MAT 203 L21 Thu Nov 20

<u>Recall:</u>

- C = curve w/ parametrization $\vec{r}(t)$, $a \le t \le b$.

  $\vec{F}$ = vector field.

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

<u>EX</u>: Let $\vec{F}(x,y,z) = \langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz \rangle$ and C is a curve with parametrization $\vec{r}(t) = \langle t^2, t, t \rangle$, $1 \le t \le e$.

Then let's compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^e \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt.$$

$$\vec{F}(\vec{r}(t)) = \vec{F}(t^2, t, t)$$

$$= \langle 2t^2 \ln(t), t^3 + t^2, 2t^2 \rangle$$

$\vec{r}_e'(t) = \langle 2t, 1, 1 \rangle$. Then

$$\int_1^e \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_1^e \langle 2t^2 \ln(t), t^3 + t^2, 2t^2 \rangle \cdot \langle 2t, 1, 1 \rangle \, dt$$

$$= \int_1^e 4t^3 \ln(t) + t^3 + t^2 + 2t^2 \, dt$$

$$= \left[\frac{t^4}{4} + t^3\right]_1^e + \int_1^e 4t^3 \ln(t)\, dt$$

$$= \left[\frac{e^4 - 1}{4} + e^3 - 1\right] + \left[t^4 \ln(t)\right]_1^e$$

$$- \int_1^e t^4 \cdot \frac{1}{t}\, dt = \left(\frac{e^4-1}{4} + e^3 - 1\right)$$

$$+ \left(e^4 \overbrace{\ln(e)}^{=1}\right) - \left[\frac{t^4}{4}\right]_1^e$$

$$= \left(\cancel{\frac{e^4-1}{4}} + e^3 - 1\right) + e^4 - \left(\cancel{\frac{e^4-1}{4}}\right)$$

$$= e^4 + e^3 - 1.$$

§6.3 Conservative Vector Fields

Recall that a vector field $\vec{F}$ is Conservative if $\vec{F} = \nabla f$ for some function f called a potential.

Also recall from Calc II:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

(fundamental theorem of Calculus)

Thm (Fundamental theorem of line integrals)

Let C be a curve with parametrization $\vec{r}(t)$, $a \le t \le b$, and let f be a differentiable function. Then

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

<u>Ex</u>: Let's Consider the previous example with

$$\vec{F}(x,y,z) = \langle 2x\ln(y), \frac{x^2}{y} + z^2, 2yz \rangle.$$

Is this vector field Conservative?

$$\vec{F} = \nabla f \iff \begin{cases} \frac{\partial f}{\partial x} = 2x\ln(y) & (1) \\ \frac{\partial f}{\partial y} = \frac{x^2}{y} + z^2 & (2) \\ \frac{\partial f}{\partial z} = 2yz & (3) \end{cases}$$

Integrate 1^{st} eq: $f(x,y,z) = x^2\ln(y) + g(y,z)$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} + \frac{\partial g}{\partial y} \overset{(2)}{=} \frac{x^2}{y} + z^2$$

$$\Rightarrow \frac{\partial g}{\partial y} = z^2 \Rightarrow g(y,z) = yz^2 + h(z).$$

So $f(x,y,z) = x^2\ln(y) + yz^2 + h(z)$.

$$\frac{\partial f}{\partial z} = 2yz + h'(z) \overset{(3)}{=} 2yz$$

$$\iff h'(z) = 0 \Rightarrow h(z) = C$$

So $f(x,y,z) = x^2 \ln(y) + yz^2 + C$

is a potential:

