MAT 203 L 12 Mon Mar 23

Recall: $f = f(x,y)$

- $\vec{v} = \langle v_x, v_y \rangle$. Directional derivative of f in the direction of $\vec{v}$ is

$$D_{\vec{v}} f = \lim_{h \to 0} \frac{f(x+hv_x, y+hv_y) - f(x,y)}{h \|\vec{v}\|}$$

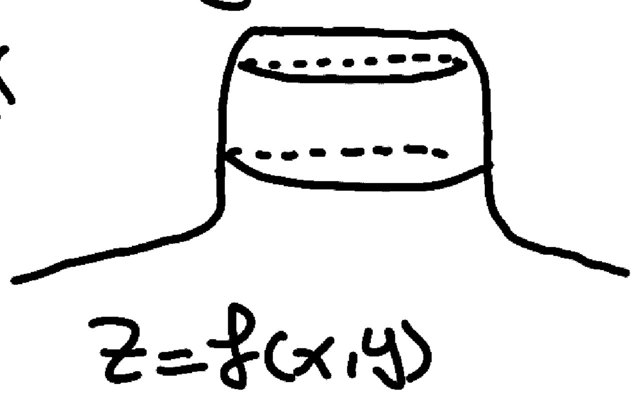
- The gradient of f is

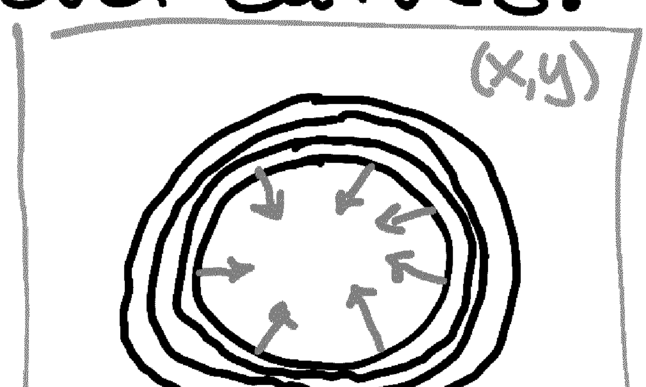
$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

- $D_{\vec{v}} f = \nabla f(x,y) \cdot \dfrac{\vec{v}}{\|\vec{v}\|}$

The gradient $\nabla f(x,y)$ is orthogonal to level curves.

Ex





§4.7 Optimization problems

Def: A point (x_0, y_0) is called a critical point for $f = f(x,y)$ if
(1) $\nabla f(x_0, y_0) = \vec{0}$, or
(2) $\nabla f(x_0, y_0)$ doesn't exist.

Ex: $f(x,y) = x^3 + 2xy - 2x - 4y$

Let's find all critical points.

$$\nabla f(x,y) = \langle 3x^2 + 2y - 2, 2x - 4 \rangle = \langle 0, 0 \rangle$$

$$\begin{cases} 3x^2 + 2y - 2 = 0 \\ 2x - 4 = 0 \Rightarrow \boxed{x = 2} \end{cases}$$

Substitute into first eq.:

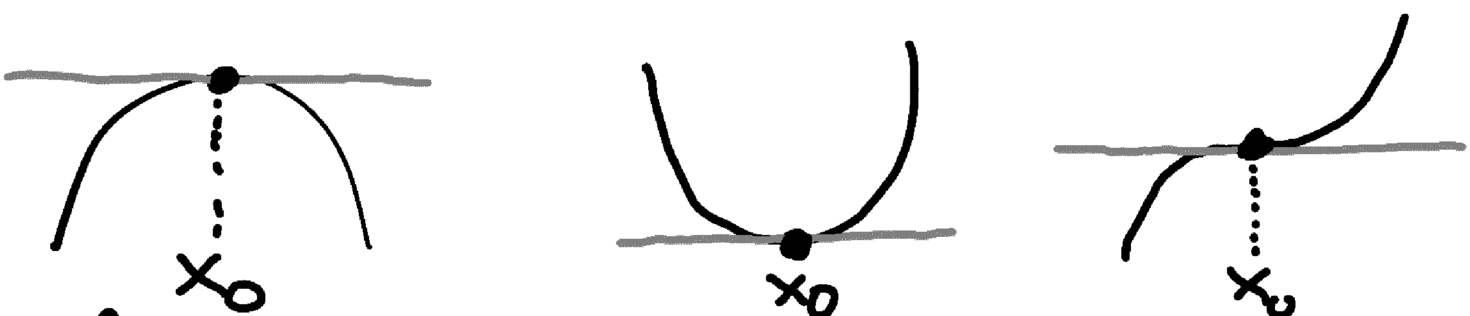
$$3 \cdot 2^2 + 2y - 2 = 0 \Rightarrow \boxed{y = -5}$$

The only critical point is $(x,y) = (2, -5)$.

Intuitively, what is a critical point?

