MAT 203 L23 Tue Dec 2

Recall: $\vec{F}(x,y,z) = \langle P, Q, R \rangle$

- Divergence:

$$(\nabla \cdot \vec{F})(x,y,z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- Curl:

$$(\nabla \times \vec{F})(x,y,z) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

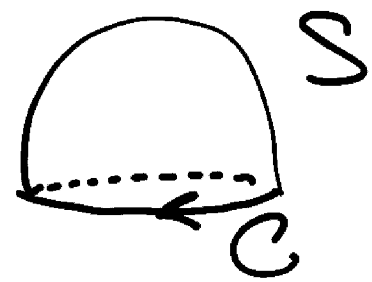
$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

- S surface with parametrization $\vec{r}(s,t)$, $(s,t) \in D$, then

$$\boxed{\iint_S \vec{F} \cdot d\vec{r} = \iint_D \vec{F}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) ds\,dt}$$

§6.7 Stokes' theorem

Let S be a surface with boundary curve C.



Assume S is parametrized by $\vec{r}(s,t)$, and that $\vec{F}$ is a vector field. Then

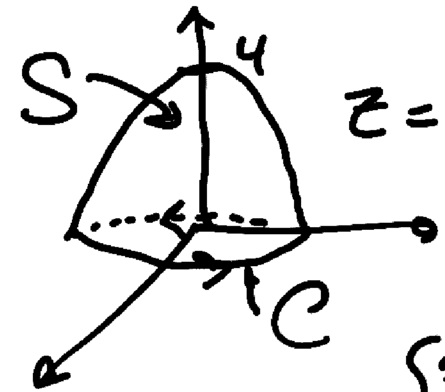
$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{r}}$$

Ex: Consider the vector field $\vec{F}(x,y,z) = \langle y, 2z, x^2\rangle$. Let S be the part of the paraboloid

$z = 4 - x^2 - y^2$ with $z \geq 0$,

with outwards normal. Let's verify Stokes' theorem in this case.

<u>Sol</u>:



$z = 4 - x^2 - y^2.$

Boundary curve: $\begin{cases} z = 4 - x^2 - y^2 \\ z = 0 \end{cases}$

$\Leftrightarrow \{x^2 + y^2 = 4, \ z = 0\}$

This is the circle centered at O with radius 2, with counterclockwise orientation by the right hand rule.

Parametrize the curve by

$$\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0 \rangle, \quad 0 \le \theta \le 2\pi$$

Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\, dt$$

$$= \int_0^{2\pi} \langle 2\sin\theta, 0, 4\cos^2\theta \rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} -4\sin^2\theta \, d\theta = -4\int_0^{2\pi} \frac{1-\cos(2\theta)}{2} d\theta$$

$$= -4\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{2\pi}$$

$$= -4\left(\frac{2\pi}{2}\right) = -4\pi.$$

Now let's compute the surface integral.

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{r}.$$

$$(\nabla \times \vec{F})(x,y,z) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & x^2 \end{vmatrix}$$

$$= \langle -2, -2x, -1 \rangle.$$

Parametrize the surface by

$$\vec{r}(x,y) = \langle x, y, 4-x^2-y^2 \rangle,$$

$$(x,y) : 4-x^2-y^2 \geq 0 \iff x^2+y^2 \leq 4.$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{r} = \iint_D (\nabla \times \vec{F})(\vec{r}(x,y)) \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right) dx\,dy$$

$$= \iint_D \langle -2, -2x, -1 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx\,dy$$

$$= \iint_D -4x - 4xy - 1 \, dx\,dy = (*)$$

$D = \{ (x,y) \mid x^2 + y^2 \leq 4 \}$. Let's Change to polar coords:

$$D = \{ (r, \theta) \mid 0 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi \}.$$

$$(*) = \int_0^{2\pi} \int_0^2 (-4r\cos\theta - 4r^2 \sin\theta\cos\theta - 1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[-\frac{4r^3}{3}\cos\theta - r^4 \sin\theta\cos\theta - \frac{r^2}{2} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} -\frac{32}{3}\cos\theta - 16\sin\theta\cos\theta - 2 \, d\theta$$

$$= \left[-\frac{32}{3}\sin\theta - 2\theta\right]_0^{2\pi} - 16\int_0^{2\pi} \sin\theta\cos\theta \, d\theta$$

$$= \left[-4\pi\right] - 16\underbrace{\left[\sin^2\theta\right]_0^{2\pi}}_{=0} = -4\pi.$$

So it matches the line integral computation above!

