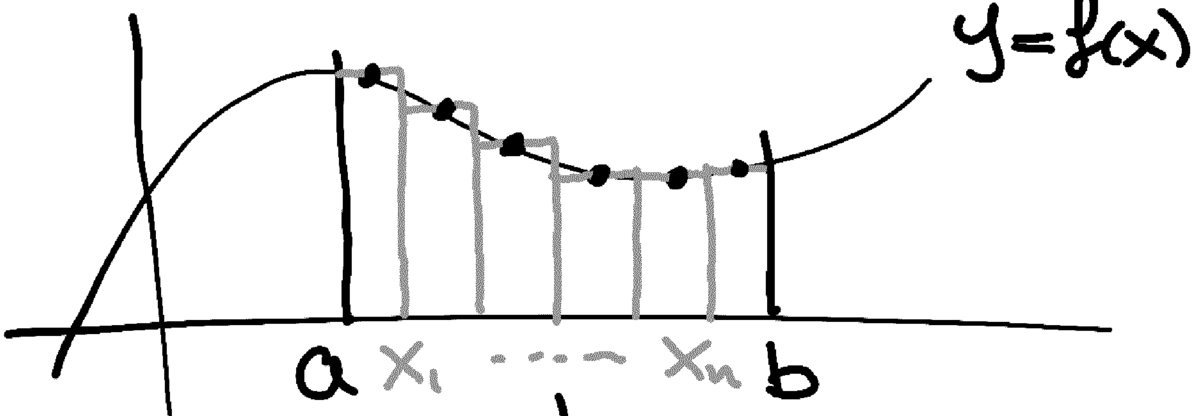
MAT 203 L 16 Mon Mar 30

## §5.1 Double integrals

### Riemann sums:




To define $\int_a^b f(x)dx$, we divide $[a,b]$ into small intervals:

$$a = x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} = b$$

Pick a point $x_i^* \in [x_i, x_{i+1}]$ for $0 \leq i \leq n$

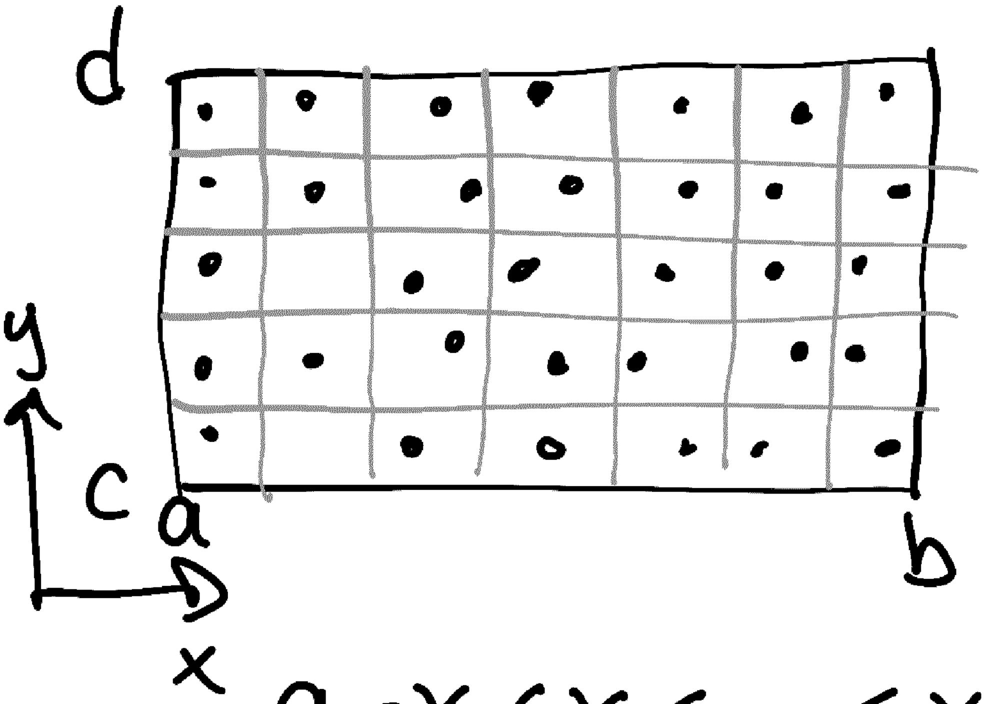
Let $\Delta x_i = x_{i+1} - x_i$

Area of each rectangle is $f(x_i^*)\Delta x_i$. Now sum up all contributions, and take the limit as the subdivision of the interval [a,b] becomes finer & finer:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x_i .$$