In single variable calculus:

If $f'(x_0) = 0$ then x_0 is a critical point. This means that the slope $= 0$.



local max   local min   saddle

In our case: $\nabla f(x_0, y_0) = \vec{0}$

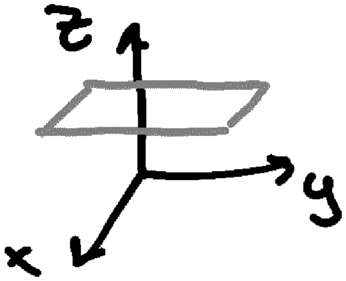
means $\begin{cases} f_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) = 0 \end{cases}$

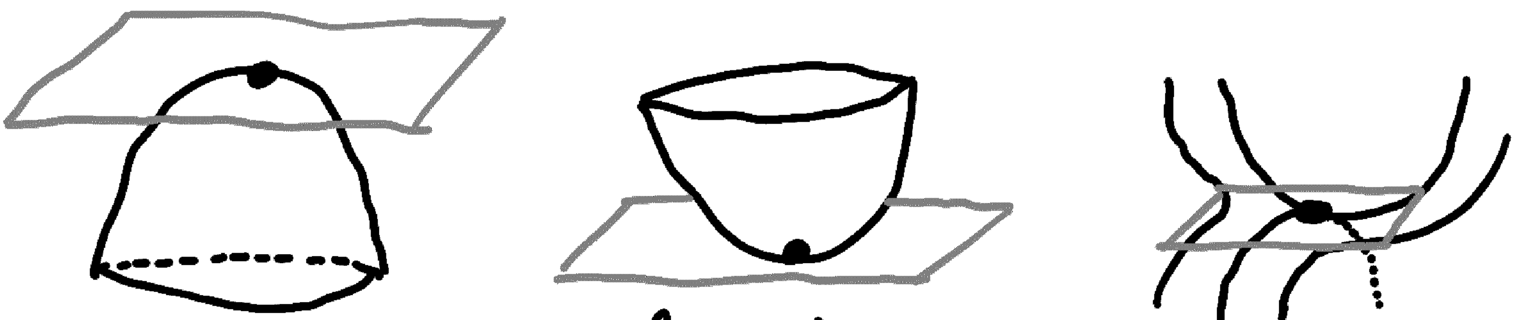
So if f has a tangent plane at $(x, y) = (x_0, y_0)$, its given by

$$\overbrace{f_x(x_0,y_0)}^{=0}(x-x_0)+\overbrace{f_y(x_0,y_0)}^{=0}(y-y_0)$$
$$+(z-f(x_0,y_0))=0$$

$$\Longleftrightarrow \quad z=f(x_0,y_0).$$

This is a horizontal plane





<u>Def</u>: Let $f(x,y)$ be continuous.

(1)
We say (x_0,y_0) is a <u>local maximum</u>

if $f(x_0,y_0) \geq f(x,y)$ for all (x,y) close to (x_0,y_0)

(2) We say (x_0,y_0) is a <u>local minimum</u> if $f(x_0,y_0) \leq f(x,y)$ for all (x,y) near (x_0,y_0).

(3) We say (x_0,y_0) is a <u>saddle point</u> if $\nabla f(x_0,y_0) = \vec{0}$ and (x_0,y_0) is neither a local max, nor a local min.

For functions of one variable, we can conveniently find min/max/saddles using $f''(x)$.

$f''(x_0)<0$ $\quad$ $f''(x_0)>0$

Theorem (Second derivative test)
Let f be differentiable. Suppose $\nabla f(x_0,y_0)=\vec{0}$. Define

$$D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-(f_{xy}(x_0,y_0))^2$$

"discriminant"

(1) If $D>0$ and $f_{xx}(x_0,y_0)>0$ then (x_0,y_0) is a local min

(2) If $D>0$ and $f_{xx}(x_0,y_0)<0$ then (x_0,y_0) is a local max

(3) If $D<0$, (x_0,y_0) is a saddle

(4) If $D=0$ the test is inconclusive.

Ex: $f(x,y)=4x^2+9y^2+8x-36y+24$

Let's find and classify all

Critical points.

$\nabla f(x,y) = \langle 8x+8, 18y-36 \rangle = \vec{0}$

$\Rightarrow \quad (x,y) = (-1,2)$

$f_{xx}(x,y) = 8, \; f_{yy}(x,y) = 18$

$f_{xy}(x,y) = 0.$

$D = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$

$\quad = 8 \cdot 18 = 144$

so $D > 0$ and $f_{xx}(-1,2) > 0$

so $(x,y) = (-1,2)$ is a local min

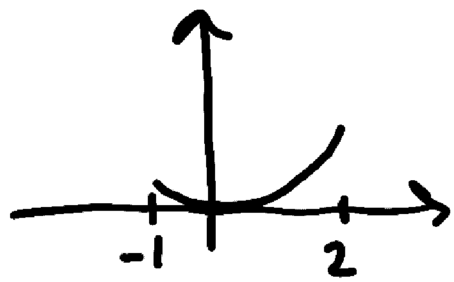
If (x_0,y_0) is a critical point, $f(x_0,y_0)$ is the critical value.

