MAT 203 L13 Wed Mar 25

Recall: • (x_0,y_0) critical point of f if $\nabla f(x_0,y_0)=0$.

• "Second derivative test":

$$D(x,y)=f_{xx}(x,y)f_{yy}(x,y)-(f_{xy}(x,y))^2$$

discriminant

Let (x_0,y_0) be a critical point.

• $D(x_0,y_0)>0$, $f_{xx}(x_0,y_0)>0$
$\Rightarrow$ (x_0,y_0) is a local min

• $D(x_0,y_0)>0$, $f_{xx}(x_0,y_0)<0$
$\Rightarrow$ (x_0,y_0) is a local max

• $D(x_0,y_0)<0$
$\Rightarrow$ (x_0,y_0) Saddle

• $D(x_0,y_0)=0$
$\Rightarrow$ Inconclusive

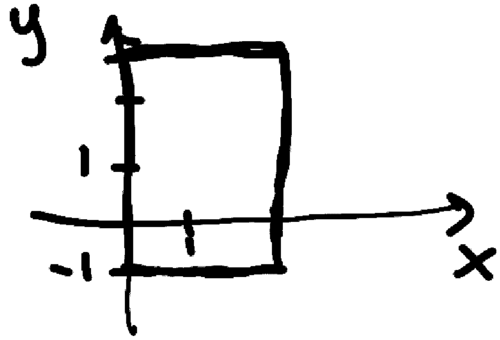
• If $f=f(x,y)$ is defined on the domain $D\subset\mathbb{R}^2$, then we find global max/min by:

1. Find critical pts (x_0,y_0)
2. Compute values $f(x_0,y_0)$
3. Find min/max along the boundary $\partial D \subset \mathbb{R}^2$
4. Compare all values.

<u>Ex</u>: Find global max/min at

$f(x,y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$

in the domain $\{(x,y) \mid 0 \le x \le 2,\ -1 \le y \le 3\}$



1. Find critical points

$\nabla f(x,y) = \langle 8x - 2y - 8, -2x + 12y + 2 \rangle$

$= \langle 0,0 \rangle$

$$\begin{cases} 8x - 2y - 8 = 0 & (1) \\ -2x + 12y + 2 = 0 & (2) \end{cases}$$

Eq ① $\Leftrightarrow 2y = 8x - 8 \Leftrightarrow y = 4x - 4$

Substitute into eq ②:

$$-2x + 12(4x-4) + 2 = 0$$

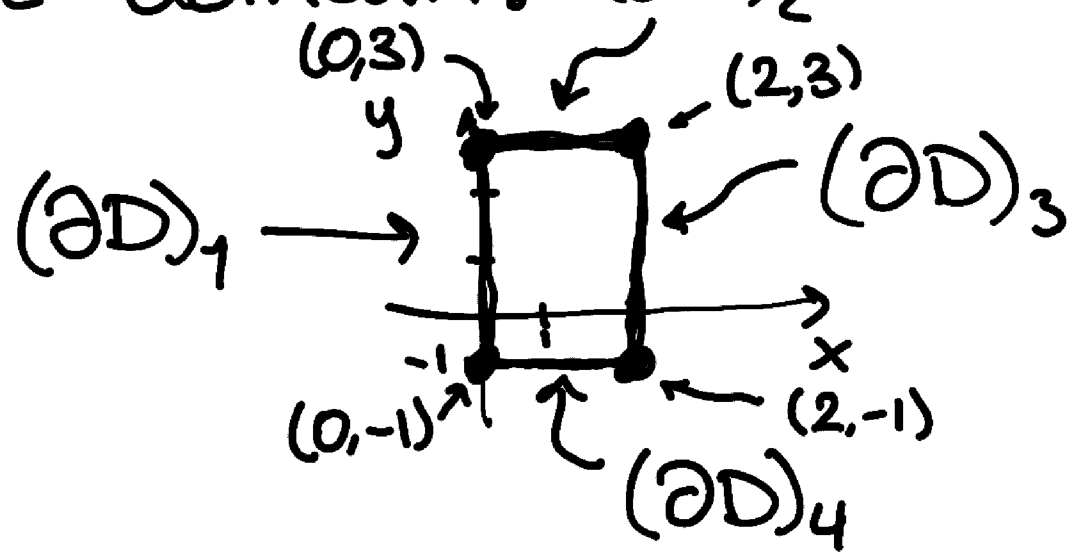
$$\Leftrightarrow -2x + 48x - 48 + 2 = 0$$