Now, we consider 2 variables:

$$[a,b]\times[c,d] = \{(x,y) \mid a\le x\le b,\ c\le x\le d\}$$




$$a = x_0 \le x_1 \le \cdots \le x_n \le x_{n+1} = b$$

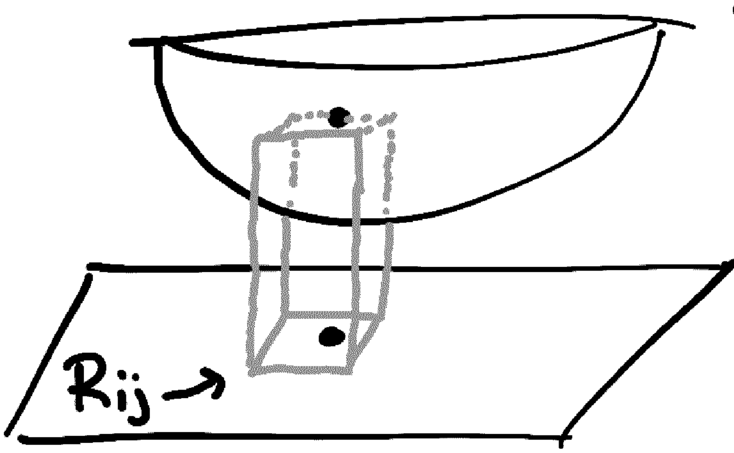
$$c = y_0 \le y_1 \le \cdots \le y_m \le y_{m+1} = d$$

$$\Delta x_i = x_{i+1} - x_i,\quad \Delta y_i = y_{i+1} - y_i$$

Let $R_{ij}$ be the rectangle

$$\{x_i \leq x \leq x_{i+1},\ y_j \leq y \leq y_{j+1}\}$$

Let $(x_{ij}^*, y_{ij}^*)$ be a point in $R_{ij}$.




$$\Delta A_{ij} = \Delta x_i \Delta y_j$$

Volume: $f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$.

$R = [a,b] \times [c,d]$ integration region

$$\boxed{\iint_R f(x,y)\,dA = \lim_{n,m \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}}$$

Sometimes we also write

$$\iint_R f(x,y)\,dx\,dy.$$

We always assume $f(x,y)$ is continuous when working with integrals.

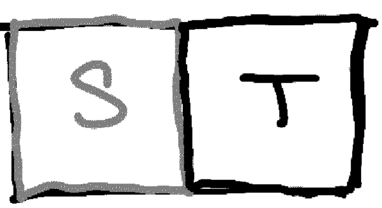
Properties:

① $\iint_R f(x,y)+g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

② If $C$ is a constant

$\iint_R C\,f(x,y)\,dA = C\iint_R f(x,y)\,dA$

③ Let $R = S\cup T$, and $S\cap T = \emptyset$

R 


$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA$

④ If $f(x,y)\geq g(x,y)$, $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

⑤ If $m\leq f(x,y)\leq M$ then

$m\cdot \text{Area}(R) \leq \iint_R f(x,y)\,dA \leq M\cdot \text{Area}(R)$

⑥ In case $f(x,y)=g(x)h(y)$, and $R=[a,b]\times[c,d]$ then

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

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How to compute?

If $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

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<u>Ex</u>: $f(x,y)=xy-3xy^2$

$R=[0,2]\times[1,2]$, compute $\iint_R f(x,y)\,dA$.

$$\iint_R f(x,y)\,dA = \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy-3xy^2\,dx\,dy$$

$$= \int_{y=1}^{y=2} \left( \left[ \frac{x^2 y}{2} - \frac{3x^2y^2}{2} \right]_{x=0}^{x=2} \right) dy$$

$$= \int_1^2 2y - 6y^2 \, dy = \left[ \frac{2y^2}{2} - \frac{6y^3}{3} \right]_1^2$$

$$= (4-16) - (1-2) = -11$$

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<u>Ex</u>: Let $R = [0, \frac{\pi}{2}] \times [0,1]$ & lets compute $\iint_R e^y \cos x \, dA$.

