MAT 203 L24 Thu Dec 4

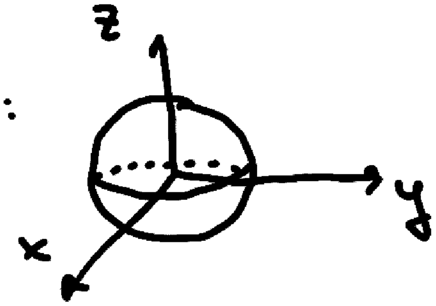
Final exam:

- Wed Dec 10, 2:15–5:00 pm
  - R01 & R02: Light Eng. 102
  - R03 & R04: Harriman Hall 137
- All sections (see schedule at course web page) except 3.3, 6.7, and 6.8.
- 9 Problems
- 1 Cheat Sheet allowed (1 physical sheet, front and back) like on midterm 2.

Selected practice problems:

Problem 49: Compute the surface integral of $\vec{F}(x,y,z) = \langle 1,1,z \rangle$ of the unit sphere centered at $(0,0,0)$.

Sol:



Parametrization:

$$\vec{r}(\theta,\varphi)=\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$$

$$0\le\theta\le 2\pi,\quad 0\le\varphi\le\pi$$

normal:

$$\vec{n}(\theta,\varphi)=\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\varphi}$$

$$=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

$$=\langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta\rangle$$

$$=\langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi\rangle$$

$$\int_S \vec{F}\cdot d\vec{r}=\int_0^{\pi}\int_0^{2\pi}\vec{F}(\vec{r}(\theta,\varphi))\cdot\vec{n}(\theta,\varphi)\,d\theta\,d\varphi$$

$$=\int_0^{\pi}\int_0^{2\pi}\langle 1,1,\cos\varphi\rangle\cdot\vec{n}(\theta,\varphi)\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} -\sin^2\varphi\cos\theta - \sin^2\varphi\sin\theta - \sin\varphi\cos^2\varphi \, d\theta\, d\varphi$$

$$= \int_0^{\pi} \Big[-\sin^2\varphi\sin\theta + \sin^2\varphi\cos\theta - \sin\varphi\cos^2\varphi\,\theta \Big]_0^{2\pi} d\varphi$$

All terms are zero except last which is:

$$-2\pi\int_0^{\pi} \sin\varphi\cos^2\varphi\, d\varphi = \left[\begin{array}{l} u=\cos\varphi \\ du=-\sin\varphi\, d\varphi \\ \varphi=0 \Rightarrow u=1 \\ \varphi=\pi \Rightarrow u=-1 \end{array}\right]$$

$$= -2\pi\int_1^{-1} -u^2\, du = -2\pi\int_{-1}^{1} u^2\, du$$

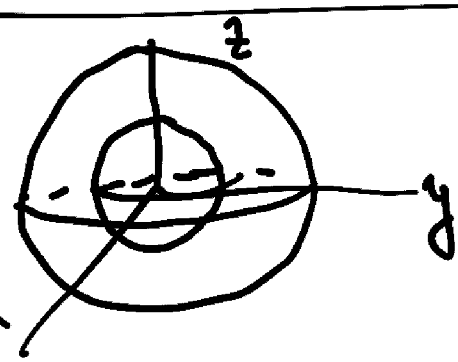
$$= -2\pi\left[\frac{u^3}{3}\right]_{-1}^{1} = -2\pi\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) = -\frac{4\pi}{3}.$$

Problem 42:

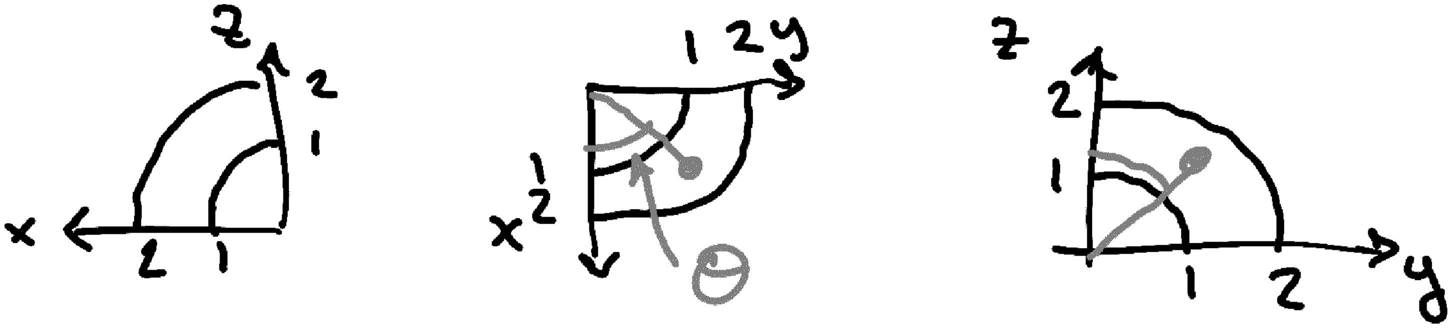
Compute $\iiint_B \frac{z\cos(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}\,dV$

where B is the region in the first octant bounded by the two spheres centered at the origin of radii 1 and 2.

Sol:



First octant is $x \geq 0$, $y \geq 0$, $z \geq 0$.



B in Spherical Coords:

$$\{(r,\theta,\varphi) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq \varphi \leq \tfrac{\pi}{2}\}$$

$x^2+y^2+z^2 = r^2, \quad z = r\cos\varphi$

$$\iiint_B \frac{z\cos\left(\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}}\,dU$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^2 \frac{r\cos\varphi\cos r}{r}\, r^2\sin\varphi\, dr\, d\varphi\, d\theta$$

$$= \left(\int_1^2 r^2\cos r\, dr\right)\underbrace{\left(\int_0^{\pi/2} d\theta\right)}_{=\pi/2}\left(\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\, d\varphi\right)$$

$$\int_0^{\pi/2}\cos\varphi\sin\varphi\, d\varphi = \int_0^{\pi/2}\frac{1}{2}\sin(2\varphi)\, d\varphi$$

$$= \frac{1}{2}\left[-\frac{\cos(2\varphi)}{2}\right]_0^{\pi/2} = -\frac{1}{2}\left(\frac{(-1)}{2}-\frac{1}{2}\right) = \frac{1}{2}$$

Finally   Partial integration

$$\int_1^2 r^2\cos r\, dr = \left[r^2\sin r\right]_1^2 - 2\int_1^2 r\sin r\, dr$$

$$= \left[r^2\sin r\right]_1^2 - 2\left(-\left[r\cos r\right]_1^2 + \int_1^2 \cos r\, dr\right)$$

$$= \left[r^2\sin r + 2r\cos r - 2\sin r\right]_1^2$$

$$= \left(4\sin(2) + 4\cos(2) - 2\sin(2)\right) - \left(\sin(1) + 2\cos(1) - 2\sin(1)\right)$$

$= 2\sin(2) + 4\cos(2) + \sin(1) - 2\cos(1)$

$= A$ (just some number)

So $\displaystyle\iiint_B \frac{z\cos(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}\, dV = \frac{A\pi}{4}.$

$\boxed{20}$ Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$?

<u>Sol</u>: Need to compute

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$. Note

$$\left|\frac{x^2y}{x^2+y^2}\right| \leq \left|\frac{(x^2+y^2)y}{x^2+y^2}\right| = |y| \to 0$$

So $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$ & f is continuous.