$$\Leftrightarrow 46x - 46 = 0 \Leftrightarrow \boxed{x = 1}$$

$$\Rightarrow \boxed{y = 0}$$

$$f(1,0) = 4 - 8 + 3 = \boxed{-1}$$

Now consider the boundary of the domain:



$$(\partial D)_1 = \{x = 0,\ -1 \le y \le 3\}$$

$$(\partial D)_2 = \{0 \le x \le 2,\ y = 3\}$$

$$(\partial D)_3 = \{x = 2,\ -1 \le y \le 3\}$$

$(\partial D)_4 = \{0 \le x \le 2,\ y = -1\}$

We check each boundary separately, and find min/max of the function along each one.

$(\partial D)_1$: $f(y) = 6y^2 + 2y + 3$, $-1 \le y \le 3$

$$f'(y) = 12y + 2 = 0 \Leftrightarrow y = -\frac{1}{6}$$

$$f\left(-\frac{1}{6}\right) = 6\left(-\frac{1}{6}\right)^2 + 2\left(-\frac{1}{6}\right) + 3$$

$$= \frac{1}{6} - \frac{2}{6} + 3 = 3 - \frac{1}{6} = \boxed{\frac{17}{6}}$$

$(\partial D)_2$: $f(x) = 4x^2 - 6x + 54 - 8x + 6 + 3$

$$= 4x^2 - 14x + 63 = 0$$

$$0 \le x \le 2$$

$$f'(x) = 8x - 14 = 0 \Leftrightarrow x = \frac{14}{8} = \frac{7}{4}$$

$$f\left(\frac{7}{4}\right) = 4 \cdot \frac{7^2}{4^2} - 14 \cdot \frac{7}{4} + 63$$

$$= \frac{49}{4} - \frac{49 \cdot 2}{4} + 63 = \boxed{-\frac{49}{4} + 63}$$

$(\partial D)_3$: $f(y) = 16 - 4y + 6y^3 - 16 + 2y + 3$

$$= 6y^2 - 2y + 3 \quad -1 \le y \le 3$$

$$f'(y) = 12y - 2 = 0 \Leftrightarrow y = \frac{1}{6}$$

$$f(\tfrac{1}{6}) = 6 \cdot \frac{1}{6^2} - \frac{2}{6} + 3 = \frac{1}{6} - \frac{2}{6} + 3$$

$$= 3 - \frac{1}{6} = \boxed{\frac{17}{6}}$$

$(\partial D)_4$: $f(x) = 4x^2 + 2x + 6 - 8x - 2 + 3$

$$= 4x^2 - 6x + 7 \qquad 0 \leq x \leq 2$$

$$f'(x) = 8x - 6 = 0 \Leftrightarrow x = \frac{6}{8} = \frac{3}{4}$$

$$f(\tfrac{3}{4}) = 4 \cdot \frac{9}{4^2} - \frac{18}{4} + 7 = \frac{36}{4} - \frac{18}{4} + 7$$