The global min/max (if it exists) is the min/max value of $f(x,y)$.

Global min/max can occur at local min/max, or at the boundary of the domain of definition.

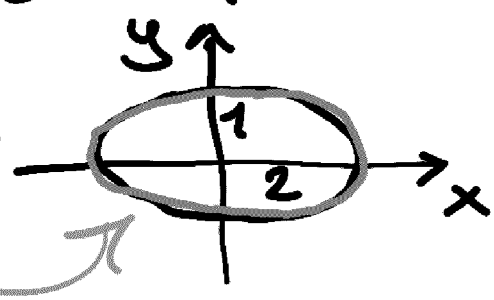
<u>Ex</u>: One variable:

$f(x) = x^2, \quad x \in [-1, 2]$.

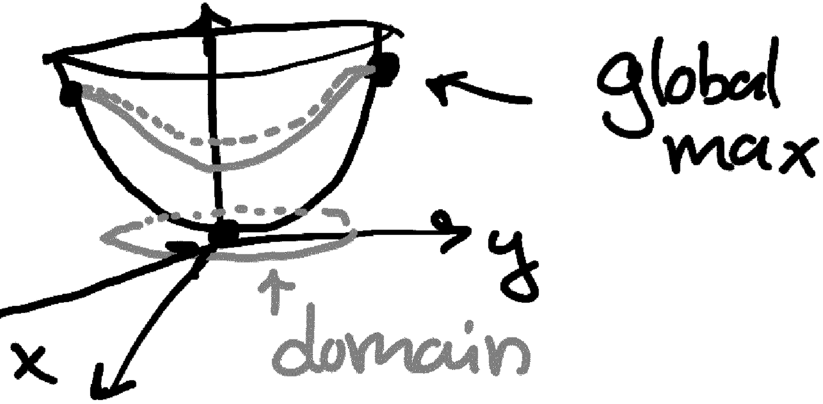


- global min is $f(0) = 0$ occurring at $x=0$ (local min)
- global max on $-1 \le x \le 2$ is $f(2) = 4$, at the boundary point $\underline{x=2}$.

<u>Ex</u>: $f(x,y) = x^2 + y^2$, on the domain $\left\{ \frac{x^2}{4} + y^2 \le 1 \right\}$



The graph:



Intuitively, the global min is $f(0,0)=0$, at the local min.

The global max can be "seen" from the graph to lie along the boundary, at $(x,y)=(\pm 2,0)$
$f(\pm 2,0)=4$.

Let's verify all of this via computation.

$$\nabla f(x,y)=\langle 2x,2y\rangle=\langle 0,0\rangle$$

$$(x,y)=(0,0)$$

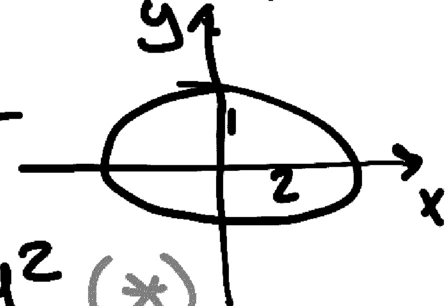
$$D=f_{xx}(x,y)f_{yy}(x,y)-(f_{xy}(x,y))^2$$

$$=2\cdot 2-0=4>0$$

$f_{xx}(0,0) = 2 > 0$

$\Rightarrow (x,y) = (0,0)$ <u>local min</u>
by the second derivative test.

Now, lets consider the boundary of the domain, which is $\frac{x^2}{4} + y^2 = 1$



$x^2 + 4y^2 = 4 \Leftrightarrow x^2 = 4 - 4y^2 \quad (*)$

The function along this boundary is therefore given by: $f(x,y) = x^2 + y^2 = 4 - 4y^2 + y^2$
$= 4 - 3y^2.$

We now need to maximize this function of <u>one</u> variable!

$g(y) = 4 - 3y^2. \quad -1 \leq y \leq 1$

To do this, we can use our Calc I knowledge.

$$g'(y) = -6y = 0$$

$$y=0, \quad g''(y) = -6 < 0$$

so $y=0$ is a local max with value $g(0)=4$.

Boundary points: $g(-1)=g(1)=1$

$$y=0 \overset{(*)}{\Longrightarrow} x^2=4 \Rightarrow \boxed{x=\pm 2}$$

So $(\pm 2,0)$ are points realizing the global max of f, at $\left\{\frac{x^2}{4}+y^2\leq 1\right\}$. $f(\pm 2,0)=4$ max

$f(0,0)=0$ min
