\,dr\,d\theta$$

So $\displaystyle\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

Ex: Compute $\displaystyle\iint_D 4(x^2+y^2)\,dx\,dy$ where

$D = \{(x,y) \mid x^2+y^2 \le 1\}$



unit disk

This region is described as

$\{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ in polar coords

$$\iint_R 4(x^2+y^2)\,dx\,dy = \iint_R 4(r^2)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 4r^3\,dr\,d\theta = \int_0^{2\pi} \left[r^4\right]_0^1 d\theta$$

$$= \int_0^{2\pi} d\theta = 2\pi.$$

Ex: Let's compute $\iint_R e^{-x^2-y^2}\,dx\,dy$

where $R = \mathbb{R}^2$ = entire xy-plane

$= (-\infty,\infty)\times(-\infty,\infty).$

On the one hand

$$\iint_R e^{-x^2-y^2}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-y^2}\,dx\,dy$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 \quad (*)$$

But this integral is hard to compute

There is no easy primitive function to $f(x) = e^{-x^2}$.

Let's compute the original integral using polar coords instead.

R in polar coords:

$$R = \{(r,\theta) \mid 0 \le r < \infty,\ 0 \le \theta \le 2\pi\}$$

$$\iint_R e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} r e^{-r^2}\,dr = \left[\begin{array}{l} u = r^2 \\ du = 2r\,dr \\ r=0 \Rightarrow u=0 \\ r\to\infty \Rightarrow u\to\infty \end{array}\right]$$

$$= \pi \int_0^{\infty} e^{-u}\,du = \pi\left[-e^{-u}\right]_0^{\infty}$$

$$= -\pi\left(\lim_{u\to\infty} \frac{1}{e^u} - 1\right) = \pi.$$

Combine this with $(*)$ to get $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi \Leftrightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$