$$\vec{F} = \nabla f.$$

Therefore

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

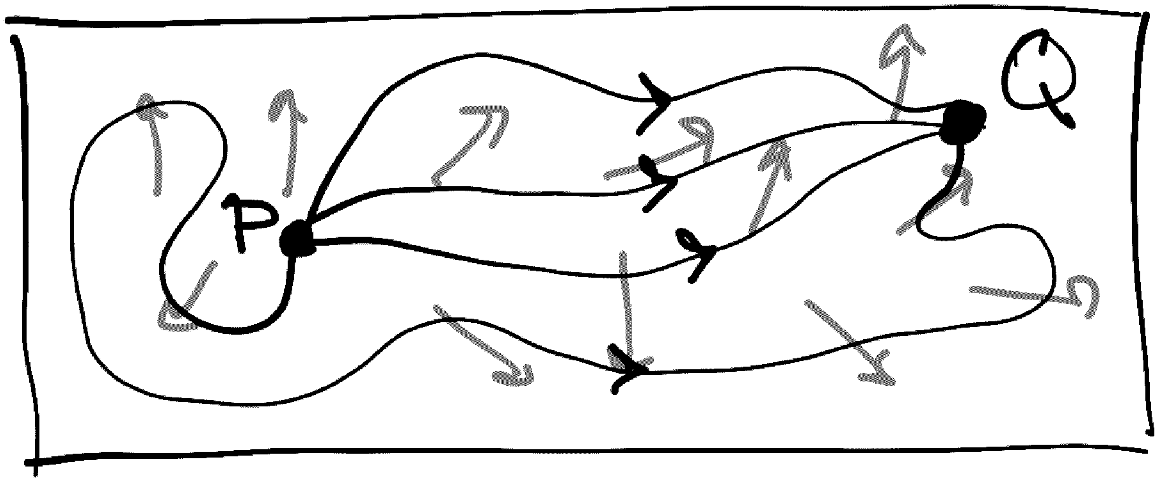
$$= f(\vec{r}(e)) - f(\vec{r}(1))$$

$$= f(e^2, e, e) - f(1,1,1)$$

$$= \left(e^4 \ln(e) + e^3 + C\right) - \left(1 \overbrace{\ln(1)}^{=0} + 1 + C\right)$$

$$= e^4 + e^3 - 1.$$

If $\vec{F}$ is conservative, it's only the endpoints of the path that matters for $\int_C \vec{F} \cdot d\vec{r}$.



Only the value at the potential f of $\vec{F}$ at endpts P and Q matter for $\int_C \nabla f \cdot d\vec{r}$

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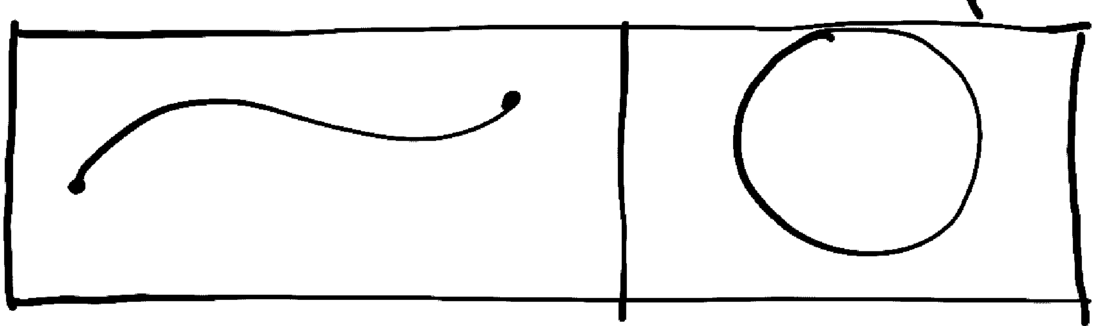
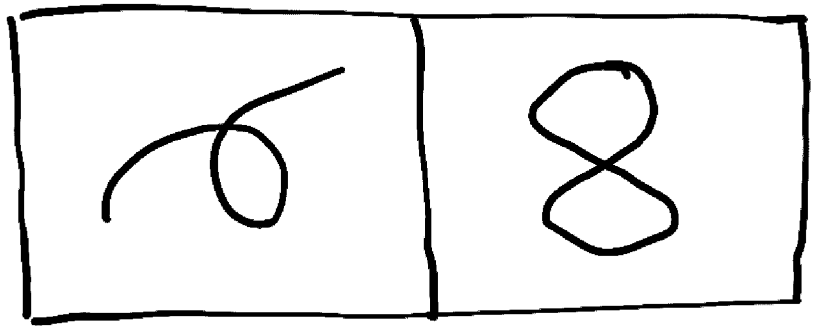
§6.4 Green's theorem

Def: A curve C in the plane is simple if it never intersects itself.

