

MAT 203 MIDTERM II

PRACTICE PROBLEMS

1. (a) (5 pts) Compute the limit $\lim_{(x,y) \rightarrow (0,1)} \frac{y^2 \sin x}{x}$

Solution. We have $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\lim_{y \rightarrow 1} y^2 = 1$, therefore

$$\lim_{(x,y) \rightarrow (0,1)} \frac{y^2 \sin x}{x} = 1.$$

□

- (b) (5 pts) Compute the directional derivative of $f(x, y) = xy^2 - yx^2$ in the direction $\vec{v} = \langle -1, 1 \rangle$.

Solution. The directional derivative may be computed as $\nabla f(x, y) \cdot \frac{\vec{v}}{\|\vec{v}\|}$. We have $\nabla f(x, y) = \langle y^2 - 2xy, 2xy - x^2 \rangle$, and therefore

$$\begin{aligned} D_{\langle -1, 1 \rangle} f(x, y) &= \langle y^2 - 2xy, 2xy - x^2 \rangle \cdot \frac{\langle -1, 1 \rangle}{\sqrt{(-1)^2 + 1^2}} \\ &= \frac{1}{\sqrt{2}} ((2xy - y^2) + (2xy - x^2)) = \frac{4xy - x^2 - y^2}{\sqrt{2}}. \end{aligned}$$

□

2. (10 pts) Find and classify all critical points of the function $f(x, y) = x^3 - 27x + 4y^2$.

Solution. We first find the critical points:

$$\nabla f(x, y) = \langle 3x^2 - 27, 8y \rangle = \langle 0, 0 \rangle \iff \begin{cases} 3x^2 - 27 = 0 \\ 8y = 0 \end{cases}$$

which leaves the two critical points $(\pm 3, 0)$. To classify them, we find the discriminant

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2,$$

and $f_{xx}(x, y)$ at these two points. We have

$$f_{xx}(x, y) = 6x, \quad f_{xy} = 0, \quad f_{yy} = 8.$$

Therefore, for the critical point $(x, y) = (3, 0)$ we have $f_{xx}(3, 0) = 18 > 0$, and the discriminant is $D(3, 0) = f_{xx}(3, 0)f_{yy}(3, 0) = 18 \cdot 8 > 0$. Therefore $(3, 0)$ is a local minimum.

For the critical point $(x, y) = (-3, 0)$, we have $f_{xx}(-3, 0) = -18 < 0$, and the discriminant is $D(-3, 0) = f_{xx}(-3, 0)f_{yy}(-3, 0) = -18 \cdot 8 < 0$. Therefore $(-3, 0)$ is a saddle. \square

3. (10 pts) Compute the double integral $\iint_D x^2 dx dy$ where D is the region in the xy -plane of all (x, y) such that $x^2 \leq y \leq 2 - x^2$.

Solution. The x -values of the intersection of the two graphs $y = x^2$ and $y = 2 - x^2$ are $x = \pm 1$, which gives

$$\begin{aligned}\iint_D x^2 dx dy &= \int_{-1}^1 \int_{x^2}^{1-x^2} x^2 dy dx \\ &= \int_{-1}^1 x^2 [y]_{x^2}^{2-x^2} dx = \int_{-1}^1 x^2 (2 - x^2 - x^2) dx \\ &= 2 \int_0^1 2x^2 - 2x^4 dx = 4 \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = 4 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{8}{15}.\end{aligned}$$

□

4. (10 pts) Compute the triple integral $\iiint_C ze^{x^2+y^2} dx dy dz$, where C is the region described by the inequalities $x^2 + y^2 \leq 4$ and $1 \leq z \leq e$.

Solution. We change to cylindrical coordinates and use $dx dy dz = r dr d\theta dz$. The region C in cylindrical coordinates is described by the inequalities $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$ and $1 \leq z \leq e$. Therefore

$$\begin{aligned}\iiint_C ze^{x^2+y^2} dx dy dz &= \int_1^e \int_0^2 \int_0^{2\pi} ze^{r^2} r d\theta dr dz \\ &= 2\pi \left(\int_0^2 r e^{r^2} dr \right) \left(\int_1^e z dz \right) \\ &= 2\pi \left[\frac{e^{r^2}}{2} \right]_{r=0}^2 \left[\frac{z^2}{2} \right]_{z=1}^e = \frac{\pi}{2} (e^4 - 1) (e^2 - 1).\end{aligned}$$

□