$$= \frac{18}{4} + 7 = \boxed{\frac{46}{4}}$$

<u>Corner points:</u>

$$f(0,-1) = 6 - 2 + 3 = \boxed{7}$$

$$f(2,-1) = 16 + 4 + 6 - 16 - 2 + 3 = \boxed{11}$$

$$f(0,3) = 54 + 6 + 3 = \boxed{63}$$

$$f(2,3) = 16 - 18 + 54 - 16 + 6 + 3 = \boxed{45}$$

Comparing all values we find global min at $(x,y) = (1,0)$ with value -1

global max at $(x,y)=(0,3)$
with value 63

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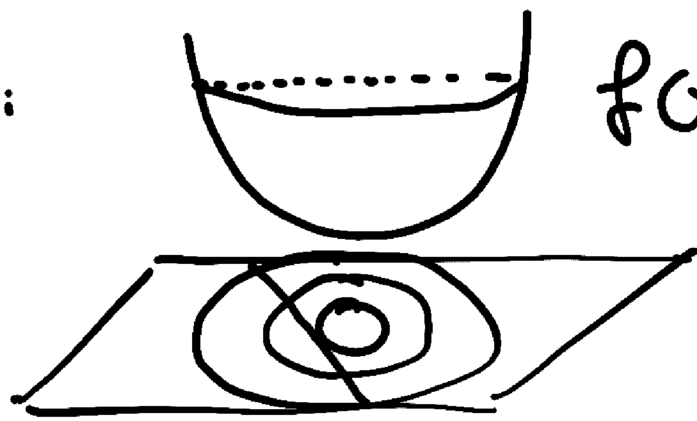
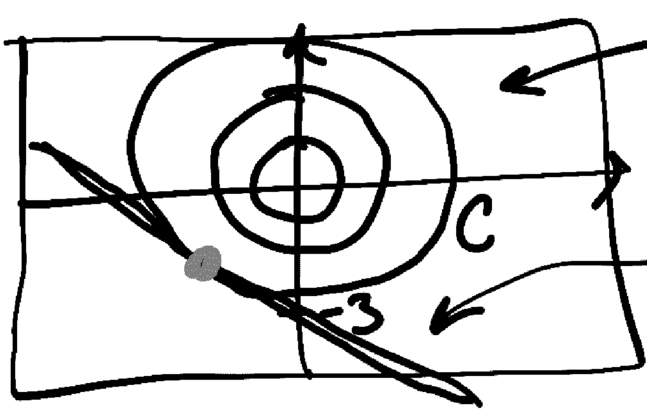
§4.8 Lagrange multipliers

The method of Lagrange multipliers is used to maximize/minimize a function with respect to one or more constraints.

Ex: $f(x,y) = x^2+y^2$



Suppose we want to minimize $f(x,y)$ with respect to the constraint $x+y+3=0$.



level curves

Constraint

The minimum is the value of the level curve that is tangent to the constraint.

$f(x,y) = x^2+y^2$, $\nabla f = \langle 2x, 2y \rangle$

$g(x,y) = x+y+3$, $\nabla g = \langle 1, 1 \rangle$

∇f normal to level curves

∇g normal to the constraint.

Constraint is tangent to level curves if $\nabla f = \lambda \nabla g$ for some λ.

Theorem: Let $f = f(x,y)$, $g = g(x,y)$. Suppose (x_0, y_0) is a local max/min of $f(x,y)$ when restricted to the curve $g(x,y) = 0$, and $\nabla g(x_0, y_0) \neq 0$. Then there is some scalar λ (="Lagrange multiplier") such that

$$\boxed{\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)}$$

How to use it:

If we want to optimize $f(x,y)$ with respect to the constraint $g(x,y)=0$ we solve

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$$

All points (x,y) that solve this are candidates for min/max.

Ex. $f(x,y) = x^2+4y^2-2x+8y$

$g(x,y) = x+2y-7$.

$\nabla f(x,y) = \langle 2x-2, 8y+8 \rangle$

$\nabla g(x,y) = \langle 1, 2 \rangle$

$$\begin{cases} \langle 2x-2, 8y+8 \rangle = \lambda \langle 1,2 \rangle \\ x+2y-7 = 0 \end{cases}$$

$$\begin{cases} 2x-2=\lambda & (1) \\ 8y+8=2\lambda & (2) \\ x+2y-7=0 & (3) \end{cases}$$

Insert (1) in (2): $\begin{cases} 8y+8=2(2x-2) & (1') \\ x=7-2y & (2') \end{cases}$

Insert (2') in (1')

$$8y+8=2(2(7-2y)-2)$$

$$\Leftrightarrow 8y+8=2(14-4y-2)=28-8y-4$$

$$\Leftrightarrow 16y=16 \Leftrightarrow \boxed{y=1}$$

$$\Rightarrow x=7-2\cdot 1=5$$

Only solution is $(x,y)=(5,1)$.

$$f(5,1)=25+4-10+8=27$$

To figure out if it's a max or min, find some other point on the line $g(x,y)=0 \Leftrightarrow x+2y=7$ & compare the values of f. For example

$x=6$ gives $y=\frac{7-6}{2}=\frac{1}{2}$

$$f(6,\tfrac{1}{2}) = 36 + 4\cdot\tfrac{1}{4} - 12 + \tfrac{8}{2} = 24 + 1 + 4 = 29 > 27$$

So $(x,y) = (5,1)$ with $f(5,1) = 27$ appears to be a <u>minimum</u>.