EX: These curves are simple:



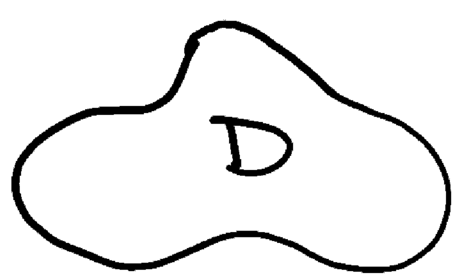
These are not:



Def: A domain in the xy-plane is called simply connected if "there are no holes."

Remark: The correct definition is that D is simply connected if any closed loop in D can be continuously shrunk to a point while remaining completely in D.

Ex: This domain is simply connected:



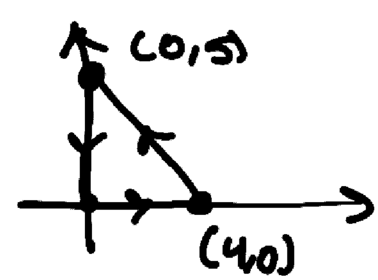
This is not:

Theorem (Green's theorem)

Let D be a simply connected domain, with boundary curve C that is simple, and oriented counterclockwise. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ be a vector field. Then

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy}$$

Ex: Consider $\int_C \vec{F} \cdot d\vec{r}$ where

C is the curve 

and $\vec{F}(x,y) = \langle y^2, 2xy+1 \rangle$.

Last time we computed $\int_C \vec{F} \cdot d\vec{r} = 0$.

Let's now compute it using

Green's theorem instead.

$$\langle y^2, 2xy+1\rangle = \langle P(x,y), Q(x,y)\rangle$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 2y, \text{ so}$$

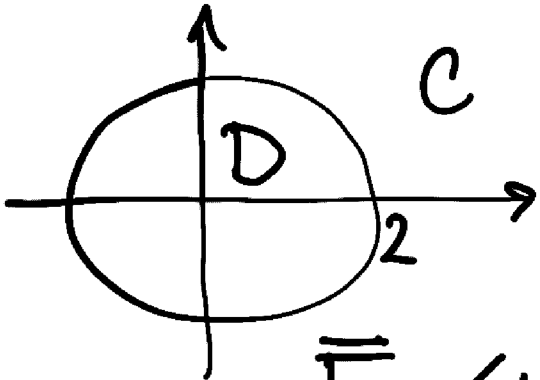
$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0}\, dx\,dy = 0.$$

<u>Ex</u>: Let $\vec{F}(x,y) = \langle y+\sin x, e^y + x^2\rangle$ and let C be the unit circle with radius 2, oriented counter-clockwise.

Let's calculate $\int_C \vec{F}\cdot d\vec{r}$.

You can try to calculate it directly via the definition, but it seems close to impossible.

We will use Green's theorem instead:



$D = \{(x,y) \mid x^2+y^2 \leq 4\}$.

$\vec{F} = \langle y+\sin x,\ e^y + x^2 \rangle = \langle P, Q \rangle$

$\frac{\partial Q}{\partial x} = 2x \qquad \frac{\partial P}{\partial y} = 1$

so $\int_C \vec{F} \cdot d\vec{r} = \iint_D 2x - 1 \, dx\,dy$.

Change to polar coords:

$D = \{(r,\theta) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$

$$\iint_D 2x-1 \, dx\,dy = \int_0^{2\pi}\int_0^2 (2r\cos\theta - 1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{2r^3}{3}\sin\theta - \frac{r^2}{2}\right]_0^2 d\theta = \int_0^{2\pi} \frac{16}{3}\sin\theta - 2\, d\theta$$

$$= \left[-\frac{16}{3}\cos\theta - 2\theta\right]_0^{2\pi}$$

$$= -\frac{16}{3}(1-1) - 4\pi = -4\pi$$