MAT 203 MIDTERM I

PRACTICE PROBLEMS

1. (a) (5 pts) Let $\vec{u} = \langle 1, 2, 3\rangle$ and $\vec{v} = \langle 4, 5, 6\rangle$. Compute $5\vec{u} - \vec{v}$.

Solution. We have

$5\vec{u}-\vec{v} = 5\langle 1, 2, 3\rangle - \langle 4, 5, 6\rangle = \langle 5, 10, 15\rangle - \langle 4, 5, 6\rangle = \langle 5-4, 10-5, 15-6\rangle = \langle 1, 5, 9\rangle$

□

(b) (5 pts) Let $\vec{u} = \langle -1, 2, 3\rangle$ and $\vec{v} = \langle 0, 2, 1\rangle$. Compute the vector $\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}\vec{v}$.

Solution. We have

$$\vec{u}\cdot\vec{v} = \langle -1, 2, 3\rangle \cdot \langle 0, 2, 1\rangle = 0 + 4 + 3 = 7$$

and

$$||\vec{v}|| = \sqrt{0^2 + 2^2 + 1^1} = \sqrt{5}.$$

Then

$$\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}\vec{v} = \frac{7}{\sqrt{5}}\langle 0, 2, 1\rangle .$$

□

2. (a) (5 pts) Find a parametrization of the line passing through the points $(0, 4)$ and $(-1, -2)$.

Solution. A direction vector is $\vec{v} = \langle 0 - (-1), 4 - (-2) \rangle = \langle 1, 6 \rangle$, and the line passes through the point $(0, 4)$. Therefore a parametrization is given by

$$\vec{\gamma}(t) = \langle 0, 4 \rangle + t \langle 1, 6 \rangle .$$

□

(b) (5 pts) Find a parametrization of the plane in space that passes through the point $(1, 0, -1)$ and has normal vector $\vec{n} = \langle 1, -1, -1 \rangle$.

Solution. To parametrize the plane, we need to find two (non-parallel) vectors that belongs to the plane. Any such vectors $\vec{u}$ and $\vec{v}$ are orthogonal to the normal vector $\vec{n}$, and therefore satisfies $\vec{u} \cdot \vec{n} = \vec{v} \cdot \vec{n} = 0$. If we write $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, we obtain the equations

$$\begin{cases} \langle u_1, u_2, u_3 \rangle \cdot \langle 1, -1, -1 \rangle = 0 \\ \langle v_1, v_2, v_3 \rangle \cdot \langle 1, -1, -1 \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} u_1 - u_2 - u_3 = 0 \\ v_1 - v_2 - v_3 = 0 \end{cases} .$$

For example, we may take $\vec{u} = \langle 2, 1, 1 \rangle$ and $\vec{v} = \langle 3, 2, 1 \rangle$. Then, a parametrization of the plane is given by

$$\vec{r}(s, t) = \langle 1, 0, -1 \rangle + s \langle 2, 1, 1 \rangle + t \langle 3, 2, 1 \rangle .$$

□

3. (10 pts) Rewrite the following hyperbolic paraboloid into standard form, and determine its center

$$3x^2 - 6x - 4y^2 - 8y - 1 - 12z + 24 = 0.$$

Solution. We complete the squares in the x and y variables.

$$\begin{aligned} 3x^2 - 6x - 4y^2 - 8y - 1 - 12z + 24 &= 3(x^2 - 2x) - 4(y^2 + 2y) - 1 - 12z + 24 \\ &= 3((x-1)^2 - 1) - 4((y+1)^2 - 1) - 1 - 12z + 24 \\ &= 3(x-1)^2 - 4(y+1)^2 - 12z + 24 = 0. \end{aligned}$$

Then we get

$$3(x-1)^2 - 4(y+1)^2 = 12z - 24 = 12(z-2) \iff \frac{(x-1)^2}{4} - \frac{(y+1)^2}{3} = z - 2.$$

The center is therefore $(1, -1, 2)$. □

4. Consider the following parametrization of a curve C in space:

$$\vec{r}(t) = \left\langle \sqrt{t}, 2t+5, t \right\rangle.$$

(a) (5 pts) Find the vector $\vec{r}''(t)$.

Solution. We have $\vec{r}'(t) = \left\langle \frac{1}{2\sqrt{t}}, 2, 1 \right\rangle$, and then

$$\vec{r}''(t) = \left\langle \frac{d}{dt}\left(\frac{1}{2}t^{-\frac{1}{2}}\right), 0, 0 \right\rangle = \left\langle \frac{1}{2}\left(-\frac{1}{2}t^{-\frac{3}{2}}\right), 0, 0 \right\rangle = \left\langle -\frac{1}{4t^{\frac{3}{2}}}, 0, 0 \right\rangle.$$

□

(b) (5 pts) Compute $\int \vec{r}(t) \cdot \vec{r}'(t)\, dt$.

Solution. From part (a) we have $\vec{r}'(t) = \left\langle \frac{1}{2\sqrt{t}}, 2, 1 \right\rangle$. We first compute the dot product

$$\vec{r}(t)\cdot\vec{r}'(t) = \left\langle \sqrt{t}, 2t+5, t \right\rangle \cdot \left\langle \frac{1}{2\sqrt{t}}, 2, 1 \right\rangle = \frac{\sqrt{t}}{2\sqrt{t}} + 2(2t+5) + t = \frac{1}{2} + 4t + 10 + t = 5t + \frac{21}{2}.$$

Then

$$\int \vec{r}(t) \cdot \vec{r}'(t)\, dt = \int 5t + \frac{21}{2}\, dt = \frac{5t^2}{2} + \frac{21t}{2} + C.$$

□