Since the integrand is $e^y \cos(x)$ $= h(y) g(x)$,

$$\iint_R e^y \cos x \, dA = \left( \int_0^1 e^y dy \right) \left( \int_0^{\pi/2} \cos x \, dx \right)$$

$$= \left[ e^y \right]_0^1 \left[ \sin x \right]_0^{\pi/2} = (e-1) \underbrace{\left( \sin \frac{\pi}{2} \right)}_{=1}$$

$$= e - 1.$$

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$\underline{Ex}$: $\displaystyle\int_{-1}^{0}\int_{0}^{1} 3x^2+2y \, dx\, dy$.

(1) $$\int_{y=-1}^{y=0}\left(\int_{x=0}^{x=1} 3x^2+2y \, dx\right)dy$$

$$= \int_{y=-1}^{y=0} \left[x^3+2xy\right]_{x=0}^{x=1} dy = \int_{-1}^{0} 1+2y \, dy$$

$$= \left[y+y^2\right]_{-1}^{0} = -\left(-1+(-1)^2\right) = 0$$

Let's compute it in the other order!

(2) $$\int_{x=0}^{x=1}\left(\int_{y=-1}^{y=0} 3x^2+2y \, dy\right)dx$$

$$= \int_{x=0}^{x=1} \left[3x^2y+y^2\right]_{y=-1}^{y=0} dx = \int_{0}^{1} -(-3x^2+1)\, dx$$

$$= \left[x^3-x\right]_{0}^{1} = 0$$

The order does not matter in general, as we mentioned above.

Ex: $R=[0,\pi]\times[1,2]$. Compute

$$\iint_R x\sin(xy)\,dx\,dy.$$

① $$\int_{y=1}^{y=2}\left(\int_{x=0}^{x=\pi} x\sin(xy)\,dx\right)dy.$$

Let's evaluate the inner integral first by integration by parts:

$$\int_{x=0}^{x=\pi} x\sin(xy)dx=\left[-\frac{x\cos(xy)}{y}\right]_{x=0}^{x=\pi}+\frac{1}{y}\int_0^{\pi}\cos(xy)dx$$

$$=-\frac{\pi\cos(\pi y)}{y}+\frac{1}{y}\underbrace{\left[\frac{\sin(xy)}{y}\right]_0^{\pi}}_{=0}$$

Now $\int_1^2 -\frac{\pi\cos(\pi y)}{y}dy$ is very hard to compute....

If we restart our computation & instead compute the integral with respect to $y$ first, it's easier.

② $$\int_{x=0}^{x=\pi}\left(\int_{y=1}^{y=2} x\sin(xy)\,dy\right)dx$$

$$= \int_{x=0}^{x=\pi}\left[\frac{-x\cos(xy)}{x}\right]_{y=1}^{y=2} dx$$

$$= \int_{0}^{\pi} -\cos(2x)+\cos(x)\,dx$$

$$= \left[\frac{-\sin(2x)}{2}+\sin(x)\right]_0^{\pi}$$

$$= \left(-\frac{\sin(2\pi)}{2}+\sin(\pi)\right)-\left(-\frac{\sin(0)}{2}+\sin 0\right)$$

$$= 0$$

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$\text{Area}(R) = \iint_R dA$ (integrand is $f(x,y)=1$)

The average value of $f(x,y)$ in the region $R$ is

$$f_{ave} = \frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dA = \frac{\iint_R f(x,y)\,dA}{\iint_R dA}$$

$$\underline{Ex}: \int_{x=1}^{x=e}\int_{y=1}^{y=2} \frac{\ln(x)}{x} \, dydx$$

$$= \left(\int_1^e \frac{\ln(x)}{x} \, dx\right)\underbrace{\left(\int_1^2 dy\right)}_{=1}$$

$$= \int_1^e \frac{\ln(x)}{x} dx = \begin{bmatrix} u = \ln(x) \\ du = \frac{1}{x}dx \\ x=1 \Rightarrow u=0 \\ x=e \Rightarrow u=1 \end{bmatrix} = \int_0^1 u \, du$$

$$= \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

$R = [1,e] \times [1,2]$

Average value of $f(x,y) = \frac{\ln(x)}{x}$ in $R$ is

$$f_{ave} = \frac{1}{\text{Area}(R)} \overbrace{\iint_R \frac{\ln(x)}{x} dx}^{=\frac{1}{2}}$$

$$= \frac{1}{(e-1)(2-1)} = \frac{1}{2(e-1)}.$$