Green's theorem

$\vec{F} = \langle P, Q \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy$$

is in fact a special case of Stokes' theorem, since

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the last component of the curl of $\langle P, Q, 0 \rangle$.

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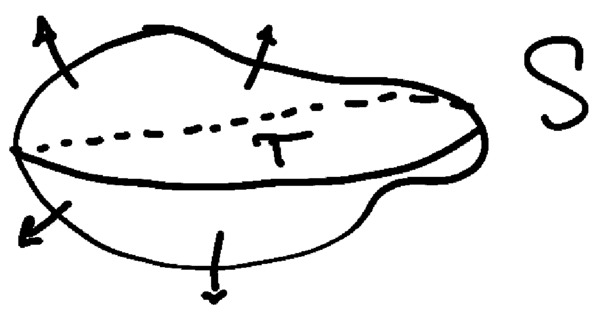
§6.8 Divergence theorem

Finally we discuss a 3-dimensional version of Green's theorem.
The divergence theorem is sometimes called Gauss' theorem.

Suppose S is a surface, with no boundary curve, in space that encloses a domain T.

eg. 

We assume S is oriented with the outwards normal.

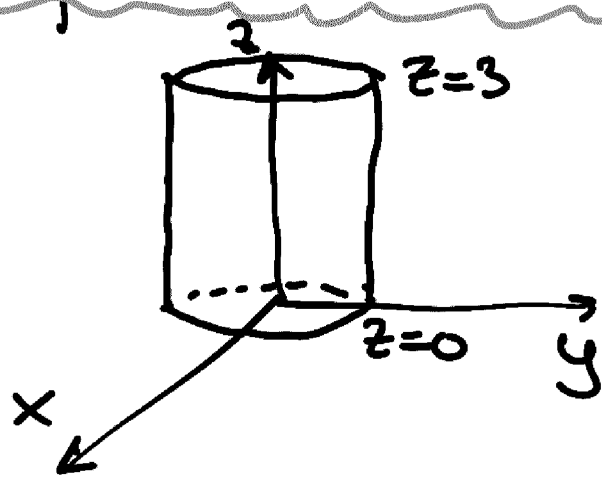
If $\vec{F}$ is a vector field, then

$$\boxed{\iint_S \vec{F}\cdot d\vec{r} = \iiint_T \nabla\cdot\vec{F}\, dx\,dy\,dz}$$

Ex: Let's verify the divergence theorem in the following example:

$\vec{F}(x,y,z) = \langle x+y+z,\ y,\ 2x-y\rangle$, and the surface S is the cylinder

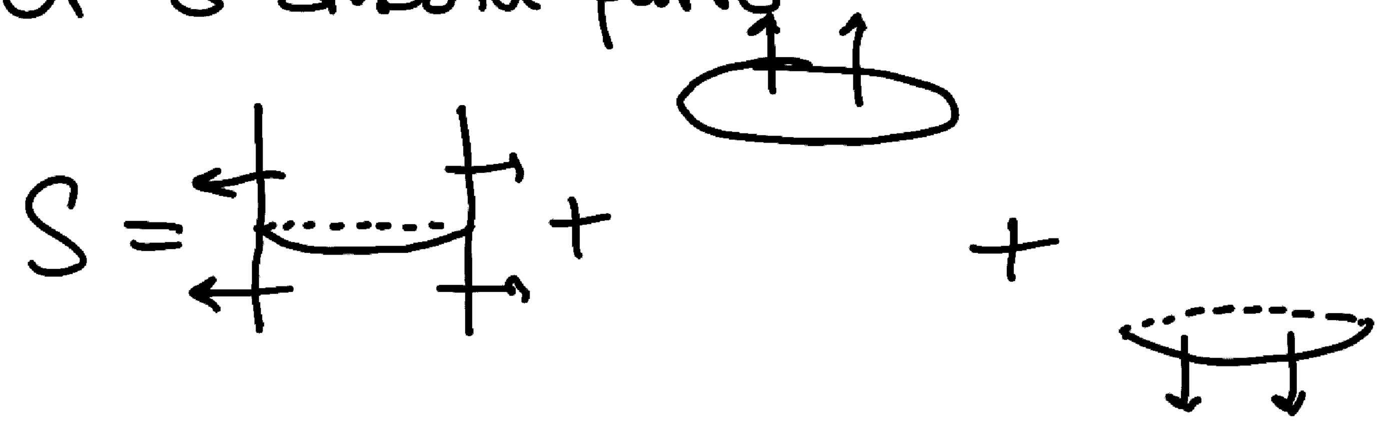
$x^2+y^2=1,\ 0\le z\le 3$ plus the top & the bottom:



We first compute the surface integral:

Because the surface consists

of 3 smooth parts



S_1 $\qquad$ S_2 $\qquad$ S_3

We need to split the integral up into pieces:

$$\boxed{\iint_{S_1} \vec{F} \cdot d\vec{r}}$$

We parametrize the cylinder:

$r = 1$ on the surface, so

$$\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$$

$$0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\iint_{S_1} \vec{F}(\vec{r}(\theta,z)) \cdot \left(\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z}\right) d\theta\, dz$$

$$= \int_0^3 \int_0^{2\pi} \langle \cos\theta + \sin\theta + z, \sin\theta, 2\cos\theta - \sin\theta \rangle$$

$$\cdot \langle \cos\theta, \sin\theta, 0 \rangle \, d\theta\, dz$$

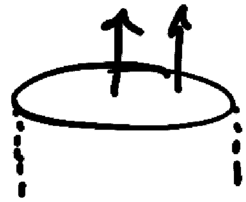
$$= \int_0^3 \int_0^{2\pi} \cos^2\theta + \sin\theta\cos\theta + z\cos\theta + \sin^2\theta \, d\theta\, dz$$

$$= \int_0^3 \int_0^{2\pi} 1 + \frac{1}{2}\sin(2\theta) + z\cos\theta \quad d\theta\, dz$$

$$= \int_0^3 \left[\theta - \frac{\cos(2\theta)}{4} + z\sin\theta\right]_0^{2\pi} dz$$

$$= \int_0^3 2\pi\, dz = 6\pi$$

Second part: $\boxed{\iint_{S_2} \vec{F} \cdot d\vec{r}.}$

Parametrization:  $z=3$

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 3\rangle$$

$$0 \le r \le 1,\ 0 \le \theta \le 2\pi$$

$$\text{Normal} = \langle 0,0,1\rangle$$

$$\iint_{S_3} \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\int_0^1 \vec{F}(\vec{r}(r,\theta))\cdot\langle 0,0,1\rangle\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \langle r\cos\theta + r\sin\theta + 3,\ r\sin\theta,\ 2r\cos\theta - r\sin\theta\rangle \cdot \langle 0,0,1\rangle\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r(2\cos\theta - \sin\theta)\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} 2\cos\theta - \sin\theta\, d\theta\right)\underbrace{\left(\int_0^1 r\, dr\right)}_{=1/2}$$

$$= \frac{1}{2}\underbrace{\Big[-2\sin\theta - \cos\theta\Big]_0^{2\pi}}_{=0} = 0$$

A very similar computation shows

$$\iint_{S_3} \vec{F} \cdot d\vec{r} = 0 .$$

So $\displaystyle\iint_S \vec{F} \cdot d\vec{r} = \iint_{S_1} \vec{F} \cdot d\vec{r} + \iint_{S_2} \vec{F} \cdot d\vec{r} + \iint_{S_3} \vec{F} \cdot d\vec{r}$

$$= 6\pi + 0 + 0 = 6\pi$$

Next, let's compute

$\displaystyle\iiint_B \nabla \cdot \vec{F} \, dx\,dy\,dz$, where

B = inside of the cylinder

$$(\nabla \cdot \vec{F})(x,y,z) = \frac{\partial(x+y+z)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(2x-y)}{\partial z}$$

$$= 1 + 1 = 2.$$

So $\displaystyle\iiint_B \nabla \cdot \vec{F} \, dx\,dy\,dz = 2 \cdot \mathrm{Vol}(\text{Cylinder}).$

$$= 2 \cdot (\pi \cdot 1^2 \cdot 3) = 6\pi .$$